Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 23, Summer 1998, 45–74

STRONGLY EXTRARESOLVABLE GROUPS AND SPACES

W. W. Comfort * and Salvador García-Ferreira †

Abstract

Throughout this Abstract, $X = \langle X, \mathcal{T} \rangle$ [resp., $G = \langle G, \mathcal{T} \rangle$] denotes a Tychonoff space [resp., a Tychonoff topological group].

Notation. (a) $\Delta(X) = \min\{|U| : \emptyset \neq U \in \mathcal{T}\}$;
(b) $nwd(X) = \min\{|A| : A \subseteq X, \text{int}_X \, \text{cl}_X A \neq \emptyset\}$;
(c) $iw(X) = \min\{w(Y)$: there is a continuous bijection $f : X \to Y\}$;
(d) $tb(G) = \min\{\kappa : \emptyset \neq U \in \mathcal{T} \Rightarrow$ there is $F \in [G]^{<\kappa}$ such that $G = FU\}$;
(e) X is *strongly extraresolvable* if there is a family $\mathcal{D}$ of dense subsets of X such that $|\mathcal{D}| > \Delta(X)$ and $|D \cap E| < nwd(X)$ whenever D and E are distinct elements of $\mathcal{D}$.

Theorems. (A) Every product of infinitely many nontrivial separable metric spaces admits a dense strongly extraresolvable subspace.
(B) If $|G|^{<tb(G)} = nwd(G) \geq \omega$, then G is strongly extraresolvable. Hence every totally bounded topological group such that $|G| = d(G) \geq \omega$ is strongly extraresolvable, every infinite totally bounded topological group admits a dense strongly extraresolv-

* This work was written in part during the visit of the second-listed author to the Department of Mathematics of Wesleyan University, during the period September, 1997—March, 1998.

† This author acknowledges with thanks the generous hospitality and support received from the Department of Mathematics of Wesleyan University.

able subgroup, and every totally bounded topological group with no proper dense subgroup is strongly extraresolvable.
(C) If $iw(X) = (iw(X))^\omega$, then $C_p(X)$ has a dense, strongly extraresolvable subgroup.
(D) Every nondiscrete locally compact group has a dense extraresolvable subset.

0. The Historical Context

It was Hewitt [26] who defined and initiated the study of what he called *resolvable* spaces, that is, spaces which admit two disjoint dense subsets. He showed [26] that every locally compact Hausdorff space without isolated points, and every metric space without isolated points, is resolvable; and every Tychonoff topology without isolated points on a set X expands to an irresolvable Tychonoff topology without isolated points on X. After Hewitt, Ceder [4] proposed the following definitions and initiated the study of their implications: For a (possibly infinite) cardinal number κ, a space is *κ-resolvable* if it admits κ-many pairwise disjoint dense subsets; and a space X is *maximally resolvable* if X is κ-resolvable with $\kappa = \Delta(X)$; here as usual $\Delta(X)$, the so-called *dispersion character* of X, is the cardinal number

$$\Delta(X) = \min\{|U| : U \text{ is open in } X,\, U \neq \emptyset\}.$$

In this notation and terminology, the following results are known: (1) for $n < \omega$ there are Tychonoff spaces $X(n)$ which are n-resolvable but not $(n+1)$-resolvable [18], [15], [22], [16]; (2) if X is n-resolvable for each $n < \omega$ then X is ω-resolvable [27]; (c) if

Mathematics Subject Classification: Primary 22A05, 54A05, 54H11
Key words: topological group, locally compact topological group; totally bounded topological group, total boundedness number; weight, density character; resolvable space, extraresolvable space, strongly extraresolvable space, κ-resolvable space

a group G admits a Hausdorff totally bounded group topology, then G is maximally resolvable in each such topology.

For further remarks on the historical development of the subject prior to 1996, the reader may consult our earlier work [8].

Malykhin [31] proposed the next substantial generalization of Hewitt's concept; he used the term *extraresolvable* to refer to spaces X with a family $\mathcal{D}$ of dense subsets such that $|\mathcal{D}| > \Delta(X)$ and every two members of $\mathcal{D}$ have nowhere dense intersection. Malykhin [31] showed that for Y analytic in a compact Hausdorff space, every countable subspace of $C_p(Y)$, the set of continuous real-valued functions on Y with the topology inherited from $\mathbb{R}^Y$, is extraresolvable; and together with co-authors [23] he observed that compact metric spaces, and compact groups, are not extraresolvable (although such spaces are maximally resolvable [17]).

Many of the spaces cited in our List of References, and the papers which they themselves cite, are devoted in part to the study of maximally resolvable and extraresolvable spaces. The following definition, clearly closely related to that of Malykhin [31], suggests a partition of these classes (into the classes of spaces which are, and those which are not, strongly extraresolvable). This concept enables us to see *why* it is that certain extraresolvable spaces are indeed extraresolvable.

Definition 0.1. A space X is *strongly extraresolvable* if there is a family $\mathcal{D}$ of dense subsets of X with $|\mathcal{D}| > \Delta(X)$ such that distinct members E and F of $\mathcal{D}$ satisfy $|E \cap F| < nwd(X)$, with $nwd(X)$ the cardinal number given by

$$nwd(X) = \min\{|A| : A \subseteq X, \operatorname{int}_X \operatorname{cl}_X A \neq \emptyset\}.$$

(We call such a family $\mathcal{D}$ a *strongly extraresolvable* family of subsets of X. Similarly a family of dense sets witnessing the extraresolvability of a space X is called an *extraresolvable* family on X.)

In the present paper we strengthen and extend certain results from the literature. For example, it was shown in [9] that every infinite totally bounded topological group G contains a dense strongly extraresolvable subgroup (in particular, every infinite compact Hausdorff group admits a dense strongly extraresolvable subgroup); here we achieve the same conclusion for every group G such that $tb(G) \leq (nwd(G))^+$ and $d(G) = (d(G))^{<tb(G)}$. Another example: That every space of the form $C_p(\beta(\alpha))$ is extraresolvable (with α discrete, $\alpha \geq \omega$) is known [1]; here we show that $C_p(X)$ is even strongly extraresolvable, not only for X of the form $X = \beta(\alpha)$ but for every (Tychonoff) space X satisfying $iw(X) = (w(\beta(X)))^\omega$. In Section 4 we analyze the effect on strong extraresolvability of passage from a space X to a space of the form $X \times A$, and conversely, when A is discrete. And in Section 5 we show that every nondiscrete locally compact group contains a dense extraresolvable subset, but not necessarily a dense extraresolvable subgroup.

Remark 0.2. Some of the results of this and of our related manuscript [9] were announced at the URLs www.unipissing.ca /topology/i/a/a/c/53.htm and www.unipissing.ca/topology /i/a/a/c/55.htm of the Topology Atlas website.

Acknowledgement 0.3. We acknowledge with thanks several helpful comments and suggestions received from Li Feng and Oscar Masaveu.

1. Notation, Terminology, Preliminaries

In this section we define notation and certain terms, and we record some preliminary results.

For X a set and κ a cardinal, we write as usual $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$; the notations $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined similarly.

For a space X, the cardinal numbers $\Delta(X)$ and $nwd(X)$ were defined in section 0. (We note that every set $A \in [X]^{<nwd(X)}$ is nowhere dense in X.) We denote by $d(X)$, $\pi w(X)$, and by $w(X)$ the density character, the π-weight, and the weight of X, respectively.

For a space X and $A \subseteq X$ we often write $\overline{A}^X$ in place of $\mathrm{cl}_X A$.

Lemma 1.1. Let X be a space. Then $nwd(X) \leq d(X)$ and $nwd(X) \leq \Delta(X)$. □

We note that the inequalities in Lemma 1.1 can be strict. For example, the real line $\mathbb{R}$ satisfies $nwd(\mathbb{R}) = \omega < \Delta(\mathbb{R}) = \mathfrak{c}$, and if Y is a separable space and Z is not separable, then the topological sum $X = Y \oplus Z$ satisfies $nwd(X) = \omega < d(X)$.

Discussion 1.2. We say as in [10] that a topological group G is *totally α-bounded* if for every nonempty open subset U of G there is $F \in [G]^{<\alpha}$ such that $G = UF$. (In this terminology the closely related concept introduced independently by Guran [25] becomes total α^+-boundedness.) A totally ω-bounded group is said simply to be *totally bounded*. It is not difficult to prove that every subgroup G of a compact group is totally bounded. Less obvious is the converse statement, a theorem of Weil [35] not needed in this paper: Every totally bounded group G is dense in a compact group K, this latter being unique in an obvious sense if first G and then K are taken Hausdorff.

We define the *total boundedness number* of a topological group G as follows:

$$tb(G) = \min\{\alpha : G \text{ is totally } \alpha\text{-bounded}\}.$$

It is not difficult to see that

$$tb(G) \leq (d(G))^+ \leq \min\{|G|^+, w(G)^+\}$$

for every topological group G. Here are some elementary but useful facts about the total boundedness number. The first three of these are recorded in [34]. For (d), note that since $C_p(X)$ is dense in the c.c.c. space $\mathbb{R}^X$ it is itself a c.c.c. space, so that $tb(C_p(X)) \leq \omega_1$; then since $C_p(X)$ contains $\mathbb{R}$ as a topological subgroup and $tb(\mathbb{R}) = \omega_1$ one has $tb(C_p(X)) = \omega_1$.

(a) If H is a subgroup of G, then $tb(H) \leq tb(G)$.

(b) If H is a dense subgroup of G, then $tb(H) = tb(G)$.

(c) If $h : G \to H$ is a continuous homomorphism from G onto H, then $tb(H) \leq tb(G)$.

(d) (cf. [2], [29]) $tb(C_p(X)) = \omega_1$ for every Tychonoff space X.

The following simple statements are basic to our investigation.

Lemma 1.3. *Let X be a space.*

(a) *If U is a nonempty open subset of X, then $nwd(X) \leq nwd(U)$; and*

(b) *If Y is a dense subset of X, then $nwd(X) \leq nwd(Y)$.*

Lemma 1.4. *Every space X has a nonempty open subset U such that $d(V) = nwd(V)$ for every nonempty open subset V of X contained in U. If X is a topological group, then U may be chosen to be a subgroup of X.*

Proof. Let U be an open subset of X such that

$d(U) = \min\{d(W) : W \text{ is a nonempty open subset of} X\}$.

Clearly $d(V) = d(U)$ for every nonempty $V \subseteq U$. If for some such V there is $A \subseteq V$ such that $|A| < d(V)$ and $W := \mathrm{int_V cl_V A} \neq \emptyset$, then W is open in X and

$$d(W) \leq |A \cap W| \leq |A| < d(V) = d(U),$$

a contradiction. Hence $nwd(V) = d(V)$, as required.

To prove the second statement choose U as above and set $H = \langle U \rangle$. Every dense subset of U generates a dense subgroup of H, so since U is open in H we have $d(H) \leq d(U) \leq d(H)$ and hence

$$d(H) = d(U) = \min\{d(W) : W \text{ is a nonempty open subset of } X\}.$$

If some $A \subseteq H$ with $|A| < d(H)$ satisfies $W := \mathrm{int_H cl_H A} \neq \emptyset$ then since H is open in X we have W open in X and

$$d(W) \leq |W \cap A| < d(H) = d(U),$$

a contradiction. □

Theorem 1.5. *Let G be a topological group and let X be a space.*

(a) ([32]) *If $tb(G) \leq (\Delta(G))^+$, then $\Delta(G) = |G|$;*

(b) *if $tb(G) \leq (nwd(G))^+$, then $d(G) = nwd(G)$ and $\Delta(G) = |G|$; and*

(c) ([1]) *$d(C_p(X)) = nwd(C_p(X))$.*

Proof. (a) There is a nonempty open subset U of G such that $|U| = \Delta(G)$, and there is $F \in [G]^{\Delta(G)}$ such that $G = UF$.
(b) Let $A \subseteq G$ satisfy $|A| = nwd(G)$ and $\overline{A}^G \supseteq U$ with U nonempty and open. If $F \in [G]^{\leq nwd(G)}$ is chosen so that $G = UF$, then FA is dense in G, and from Lemma 1.1 follows

$$nwd(G) \leq d(G) \leq |FA| \leq nwd(G) \cdot nwd(G) = nwd(G).$$

(c) We give a proof different from the one in [1]. It is known that every nonempty open subset of $C_p(X)$ contains a open (basic) subset homeomorphic to the entire space $C_p(X)$ (see [29] and [2]). Now use Lemma 1.4 to find an open subset U of $C_p(X)$ such that $d(V) = nwd(V)$ for every nonempty open $V \subseteq U$, choose such V homeomorphic to $C_p(X)$ itself, and note that

$$d(C_p(X)) = d(V) = nwd(V) = nwd(C_p(X)). \qquad \square$$

We note in particular that the relations $\Delta(G) = |G|$ and $d(G) = nwd(G)$ hold for every totally bounded group G.

Except as noted, the remarks and definitions given so far are meaningful and valid for groups and spaces even when these are not subjected to any separation axioms. The interested reader will have no difficulty determining which of the theorems below apply in a similarly unrestricted setting (see Theorem 2.3 for an example). In the interest of simplicity, however, we assume henceforth that our hypothesized spaces are regular T_1-spaces. As is well known, in the context of topological groups the T_1 separation axiom ensures in addition the Tychonoff property.

The following familiar results are basic in the theory of compact groups; see for example [7] for appropriate references to the literature.

Theorem 1.6. *Let K be an infinite compact Hausdorff group with $w(K) = \alpha$. Then*

(a) $|K| = 2^\alpha$ *and* $d(K) = \log(\alpha)$*; and*

(b) *every dense subgroup G of K satisfies $w(G) = w(K)$.* □

Frequently for convenience, and without explicit mention, we assume in what follows that a hypothesized extraresolvable family $\mathcal{D}$ on a space X satisfies $|\mathcal{D}| = (\Delta(X))^+$. In connection with this convention we note the following simple result, which plays no formal role in our work.

Theorem 1.7. *Let X be a space and let $\mathcal{D}$ be an extraresolvable family on X. Then $|\mathcal{D}| \leq 2^{\Delta(X)}$.*

Proof. Suppose otherwise, and let U be an open subset of X such that $|U| = \Delta(X)$. The map $\mathcal{D} \to \mathcal{P}(U)$ given by $D \to D \cap U$ cannot be one-to-one, so there are distinct $D, E \in \mathcal{D}$ such that $D \cap U = E \cap U$. This yields

$$\emptyset \neq U \subseteq \overline{U} = \overline{D \cap E \cap U} \subseteq \overline{D \cap E},$$

contrary to the fact that $D \cap E$ is nowhere dense in X. □

2. Some Extraresolvable Spaces

The following lemma is taken from [9].

Lemma 2.1. *Let $\alpha \geq \omega$ and let $\{A_\xi : \xi < \alpha\}$ be a (faithfully indexed) family of pairwise disjoint sets with each $|A_\xi| = \alpha$ and let $X = \bigcup_{\xi<\alpha} A_\xi$. Then there is a family $\mathcal{E} \subseteq \mathcal{P}(X)$ such that*

(i) $|\mathcal{E}| = \alpha^+$*;*

(ii) $|E| = \alpha$ *for each* $E \in \mathcal{E}$*;*

(iii) $|E \cap A_\xi| = 1$ *whenever* $E \in \mathcal{E}$, $\xi < \alpha$;

and

(iv) $|D \cap E| < \alpha$ *whenever* $D \in \mathcal{E}$, $E \in \mathcal{E}$, *and* $D \neq E$.

The following special case of Theorem 2.1, needed in §3, is familiar (see for example [20](4.1) or [24](12.8 and Exercise 12.B)).

Corollary 2.2. *Let* X *be a set with* $|X| = \alpha \geq \omega$. *Then there is* $\mathcal{E} \subseteq [X]^\alpha$ *such that* $|\mathcal{E}| = \alpha^+$ *and* $|D \cap E| < \alpha$ *whenever* $D \in \mathcal{E}$, $E \in \mathcal{E}$, *and* $D \neq E$.

Proof. It is enough to choose a pairwise disjoint subfamily $\{A_\xi : \xi < \alpha\}$ of $[X]^\alpha$, and to apply Lemma 2.1. □

That every X satisfying $\pi w(X) \leq \Delta(X)$ is maximally resolvable is (a special case of) the fundamental observation of El′kin [17]; we prove (a) here as an aid to (b) and (c).

Theorem 2.3. *Let* X *be a space such that* $\omega \leq \pi w(X) \leq \Delta(X)$. *Then*

(a) X *is maximally resolvable;*

(b) *there is a family* $\mathcal{E}$ *of dense subsets of* X *such that* $|\mathcal{E}| = (\Delta(X))^+$ *and such that* $|D \cap E| < \Delta(X)$ *whenever* $D \in \mathcal{E}$, $E \in \mathcal{E}$, *and* $D \neq E$; *and*

(c) *if in addition* $nwd(X) = \Delta(X)$, *then* X *is strongly extraresolvable.*

Proof. Let $\{B_\xi : \xi < \Delta(X)\}$ be a not necessarily faithfully indexed π-base for X. Since each $|B_\xi| \geq \Delta(X)$, there is by the so-called disjoint refinement lemma a pairwise disjoint family $\{A_\xi : \xi < \Delta(X)\}$ of subsets of X such that $A_\xi \subseteq B_\xi$ and $|A_\xi| = \Delta(X)$ for each $\xi < \Delta(X)$. (See for example [12](7.5) for a proof of this familiar fact.) The family $\mathcal{F}$ defined in (the proof of) Lemma 2.1 witnesses (a); (b) is immediate from Lemma 2.1; and (c) is immediate from (b). □

Remark 2.4. An argument noted in [19] and [23] shows that if the condition $nwd(X) = \Delta(X)$ fails then a space X as in Theorem 2.3 can fail to be extraresolvable. Indeed for examples to this effect of arbitrary preassigned weight $\alpha \geq \omega$, let K be a compact group with $w(K) = \alpha \geq \omega$, so that

$$d(K) \leq w(K) < |K| = \Delta(K) = 2^\alpha$$

by Theorem 1.4(a). Then K is maximally resolvable (cf. [4] (Theorem 3) or [32] or Theorem 2.3(a) above). No family $\{E_\eta : \eta < (2^\alpha)^+\}$ of dense subsets of K can have $E_\eta \cap E_{\eta'}$ nowhere dense whenever $\eta < \eta' < (2^\alpha)^+$: For each E_η there is a dense set $D_\eta \in [E_\eta]^\alpha \subseteq [K]^\alpha$, and since $|[K]^\alpha| = (2^\alpha)^\alpha = 2^\alpha < (2^\alpha)^+$ there exist $\eta < \eta' < (2^\alpha)^+$ such that $D_\eta = D_{\eta'} \subseteq E_\eta \cap E_{\eta'}$, a contradiction. Similar reasoning shows that the real line $\mathbb{R}$ is not extraresolvable, although $\mathbb{R}$ is maximally resolvable (by Theorem 2.3).

Notation 2.5. In what follows, given a set $\{D_i : i \in I\}$ of sets and a distinguished point $p = \langle p_i : i \in I\rangle \in \Pi_{i\in I} D_i$, for $x \in \Pi_{i\in I} D_i$ we write $s(x) = \{i \in I : x_i \neq p_i\}$ and we set

$$\sigma(p, \Pi_{i\in I} D_i) = \oplus_{i\in I} D_i = \{x \in \Pi_{i\in I} D_i : |s(x)| < \omega\}.$$

(The set $s(x)$ is called the *support* of x.)

Theorem 2.6. *Let $\alpha \geq \omega$, let $\{Y_\xi : \xi < \alpha\}$ be a set of spaces without isolated points with each $|Y_\xi| > 1$ and $w(Y_\xi) \leq \alpha$, and set $Y = \Pi_{\xi<\alpha} Y_\xi$. Let D_ξ be dense in Y_ξ with $|D_\xi| \leq \alpha$, fix $p_\xi \in D_\xi$, and set $D = \sigma(p, \Pi_{\xi<\alpha} D_\xi)$ with $p = \langle p_\xi : \xi < \alpha\rangle$. Then*

(a) *D is a maximally resolvable and strongly extraresolvable dense subspace of Y; and*

(b) *if in addition each Y_ξ is a topological group with identity p_ξ, and if D_ξ is a subgroup of Y_ξ, then D is a topological subgroup of Y.*

Proof. (a) It being clear that D is dense in Y, it suffices to check that $w(D) = \Delta(D) = nwd(D) = \alpha$ (for then Theorem 2.3. applies, with D replacing X). That $w(D) = \Delta(D) = \alpha$ is clear. If $A \in [D]^{<\alpha}$ then with $S := \cup\{s(x) : x \in A\}$ we have $|S| < \alpha$, so there is $\xi \in \alpha\backslash S$; since A and hence $\overline{A}^D$ is contained in the (closed) nowhere dense subset $\pi_\xi^{-1}(\{p_\xi\}) \cap D$ of D we conclude that A itself is nowhere dense in D. Thus $\alpha \leq nwd(D) \leq |D| = \alpha$, as required.

(b) is obvious. □

Corollary 2.7. *Let $\alpha \geq \omega$, let $\{X_\eta : \eta < \alpha\}$ be a set of spaces with each $|X_\eta| > 1$ and $w(X_\eta) \leq \alpha$, and set $X = \Pi_{\eta<\alpha} X_\eta$. Then*

(a) *X has a maximally resolvable and strongly extraresolvable dense subspace D; and*

(b) *if in addition each X_η is a topological group then D may be chosen to be a topological subgroup of X.*

Proof. Let $\{A_\xi : \xi < \alpha\}$ be a partition of α into α-many infinite sets, for $\xi < \alpha$ set $Y_\xi = \Pi_{\eta\in A_\xi} X_\eta$, and apply Theorem 2.6 to the set $\{Y_\xi : \xi < \alpha\}$. □

Corollary 2.7 applies to many product spaces. We content ourselves with recording an important special case.

Corollary 2.8. (a) *Every product of infinitely many nontrivial separable metric spaces admits a dense subspace which is both maximally resolvable and strongly extraresolvable; and*

(b) *Every product of infinitely many nontrivial separable metrizable topological groups admits a dense subgroup which is both maximally resolvable and strongly extraresolvable.* □

Remark 2.9. It is a principal result of the paper [9] that every infinite totally bounded topological group G such that $|G| = d(G)$ is strongly extraresolvable. We shall extend this result in Theorem 3.4 below. Thus from the work of Malykhin and Protasov [32] it follows that every infinite totally bounded topological group G such that $d(G) = w(G) = |G|$ is maximally resolvable and strongly extraresolvable. We remark that groups G as in the statement exist in profusion. For an example of arbitrary preassigned cardinality $\alpha \geq \omega$ let $G = \oplus_{\xi<\alpha} \mathbb{Z}_2$ with $\mathbb{Z}_2 = \{+1, -1\}$ the two-element multiplicative Abelian group with identity $+1$, and give G the topology inherited from $\mathbb{Z}_2^\alpha$. Clearly $d(G) = |G| = \alpha$, and $w(G) = \alpha$ follows from Theorem 1.6(b). □

3. Totally Bounded Groups

For a group G and an infinite cardinal α, throughout this section for notational convenience we write

$$\mathcal{F} = \mathcal{F}_\kappa = \mathcal{F}_\kappa(G) = \{F \in [G]^{<\kappa} : F = F^{-1} \neq \emptyset\}.$$

In the following three preliminary lemmas we revisit the disjoint refinement lemma and arguments given by Malykhin and Protasov [32].

Lemma 3.1. *For each topological group G and for each infinite cardinal κ, the following conditions are equivalent.*

(a) *G is totally κ-bounded; and*

(b) *for each function $a \in G^{\mathcal{F}_\kappa}$ the set $D := \cup\{a_F \cdot F : F \in \mathcal{F}\}$ is dense in G.*

Proof. (a) ⇒ (b). For each nonempty open subset U of G there is $F \in \mathcal{F}_\kappa$ such that $U \cdot F = G$. Then with $a_F \in uF$ $(u \in U)$ we have $u \in (a_F \cdot F^{-1}) \cap U = (a_F \cdot F) \cap U \subseteq D \cap U$, as required.

(b) ⇒ (a). If (a) fails then for some nonempty open subset U of G, for every $F \in \mathcal{F}_\kappa$ there is $a_F \in G \backslash (UF)$. Then from (b) there is $F \in \mathcal{F}_\alpha$ such that $U \cap (a_F \cdot F) \neq \emptyset$ and we have the contradiction $a_F \in UF^{-1} = UF$. □

Lemma 3.2. *Let $\alpha \geq \kappa \geq \omega$ and let $\alpha = \alpha^{<\kappa}$. Then*

(a) *either $\kappa < \alpha$ or $\kappa = \alpha$ and α is regular; and*

(b) *if G is a group such that $|G| = \alpha$ and if $S = A \cup (\cup\{a \cdot F_a : a \in A\})$ with $A \in [G]^{<\alpha}$ and $\{F_a : a \in A\} \subseteq [G]^{<\kappa}$, then $|S| < \alpha$.*

Proof. If (a) fails then $\kappa = \alpha$ and α is singular and we have the contradiction

$$\alpha = \alpha^{<\kappa} = \alpha^{<\alpha} \geq \alpha^{\mathrm{cf}(\alpha)} > \alpha.$$

Statement (b) is then immediate from (a). □

The next Lemma and Theorem are generalizations of Lemma 2.3 and Theorem 2.4 from [9], respectively.

Lemma 3.3. *Let $\alpha \geq \kappa \geq \omega$ and let G be a group such that $|G| = |G|^{<\kappa} = \alpha$. Then for every indexing $\{F_\xi : \xi < \alpha\}$ of $\mathcal{F}_\kappa(G)$ there is a family $\{A_\xi : \xi < \alpha\} \subseteq \mathcal{P}(G)$ such that*

(i) *each $\xi < \alpha$ satisfies $|A_\xi| = \alpha$;*

(ii) *distinct $\xi, \xi' < \alpha$ satisfy $A_\xi \cap A_{\xi'} = \emptyset$;*

(iii) *distinct $\xi, \xi' < \alpha$ satisfy $A_\xi \cdot F_\xi \cap A_{\xi'} \cdot F_{\xi'} = \emptyset$; and*

(iv) *distinct $a, a' \in A_\xi$ satisfy $(a \cdot F_\xi) \cap (a' \cdot F_\xi) = \emptyset$.*

Proof. Write $L := \{\langle \eta, \xi \rangle : \xi \leq \eta < \alpha\}$ and give L the usual lexicographic ordering: $\langle \eta', \xi' \rangle < \langle \eta, \xi \rangle$ if either (a) $\eta' < \eta$, or (b) $\eta' = \eta$ and $\xi' < \xi$. By recursion, for $\langle \eta, \xi \rangle \in L$ we will define $S(\eta, \xi) \subseteq G$ and $a_{\eta,\xi} \in G$.

Let $S(0, 0) = \emptyset$ and let $a_{0,0}$ be arbitrary in G.

Now let $\langle\eta,\xi\rangle \in L$, suppose that $S(\eta',\xi')$ and $a_{\eta',\xi'}$ have been defined for all $\langle\eta',\xi'\rangle < \langle\eta,\xi\rangle$, and define

$$S(\eta,\xi) = \{a_{\eta',\xi'} : \langle\eta',\xi'\rangle < \langle\eta,\xi\rangle\} \cup \\ (\cup\{a_{\eta',\xi'} \cdot F_{\xi'} \cdot F_\xi : \langle\eta',\xi'\rangle < \langle\eta,\xi\rangle\}).$$

It follows from Lemma 3.2, taking $A = \{a_{\eta',\xi'} : \langle\eta',\xi'\rangle < \langle\eta,\xi\rangle\}$ and $F_a = F_{\xi'} \cup F_\xi$ for $a \in A$, that $|S(\eta,\xi)| < \alpha$. Thus there is $a_{\eta,\xi} \in G\backslash S(\eta,\xi)$.

The definitions of $S(\eta,\xi)$ and $a_{\eta,\xi}$ for $\langle\eta,\xi\rangle \in L$ are complete. For $\xi < \alpha$ we define $A_\xi = \{a_{\eta,\xi} : \xi \leq \eta < \alpha\}$ and we verify (i), (ii), (iii) and (iv).

(i) For $\eta' < \eta < \alpha$, from $\langle\eta',\xi\rangle < \langle\eta,\xi\rangle$ we have $a_{\eta',\xi} \in S(\eta,\xi)$ and hence $a_{\eta',\xi} \neq a_{\eta,\xi}$. Thus $|A_\xi| = \alpha$.

(ii) and (iii). Let $\xi \neq \xi'$ and let $a_{\eta',\xi'} \in A_{\xi'}$ and $a_{\eta,\xi} \in A_\xi$. Then, assuming without loss of generality that $\langle\eta',\xi'\rangle < \langle\eta,\xi\rangle$, we have $a_{\eta',\xi'} \in S(\eta,\xi)$ and $a_{\eta',\xi'} \cdot F_{\xi'} \cdot F_\xi \subseteq S(\eta,\xi)$. Then from $a_{\eta,\xi} \notin S(\eta,\xi)$ follows

$a_{\eta,\xi} \neq a_{\eta',\xi'}$ and

$(a_{\eta,\xi} \cdot F_\xi) \cap (a_{\eta',\xi'} \cdot F_{\xi'}) = (a_{\eta,\xi} \cdot F_\xi^{-1}) \cap (a_{\eta',\xi'} \cdot F_{\xi'}) = \emptyset.$

(iv) Suppose without loss of generality that $a = a_{\eta,\xi}$ and $a' = a_{\eta',\xi}$ with $\eta' < \eta$. Then $\langle\eta',\xi\rangle < \langle\eta,\xi\rangle$, and from $a_{\eta',\xi} \cdot F_\xi \cdot F_\xi \subseteq S(\eta,\xi)$ and $a_{\eta,\xi} \notin S(\eta,\xi)$ follows

$(a \cdot F_\xi) \cap (a' \cdot F_\xi) = (a_{\eta,\xi} \cdot F_\xi^{-1}) \cap (a_{\eta',\xi} \cdot F_\xi) = \emptyset.$ □

Theorem 3.4. *Let G be a topological group such that*

$$|G|^{<tb(G)} = |G| = nwd(G) = \alpha \geq \omega.$$

Then G is strongly extraresolvable.

Proof. Clearly from $|G|^{<tb(G)} = |G| = nwd(G) \geq \omega$ follows $tb(G) \leq nwd(G)$, so Theorem 1.5 gives

$$\Delta(G) = |G| = \alpha \text{ and } nwd(G) = d(G) = \alpha.$$

Let $\{F_\xi : \xi < \alpha\}$ be an indexing of $\mathcal{F}_{tb(G)}(G)$, let $\{A_\xi : \xi < \alpha\} \subseteq [G]^\alpha$ be as given by Lemma 3.3, and (using Lemma 2.1) let $\mathcal{E} = \{E_\eta : \eta < \alpha^+\}$ be a faithfully indexed subset of $[G]^\alpha$ such that $|E_\eta \cap A_\xi| = 1$ for $\eta < \alpha^+$, $\xi < \alpha$, and $|E_\eta \cap E_{\eta'}| < \alpha$ whenever $\eta < \eta' < \alpha^+$. Abandoning earlier numeration, for notational convenience we now let $E_\eta \cap A_\xi = \{a_{\eta,\xi}\}$, and we set

$$D_\eta = \cup\{a_{\eta,\xi} \cdot F_\xi : \xi < \alpha\} \text{ for } \eta < \alpha^+.$$

Since D_η is dense in G by Lemma 2.2, it remains only to check that $|D_{\eta'} \cap D_\eta| < \alpha$ whenever $\eta' < \eta < \alpha$. We have

(*) $D_{\eta'} \cap D_\eta = [\cup_{\xi'<\alpha} a_{\eta',\xi'} \cdot F_{\xi'}] \cap [\cup_{\xi<\alpha} a_{\eta,\xi} \cdot F_\xi]$.

When $\xi \neq \xi'$ we have $a_{\eta',\xi'} \cdot F_{\xi'} \cap a_{\eta,\xi} \cdot F_\xi = \emptyset$ by condition (iii) of Lemma 2.4, so (*) reduces to

(**) $D_{\eta'} \cap D_\eta = \cup_{\xi<\alpha}[(a_{\eta',\xi} \cdot F_\xi) \cap (a_{\eta,\xi} \cdot F_\xi)]$.

If $\xi < \alpha$ satisfies $(a_{\eta',\xi} \cdot F_\xi) \cap (a_{\eta,\xi} \cdot F_\xi) \neq \emptyset$ then from condition (iv) of Lemma 2.4 we have $a_{\eta',\xi} = a_{\eta,\xi} \in E_{\eta'} \cap E_\eta$. Thus relation (**) expresses $D_{\eta'} \cap D_\eta$ as the union of $|E_{\eta'} \cap E_\eta|$-many sets, each of cardinality strictly less than $tb(G)$. Since $|E_{\eta'} \cap E_\eta| < \alpha$, we conclude from Lemma 3.2(a) that $|D_{\eta'} \cap D_\eta| < \alpha$, as required. □

As usual for a space X we write

$$iw(X) = \min\{w(Y) : \text{there is a continuous bijection } f : X \to Y\}.$$

It is known that $d(C_p(X)) = iw(X)$ for every Tychonoff space X (for a proof see [2] and [29]). If X is a compact space, then $d(C_p(X)) = w(X)$, so (as noted in [6] and [11]) we have $|C_p(X)| = w(\beta(X))^\omega$ for arbitrary (Tychonoff) X.

Now we collect some consequences of Theorem 3.4.

Corollary 3.5. *Let G be an infinite topological group and let X be an infinite Tychonoff space.*

(a) *If $tb(G) \leq (nwd(G))^+$ and $|G| = d(G) = |G|^{<tb(G)}$, then G is strongly extraresolvable.*

(b) *If* $tb(G) \leq (nwd(G))^+$ *and* $d(G)^{<tb(G)} = d(G)$*, then* G *admits a dense strongly extraresolvable subgroup.*

(c) ([9]) *If* G *is totally bounded and* $|G| = d(G) \geq \omega$*, then* G *is strongly extraresolvable.*

(d)([9]) *If totally bounded,* G *admits a dense strongly extraresolvable subgroup.*

(e) ([9]) *If* G *is totally bounded and admits no proper dense subgroup, then* G *is strongly extraresolvable.*

(f) *If* $iw(X) = (w(\beta X))^\omega$*, then* $C_p(X)$ *is strongly extraresolvable.*

(g) *If* $iw(X) = (iw(X))^\omega$*, then* $C_p(X)$ *contains a dense, strongly extraresolvable subgroup.*

(h) *If* X *is compact and* $(w(X))^\omega = w(X)$*, then* $C_p(X)$ *is strongly extraresolvable.*

(i) ([1]) $C_p(\beta(\alpha))$ *is strongly extraresolvable for every cardinal* $\alpha \geq \omega$.

Proof. (a) From the hypothesis and theorem 1.5(b) it follows that $|G|^{<tb(G)} = |G| = nwd(G)$, so Theorem 1.5(b) applies.

(b) Let H be a dense subgroup of G such that $|H| = d(G)$. Then from Lemma 1.3(b) we have $nwd(G) \leq nwd(H) \leq d(H) = |H| = d(G)$ and $tb(H) = tb(G) \leq (nwd(G))^+ \leq (nwd(H))^+$ so (b) follows from (a) (applied to H in place of G).

Statements (c), (d) and (e) follow from (a) and (b).

(f) and (g) follow from (a) and (b) respectively, together with the remarks preceding the statement of the present Corollary.

(h) and (i) are immediate from (f). □

Sections (f), (h) and (i) of Corollary 3.5 are in contrast with a principal result from [1], where it is shown for many spaces X—for example, for $X = \mathbb{R}$, $X = \mathbb{Q}$, $X = \mathbb{R}\backslash\mathbb{Q}$ and $X = [0, 1]$—that

$C_p(X)$ is not extraresolvable. See also 6.7(c) below for a related question.

Remark 3.6. The interested reader might consult [13] for a (presumably incomplete) list of Abelian groups which satisfy the hypotheses of Corollary 3.5(e). Long before the appearance of [13] it was known that every infinite Abelian G, in the topology induced by $\mathrm{Hom}(G, \mathbb{T})$, has the property that each subgroup is closed. (Equipped with this topology, the so-called Bohr topology, G is often denoted $G^{\#}$; this notation and many early relevant results and questions are due to van Douwen [14].) Because of the interest which the groups $G^{\#}$ have generated, we record this special instance of Corollary 3.5(e), which was also proved in [9] explicitly: If G is an infinite Abelian group, then $G^{\#}$ is strongly extraresolvable.

Remarks 3.7. (a) Corollary 3.5.(b) cannot be directly deduced from the result cited in Remark 2.9: Fix $\alpha \geq \omega$, let K be any compact group with $w(K) = 2^\alpha$, and using Theorem 1.6(a) let G be a dense subgroup of K such that $|G| = d(K) = \log(2^\alpha) \leq \alpha$. Then every dense subgroup H of G satisfies $|H| \leq \alpha$ and $w(H) = 2^\alpha$, so Theorem 2.9 cannot apply to H.

(b) Our results concerning totally bounded groups improve certain results from [9]. Let us note however that the interesting theorem of Maykhin and Protasov [32], asserting that every infinite totally bounded topological group is maximally resolvable, cannot be improved to show that each such group is strongly extraresolvable; indeed we have noted already in Remark 2.4 the argument from [19] and [23] showing that an infinite compact group cannot (even) be extraresolvable.

In the interest of completeness we conclude this section with two observations valid for every (not necessarily totally bounded) topological group.

Theorem 3.8. *Every extraresolvable group G contains an open-and-closed extraresolvable subgroup H such that $|H| = \Delta(H)$;*

if G is strongly extraresolvable then H may be chosen strongly extraresolvable.

Proof. Let U be an open subset of G such that $|U| = \Delta(G)$, and let H be the subgroup of G generated by U. Then $\Delta(H) = |H| = |U| = \Delta(G)$ and $nwd(H) = nwd(G)$, and if $\mathcal{D}$ is a (strongly) extraresolvable family on G then $\{D \cap H : D \in \mathcal{D}\}$ is a (strongly) extraresolvable family on H. □

Theorem 3.9. *Let G be a topological group with a dense extraresolvable subset X such that $\Delta(X) = |X|$, and let H be the subgroup of G generated by X. Then H is a (dense) extraresolvable subgroup of G; indeed every extraresolvable family on X is an extraresolvable family on H.*

Proof. The second assertion, which implies the first, is immediate from the relations $\Delta(H) \leq |H| = |X| = \Delta(X)$. □

4. Spaces $X \times A$ with A Discrete

In sections 5 and 6 respectively we consider locally compact groups and we compare in strength some of the resolvability properties we have been considering. The following result will be helpful in both these endeavors. We use throughout this Section the facts that for X and A as hypothesized we have

(i) $nwd(X \times A) = nwd(X)$; and

(ii) $\Delta(X \times A) = \Delta(X)$.

Theorem 4.1.*Let X be a space with no isolated points and let A be a discrete space. Then*

(a) *$X \times A$ is extraresolvable $\Longleftrightarrow$ X is extraresolvable;*

(b) *if $X \times A$ is strongly extraresolvable, then X is strongly extraresolvable and $|A| \leq (\Delta(X))^+$;*

(b′) *If X is strongly extraresolvable and $|A| < nwd(X)$, then $X \times A$ is strongly extraresolvable;*

(c) *$X \times A$ has a dense extraresolvable subset $\Longleftrightarrow X$ has a dense extraresolvable subset;*

(d) *if $X \times A$ has a dense strongly extraresolvable subset, then X has a dense strongly extraresolvable subset and $|A| \leq (\Delta(X))^+$; and*

(d′) *If X has a dense strongly extraresolvable subset D and $|A| < nwd(D)$, then $X \times A$ has a dense strongly extraresolvable subset.*

Proof. (a)($\Rightarrow$) and (b). If $\mathcal{E}$ is an extraresolvable family on $X \times A$ then for $a \in A$ chosen arbitrarily the family $\{E \cap \pi_A^{-1}(\{a\}) : E \in \mathcal{E}\}$ is an extraresolvable family on $\pi_A^{-1}(\{a\})$; this latter space is homeomorphic to X. If in addition $|E \cap F| < nwd(X \times A)$ for distinct E, $F \in \mathcal{E}$ then

$$|(E \cap \pi_A^{-1}(\{a\})) \cap (F \cap \pi_A^{-1}(\{a\}))| < nwd(X \times A) = nwd(X)$$

for such E and F. To see that in this case the condition $|A| > (\Delta(X))^+$ is impossible, for distinct $E, F \in \mathcal{E}$ let

$$B(E,F) = \{a \in A : E \cap F \cap \pi_A^{-1}(\{a\}) \neq \emptyset\}$$

and set $B = \cup\{B(E,F) : E, F \in \mathcal{E}, E \neq F\}$. Since each $|B(E,F)| < nwd(X) \leq \Delta(X)$ we have $|B| \leq \Delta(X) \cdot |\mathcal{E}| \cdot |\mathcal{E}| = (\Delta(X))^+$, so if $|A| > (\Delta(X))^+$ there is $b \in A \backslash B$; then $\{E \cap \pi_A^{-1}(\{b\}) : E \in \mathcal{E}\}$ is a collection of $(\Delta(X))^+$-many pairwise disjoint (dense) subsets of $\pi_A^{-1}(\{b\})$, a contradiction.

(a)($\Leftarrow$) and (b′). If $\mathcal{E}$ is an extraresolvable family on X then the family $\mathcal{F} := \{E \times A : E \in \mathcal{E}\}$ is an extraresolvable family on $X \times A$ since: (1) each $E \times A$ is dense in $X \times A$; (2) $|\mathcal{F}| = |\mathcal{E}| = (\Delta(X))^+$; and (3) if E and F are distinct elements of $\mathcal{E}$ then

$$\begin{aligned} \mathrm{int}_{X \times A}\,\mathrm{cl}_{X \times A}((E \times A) \;\cap\; (F \times A)) &\\ &= \mathrm{int}_{X \times A}\,\mathrm{cl}_{X \times A}((E \cap F) \times A) \\ &= \cup_{a \in A}\,\mathrm{int}_D\,\mathrm{cl}_D((E \cap F) \times \{a\}) \\ &= \cup_{a \in A}\,\emptyset = \emptyset. \end{aligned}$$

If in addition $\mathcal{E}$ is strongly extraresolvable on X and $|A| < nwd(X)$ then $\mathcal{F}$ is strongly extraresolvable on $X \times A$, since $|(E \times A) \cap (F \times A)| = |(E \cap F)| \cdot |A| < nwd(X)$.

(c)($\Rightarrow$) and (d). Let D be a dense extraresolvable subspace of $X \times A$ and let U be open in $X \times A$ with $|U \cap D| = \Delta(D) = \alpha \geq \omega$. Fix $a \in A$ so that $U \cap D \cap \pi_A^{-1}(\{a\}) \neq \emptyset$, and note that $\Delta(D) = \Delta(D \cap \pi_A^{-1}\{a\})$. Let $\mathcal{D}$ be an extraresolvable family on D with $|\mathcal{D}| = \alpha^+$, for $E \in \mathcal{D}$ let $F(E) = E \cap \pi_A^{-1}(\{a\})$, and set $\mathcal{F} = \{F(E) : E \in \mathcal{D}\}$. Then $\mathcal{F}$ is a family of dense subsets of $\pi_A^{-1}(\{a\})$, and if E_0 and E_1 are distinct elements of $\mathcal{D}$ then $F(E_0) \cap F(E_1)$ is nowhere dense in $D \cap \pi_A^{-1}(\{a\})$ since $E_0 \cap E_1$ is nowhere dense in D. Thus $\mathcal{F}$ is an extraresolvable family (of cardinality α^+) on the dense subset $D \cap \pi_A^{-1}(\{a\})$ of $\pi_A^{-1}(\{a\})$; again, this latter space is homeomorphic to X.

The proof of (d) parallels that of (b), with $\mathcal{F}$ replacing $\mathcal{E}$ and $\pi_A^{-1}(\{c\}) \cap D$ replacing $\pi_A^{-1}(\{c\})$ for $c \in A$.

(c)($\Leftarrow$) and (d) are immediate from (a)($\Leftarrow$) and (b), with D replacing X. □

Although maximal resolvability is peripheral to our principal interests here, we note in passing that Theorem 4.1 has a natural analogue in that context.

Theorem 4.2. *Let X be a space with no isolated points and let A be a discrete space. Then $X \times A$ is maximally resolvable $\Longleftrightarrow$ X is maximally resolvable.*

Proof. With the obvious necessary modifications, the proof proceeds in parallel with that of Theorem 4.1(a). □

Lemma 4.3. *If X is strongly extraresolvable and U is an open subset of X with $\Delta(X) = \Delta(U)$, then U is also strongly extraresolvable.*

Proof. If $\mathcal{E}$ is a strongly extraresaolvable family for X, then $\{E \cap U : E \in \mathcal{E}\}$ is a strongly extraresolvable family for U. □

It follows from Lemma 4.3 that if $X \times A$ is strongly extraresolvable for some discrete space A, then X is strongly extraresolvable as well. A space X is strongly extraresolvable iff $X \times \{n\}$ is strongly extraresolvable for every positive integer n. A more general result is the following.

Theorem 4.4. *Let X be a space with no isolated points and let A be a discrete space. If $X \times A$ is strongly extraresolvable space, then $X \times B$ is strongly extraresolvable for every discrete space B with $|A| \geq |B|$.*

Proof. We may take $B \subseteq A$. Since $X \times B$ is an open subset of $X \times A$ and $\Delta(X \times A) = \Delta(X \times B)$, Lemma 4.3 applies. □

We remark that if $X \times A$ is strongly extraresolvable for a discrete space A, then it follows from Theorem 4.1(b) and (b′) that $|A| \leq \Delta(X)^+$ and that $X \times B$ is strongly extraresolvable for every discrete space B with $|B| < nwd(X)$.

Theorem 4.5. *If $w(X) = \Delta(X) = nwd(X)$ and A is a discrete space with $|A| = nwd(X)$, then $X \times A$ is strongly extraresolvable.*

Proof. With $\alpha = \Delta(X)$ we have

$$\begin{aligned} \alpha = w(X \times A) &= \Delta(X) = \Delta(X \times A) \\ &= nwd(X) = nwd(X \times A), \end{aligned}$$

so $X \times A$ is strongly extraresolvable by Lemma 2.3. □

Example 4.6. Consider the beth cardinal $\beth_{\omega_1}$ and the ordered space $[0, \beth_{\omega_1})$. It is shown in [1](1.5) that

$$\begin{aligned} \beth_{\omega_1} = w(C_p([0, \beth_{\omega_1}))) &= (\beth_{\omega_1})^\omega \\ &= \Delta(C_p([0, \beth_{\omega_1}))) \\ &= nwd(C_p([0, \beth_{\omega_1}))). \end{aligned}$$

It then follows from Theorem 4.5 that $C_p([0, \beth_{\omega_1})) \times \beth_{\omega_1}$ is strongly extraresolvable, where $[0, \beth_{\omega_1})$ is equipped with the order topology and $\beth_{\omega_1}$ with the discrete topology.

5. Concerning Locally Compact Groups

It is immediate from Theorem 3.5 that every infinite compact Hausdorff group has a dense (strongly) extraresolvable subgroup. It becomes natural then to inquire whether the same conclusion holds for infinite locally compact groups. In what follows for a space X we denote by $\kappa(X)$ the usual *compact-covering number* of X defined by the relation

$$\kappa(X) = \min\{|\mathcal{K}| : X = \cup\mathcal{K}, \text{ each } K \in \mathcal{K} \text{ is compact}\}.$$

Theorem 5.1. *Let H be an open-and-closed subgroup of a nondiscrete group G. Then G has a dense extraresolvable subset if and only if H has a dense extraresolvable subset.*

Proof. The space G is homeomorphic to the $H \times A$ with A the discrete space G/H, so Theorem 4.1 applies. □

Theorem 5.2. *Let G be a non-discrete locally compact topological group. Then*

(a) *G has a dense extraresolvable subset;*

(b) *if $\kappa(G) < nwd(G)$ then G has a dense strongly extraresolvable subset; and*

(c) *if $\kappa(G) > (\Delta(G))^+$ then G has no dense strongly extraresolvable subset.*

Proof. (a) It has been shown by Cleary and Morris [5], using structure theorems due to G. Hochschild, K. Iwasawa and P. S. Mostert, that G is homeomorphic to a space of the form $\mathbb{R}^n \times K \times A$ with $n < \omega$, K a compact group, and A a discrete space. According to Lemma 4.1 it suffices to show that $\mathbb{R}^n \times K$ has a dense extraresolvable subset. If $w(K) \leq \omega$ it is enough to choose any countable dense subset of $\mathbb{R}^n \times K$; each such set X satisfies $w(X) = |X| = \Delta(X) = d(X) = nwd(X) = \omega$, hence is extraresolvable by Theorem 2.3. If $w(K) \geq \omega$ let H be a dense subgroup of K such that $|H| = d(K) = d(H)$, let

$\alpha = |H| = \Delta(H) \geq \omega$, and using Theorem 3.4 let $\mathcal{D}$ be an extraresolvable family on H. Let C be a countable dense subset of $\mathbb{R}^n$, set $S = H \times C$, and let $\mathcal{C} = \{D \times C : D \in \mathcal{D}\}$. Then $|\mathcal{C}| = |\mathcal{D}| = \alpha^+ = \Delta(S)$, the elements of $\mathcal{C}$ are dense in S, and if D and E are distinct elements of $\mathcal{D}$ then $(D \times C) \cap (E \times C) = (D \cap E) \times C$ is nowhere dense in S because $D \cap E$ is nowhere dense in H.

(b) and (c). From $\kappa(\mathbb{R}^n \times K) = \omega$ follows $\kappa(G) = |A| \cdot \omega$, so the statements follow from parts (d$'$) and (d) respectively of Theorem 4.1. □

Theorem 5.3. *For every $\alpha \geq \omega$ there is a nondiscrete locally compact group G such that $w(G) = |G| = 2^\alpha$ and G has no dense extraresolvable subgroup.*

Proof. Given $\alpha \geq \omega$, let G be a group of the form $(\mathbb{Z}(p^\infty))^\alpha$ (p a prime) topologized with the coarsest topology such that the group $K = (\mathbb{Z}(p))^\alpha$ in its usual compact topology, together with each of its cosets, is open-and-closed in G. It is easy to see that G is a locally compact group such that $w(G) = w(K) \cdot |G/K| = \alpha \cdot 2^\alpha = 2^\alpha$. It is a theorem of Rajagopalan and Subrahmanian [33] that G has no proper dense subgroup, so for us it suffices to show that G itself is not extraresolvable. It is clear (using $\Delta(G) = \Delta(K) = 2^\alpha$) that if G is extraresolvable then K is extraresolvable, contrary to the argument given in Remark 2.4. □

Remark 5.4. It is interesting to note that although (according to Theorem 5.2(a)) the locally compact groups G of Theorem 5.3 do admit a dense extraresolvable subset X, one cannot arrange in addition that $|X| = \Delta(X)$ (cf. Theorem 3.11).

6. Relations Between the Properties

Remark 6.1. As noted in Remark 2.4, there are maximally resolvable spaces—indeed, compact groups of arbitrary preassigned weight—which are not extraresolvable (hence, not strongly extraresolvable). The apparently innocent converse question, whether every strongly extraresolvable space (perhaps even every extraresolvable space) is maximally resolvable, proves to be unexpectedly thorny. According to the simple argument given in the next paragraph, a counterexample will be (uncountable and) ω-resolvable. The existence of ω-resolvable T_1-spaces which are not maximally resolvable has been established in ZFC by El′kin [19] and Malykhin [30]; using as a point of departure a certain space whose existence is known [28] to be equiconsistent with the existence of measurable cardinals, Eckertson [16] has found Tychonoff examples. We do not know whether the existence of Tychonoff (strongly) extraresolvable spaces which are not maximally resolvable can be established in ZFC. In 6.2 and 6.3 we note that such a space X must have either $nwd(X) < \Delta(X)$ or $\Delta(X)$ singular.

To see as in [23](1.4) that every extraresolvable space X is ω-resolvable let $\{D_n : n < \omega\}$ be a countable subfamily of an extraresolvable family on X and for $n < \omega$ let $E_n = D_n \setminus \cup_{k<n} (D_n \cap D_k)$; then $\{E_n : n < \omega\}$ is a sequence of pairwise disjoint dense subsets of X, as required. This reasoning adapts readily to give the following result.

Theorem 6.2. *Let X be a strongly extraresolvable space. Then X is cf$(nwd(X))$-resolvable.*

Proof. Let $\alpha = \mathrm{cf}(nwd(X)) \leq nwd(X) \leq \Delta(X)$, let $\{D_\eta : \eta < \Delta(X)\}$ be a family of dense subsets of X such that $|D_\eta \cap D_{\eta'}| < nwd(X)$ whenever $\eta < \eta' < \alpha$, and for $\eta < \alpha$ set $E_\eta = D_\eta \setminus \cup_{\zeta<\eta} (D_\zeta \cap D_\eta)$. The cardinality assumptions guarantee for each $\eta < \alpha$ that $| \cup_{\zeta<\eta} (D_\zeta \cap D_\eta)| < nwd(X)$, so $\{E_\eta : \eta < \alpha\}$ is a faithfully indexed family of α-many pairwise disjoint dense subsets of X. □

Corollary 6.3. *Let X be a strongly extraresolvable space such that $nwd(X) = \Delta(X)$ and $\Delta(X)$ is regular. Then X is maximally resolvable.* □

Theorem 6.4. *Assume GCH. Let X be a strongly extraresolvable space and let U be an open subset of X such that $|U| = \Delta(X)$. Then*

(a) *cf*$(\Delta(U)) \leq$ *nwd*(U)*; and*

(b) *if nwd*$(U) < \Delta(U)$*, then U is κ-resolvable for each $\kappa < \Delta(U)$.*

Proof. (a) Suppose instead that $nwd(U) <$ cf$(\Delta(U))$, and let $\mathcal{D} = \{D_\eta : \eta < (\Delta(U))^+\}$ be a strongly extraresolvable family on U. According to a theorem of Erdős, Milner and Rado [21], as given by Williams [36](6.2.6), there are $\mathcal{E} \in [\mathcal{D}]^{(\Delta(U))^+}$ and a set $N \subseteq U$ such that $D_\eta \cap D_{\eta'} = N$ whenever D_η and $D_{\eta'}$ are distinct elements of $\mathcal{E}$. (The theorem of Williams [36] guarantees such $\mathcal{E}$ and N provided (i) $|\mathcal{D}| = (\Delta(U))^+$; (ii) each $D_\eta \in \mathcal{D}$ satisfies $|D_\eta| = \Delta(U)$; and (iii) distinct $D_\eta, D_{\eta'} \in \mathcal{D}$ satisfy $|D_\eta \cap D_{\eta'}| < nwd(U)$. In the present case (i) and (iii) are clearly satisfied. To achieve (ii) it is enough to replace each $D_\eta \in \mathcal{D}$ by the set $D'_\eta := D_\eta \cup (\{\eta\} \times \Delta(U))$; then writing $\mathcal{D}' = \{D'_\eta : \eta < (\Delta(U))^+\}$ and choosing $\mathcal{E}' \in [\mathcal{D}']^{(\Delta(U))^+}$ and N so that distinct elements $D'_\eta, D'_{\eta'}$ of $\mathcal{E}'$ satisfy $D'_\eta \cap D'_{\eta'} = N$, the set $\mathcal{E} := \{D_\eta : D'_\eta \in \mathcal{E}'\}$ is as asserted.) The family $\{D_\eta \backslash N : D_\eta \in \mathcal{E}\}$ is then a family of $(\Delta(U))^+$-many pairwise disjoint dense subsets of U, a contradiction.

(b) Given such κ, and $\mathcal{D}$ as in part (a), the cited theorem from [36](6.2.6) now gives $\mathcal{E} \in [\mathcal{D}]^\kappa$ and $N \subseteq U$ such that $D_\eta \cap D_{\eta'} = N$ whenever D_η and $D_{\eta'}$ are distinct elements of $\mathcal{E}$; then $\{D_\eta \backslash N : D_\eta \in \mathcal{E}\}$ is a family of κ-many pairwise disjoint dense subsets of U, as required. □

Theorem 6.5. *Assume GCH. Let X be a strongly extraresolvable space and let U be an open subset of X such that $|U| = \Delta(X)$. Then U is maximally resolvable in each of the following cases.*

(a) $\Delta(U)$ *is regular;*

(b) $nwd(U) < \Delta(U)$ *and* $cf(\Delta(U)) = \omega$.

Proof. (a) From Theorem 6.4(a) and Lemma 1.1 we have $nwd(U) = \Delta(U)$ with $\Delta(U)$ regular, so Corollary 6.3 applies.
(b) According to Theorem 6.4(b) the space U is κ-resolvable for each $\kappa < \Delta(U)$. The theorem of Bhaskara Rao [3], which is a generalization to arbitrary α with $\mathrm{cf}(\alpha) = \omega$ of the theorem of Illanes [27] cited above in §0 (the case $\alpha = \omega$), now yields: U is $\Delta(U)$-resolvable, i.e., is maximally resolvable. □

Remarks 6.6. (a) The upshot of the foregoing considerations is that (assuming GCH) a strongly extraresolvable space which is not maximally resolvable must contain an open subset set U (with $\Delta(X) = \Delta(U)$) satisfying these three conditions: (1) $\Delta(U) < w(U)$; (2) $\Delta(U)$ is singular; and (3) either $\Delta(U) = nwd(U)$, or $nwd(U) < \Delta(U)$ and $\mathrm{cf}(\Delta(U)) > \omega$. These conditions are not sufficient, however, since even in ZFC there are strongly extraresolvable, maximally resolvable homogeneous spaces X with $\Delta(X) = |X| = d(X) = nwd(X) < w(X)$ and $\Delta(X)$ singular: According to Theorems 1.3 and 3.4 it is enough for an arbitrary singular cardinal κ and $\alpha_0 \geq \omega$ to set $\alpha = \beth_\kappa(\alpha_0)$—that is, set $\alpha_{\xi+1} = 2^{\alpha_\xi}$ for $\xi < \kappa$, let $\alpha_\xi = \Sigma_{\eta<\xi}\, \alpha_\eta$ for limit ordinals $\xi \leq \kappa$, and let $\alpha = \alpha_\kappa$; then for K a compact group of weight 2^α every dense subgroup G of K with $|G| = d(K) = \alpha = \log(2^\alpha)$ satisfies $\Delta(G) = nwd(G) = \alpha < w(G)$ (and also $\mathrm{cf}(\Delta(G)) = \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)$).
(b) We note finally that the generalization of extraresolvability introduced in this paper, which we have called strong extraresolvability, is indeed a proper generalization. To construct an extraresolvable space Y which is not strongly extraresolvable, in fact which has no dense strongly extraresolvable subspace, it is by Theorem 4.1 enough to begin with any extraresolvable space X and to take $Y = X \times A$ with A discrete and $|A| > (\Delta(X))^+$; evidently if in addition X and A are chosen to be topological groups, the space Y so constructed is itself a topological group.

(c) According to Theorem 4.2 the same argument, beginning now with an arbitrary maximally resolvable space X, furnishes an example of a space $Y = X \times A$ (which may again be chosen to be a topological group) which is maximally resolvable but not strongly extraresolvable.

Questions 6.7. Although the results of this section serve to distinguish adequately between the various properties we have been considering, there remain several questions in this corner of mathematics which we have not been able to settle. We cite some of these now.
(a) Is there a countable extraresolvable (Tychonoff) space which is not strongly extraresolvable?
(b) Is there a (strongly) extraresolvable space which is not maximally resolvable?
(c) Does $C_p(X)$ have a dense strongly extraresolvable subgroup, for every Tychonoff space X? What about if X is compact and $\mathrm{cf}(\mathrm{w}(\mathrm{X})) = \omega$?
(d) Are there cardinal functions s and r as follows: Let X be a space and A be discrete. Then
(i) $X \times A$ is strongly extraresolvable if and only if X is strongly extraresolvable and $|A| \leq s(X)$; and
(ii) $X \times A$ has a dense strongly extraresolvable subset if and only if X has a dense strongly extraresolvable subset and $|A| \leq r(X)$.

Of course, Questions (d)(i) and (d)(ii) are motivated by Theorems 4.1(b,b′) and 4.1(d,d′) respectively.

We hope to return to these questions in a later communication.

References

[1] O. T. Alas, S. García-Ferreira and A. H. Tomita, *The Extraresolvability of Some Function Spaces*, Glasnik Math. **34** (54) 1999, 23-35.

[2] A. V. Arkhangel′skii, *Topological Function Spaces, Mathematics and its Applications* (Soviet Series), Kluwer Academic Publishers, Netherlands, 1992.

[3] K. P. S. Bhaskara Rao, *On* $\aleph$*-Resolvability*. Manuscript privately circulated, 1994.

[4] J. G. Ceder, *On maximally resolvable spaces*, Fund. Math., **55** (1964), 87–93.

[5] Joan Cleary and Sidney A. Morris, *Topologies on Locally Compact Groups*, Bull. Australian Math. Soc., **38** (1988), 105–111.

[6] W. W. Comfort, *A Survey of Cardinal Invariants*, Top. Appl., **1** (1971), 163–199.

[7] W. W. Comfort, *Topological Groups*, In: Handbook of Set-theoretic Topology, Kenneth Kunen and Jerry E. Vaughan (eds.), North-Holland, Amsterdam, (1984), 1143–1263.

[8] W. W. Comfort and S. García-Ferreira, *Resolvability: A Selective Survey and Some New Results*, Topology Appl., **74** (1996), 149–167.

[9] W. W. Comfort and S. García-Ferreira, *Dense Subsets of Maximally Almost Periodic Groups*, Proc. Amer. Math. Soc. (1999). To appear.

[10] W. W. Comfort and D. L. Grant, *Cardinal Invariants, Pseudocompactness and Minimality: Some Recent Advances in The Topological Theory of Topological Groups*, Topology Proc., **6** (1981), 227–265.

[11] W. W. Comfort and A. W. Hager, *Estimates for The Number of Real-Valued Continuous Functions*, Trans. Amer. Math. Soc., **150** (1970), 619–631.

[12] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, **221**, Springer-Verlag. Berlin, 1974.

[13] W. W. Comfort and Jan van Mill, *Groups With Only Resolvable Group Topologies*, Proc. Amer. Math. Soc., **120** (1993), 687–696.

[14] E. K. van Douwen, *The Maximal Totally Bounded Group Topology on G and The Biggest Minimal G-space for Abelian Groups G*, Topology Appl., **34** (1990), 69–91.

[15] Eric K. van Douwen, *Applications of Maximal Topologies*, Topology Appl., **51** (1993), 125–139.

[16] Frederick W. Eckertson, *Resolvable, Not Maximally Resolvable Spaces*, Topology Appl., **79** (1997), 1–11.

[17] A. G. El′kin, *On the Maximal Resolvability of Products of Topological Spaces*, Soviet Math. Dokl., **10** (1969), 659–662, Russian original in: Доклады Акад. Наук СССР 186:4 (1969), 765–768.

[18] A. G. El′kin, *Resolvable Spaces Which Are Not Maximally Resolvable*, Moscow Univ. Math. Bull., **24** (1969), 116–118, Russian original in: Вестник Моск. Гос. Унив, Сер. И, Мат. Мех. 24: 4 (1969), 66–70.

[19] A. G. El′kin, *Ultrafilters and Undecomposable Spaces*, Moscow Univ. Math. Bull., **24** (1969), 37–40. Russian original in: Вестник Моск. Гос. Унив, Сер. И, Мат. Мех. 24: 5 (1969), 51–56.

[20] P. Erdős, L. Gillman, and M. Henriksen, *An Isomorphism Theorem for Real-Closed Fields*, Ann. of Math., **61** (1955), 542–554.

[21] P. Erdős, E. C. Milner, and R. Rado, *Intersection Theorems for Systems of Sets III*, J. Australian Math. Soc., **13** (1974), 22–40.

[22] Li Feng and Oscar Masaveu, *Exactly n-Resolvable Spaces and ω-Resolvability*, 1995, Math. Japonica, **50** 1999.

[23] S. García-Ferreira, V. I. Malykhin and A. Tomita, *Extraresolvable Spaces*, to appear in Topology Appl..

[24] Leonard Gillman and Meyer Jerison, *Rings of Continuous Functions*, D. Van Nostrand Co.: New York, 1960.

[25] I. I. Guran, *On Topologial Groups Close to be Lindelöf*, Soviet Math. Dokl., **23** (1981), 173–175, Russian original in: Доклады Акад. Наук СССР 256:6 (1981), 1305–1307.

[26] Edwin Hewitt, *A Problem of Set-Theoretic Topology*, Duke Math. J., **10** (1943), 309–333.

[27] Alejandro Illanes, *Finite and ω-Resolvability*, Proc. Amer. Math. Soc., **124** (1996), 1243–1246.

[28] Kenneth Kunen, Andrej Szymański, and Franklin Tall, *Baire Irresolvable Spaces and Ideal Theory*, Pr. Nauk. Uniw. Śląsk. Katowiceh, Ann. Math. Silesianae, **2(14)** (1986), 98–107.

[29] R. A. McCoy and Ibula Ntantu, *Topological Properties of Spaces of Continuous Functions*, Lecture Notes in Mathematics, Springer-Verlag, **1315**, 1988.

[30] V. I. Malykhin, *On Resolvable and Maximal Spaces*, Soviet Math. Dokl., **15** (1974), 1452–1457. Russian original in: Доклады Акад. Наук СССР 218 (1974), 1017–1020.

[31] V. I. Malykhin, *Irresolvability is not Descriptively Good*, Manuscript submitted for publication, 1998.

[32] V. I. Malykhin and I. V. Protasov, *Maximal Resolvability of Bounded Groups*, Topology Appl., **73** (1996), 227–232.

[33] M. Rajagopalan and H. Subrahmanian, *Dense subgroups of locally compact groups*, Colloq. Math. **35** (1976), 289–292.

[34] M. Tkačenko, *Introduction to Topologial Groups*, Top. Appl., **86** (1998), 179–231.

[35] André Weil, *Sur les Espaces à Structure Uniforme et Sur la Topologie Générale* , Publ. Math. Univ., Strasbourg, Hermann & Cie, Paris, 1937.

[36] Neil H. Williams, *Combinatorial Set Theory*, Studies in Logic and the Foundations of Mathematics, **91**, North-Holland Publishing Company: Amsterdam–New York–Oxford, 1977.

Wesleyan University, Middletown, CT, 06459 USA

E-mail address: twcomfort@wesleyan.edu

Instituto de Matemáticas, Ciudad Universitaria (UNAM), 04510 México D.F., México

E-mail address: tgarcia@servidor.unam.mx

E-mail address: tsgarcia@zeus.ccu.umich.mx