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THE SAMUEL COMPACTIFICATION OF A QUASI-UNIFORM FRAME

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Abstract

This paper presents a new description of the completion of a quasi-uniform frame. The completion is used to extend the notion of the Samuel compactification to quasi-uniform frames.

1. Introduction

In [10, Theorem 3.4] it was proved that each quasi-uniform frame has a unique completion. The proof of that Theorem used the fact that each uniform frame has a unique completion. In this note we construct the completion of a quasi-uniform frame without assuming the existence of the completion of a uniform frame. As a consequence , we obtain the completion of a uniform frame as a special case of the completion of a quasi-uniform frame. This is analogous to the presentation of the bicompletion of a quasi-uniform space given in [8, p. 60, ff]. We utilize a frame constructed by B. Banaschewski and A. Pultr [3] to carry the completion of the given quasi-uniform frame.

In the final section we show that the category of totally bounded quasi-uniform frames forms a coreflective subcategory of the category of quasi-uniform frames. We extend the notion of Samuel compactification to quasi-uniform frames.

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2. Preliminaries

A frame $(L, \leq)$ is a complete lattice that satisfies the frame distributive law: for any $x \in L$ and any $A \subset L$,

$$x \wedge \bigvee A = \bigvee \{x \wedge a : a \in A\}.$$

A function between frames is a join homomorphism provided that it preserves arbitrary joins. A join homomorphism that preserves finite meets is called a frame homomorphism. Let f and g be order preserving functions from L to L. We write $f \leq g$ provided that $f(a) \leq g(a)$ for each $a \in L$. This obviously defines a partial order on the set of all order preserving functions from L to L. In any frame L, we use 0 to denote $\bigvee \varnothing$ and e to denote $\bigvee L$. A function f between frames with the property that $f(a) = 0$ implies that $a = 0$ is said to be dense. Let L and M be frames and let $h : L \to M$ be a frame homomorphism. The function $h_* : M \to L$ defined by $h_*(m) = \bigvee \{l \in L : h(l) \leq m\}$, $m \in M$, is the right adjoint of h. If h maps onto M, then $hh_* = id_M$. For further information on frames the reader is referred to [11].

Let L be a frame and let $a, b \in L$. The function $a \sharp b : L \to L$ is defined in [7] by:

$$a \sharp b(x) = \begin{cases} b & \text{if } a \wedge x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is proved in [10] and will be used in section 3.

Proposition 2.1. *Let L and M be frames and let $h : L \to M$ be a dense frame homorphism from L onto M. If $a \sharp b \leq \bigvee \{x_\alpha \sharp y_\alpha : \alpha \in A\}$ then $h(a) \sharp h(b) \leq \bigvee \{h(x_\alpha) \sharp h(y_\alpha) : \alpha \in A\}$.*

The following definitions and notations are taken from [4], [5], [6], [7], and [10]. Let L be a frame, let $x \in L$ and let $u : L \to L$. Then x is u-small provided that $x \sharp x \leq u$. The set of all

u-small elements of L is denoted by S_u. If u is an order preserving function and $\bigvee S_u = e$, then u is said to be a Δ-map. A quasi-uniform base on L is a nonempty collection $\mathbf{B}$ of Δ-maps that satisfy:

(1) For each $u \in \mathbf{B}$ there exists $v \in \mathbf{B}$ such that $v \circ v \leq u$, and

(2) For u, $v \in \mathbf{B}$ there is a join homomorphism w and a $z \in \mathbf{B}$ such that $z \leq w \leq u \wedge v$.

The frame quasi-uniformity $\mathbf{U}$ for which $\mathbf{B}$ is a base is the collection of all order preserving functions $w : L \to L$ for which there exists a $u \in \mathbf{B}$ with $u \leq w$. The frame of $\mathbf{U}$, $Fr(\mathbf{U})$, is the subframe of L to which a belongs provided that $a = \bigvee\{b \in L : u(b) \leq a$ for some $u \in \mathbf{U}\}$. Let $u : L \to L$ be a function. Then u is symmetric provided that for each $x, y \in L$, $u(x) \wedge y = 0$ if and only if $u(y) \wedge x = 0$. A quasi-uniformity $\mathbf{U}$ on a frame L is a uniformity [4] provided that $\mathbf{U}$ has a base of symmetric entourages and $F_r(\mathbf{U}) = \mathbf{L}$. The pair $(L, \mathbf{U})$ is called a uniform frame. If $(L, \mathbf{U})$ is a uniform frame, then $\{S_u : u \in \mathbf{U}\}$ is a base for a covering uniformity which is denoted by $\mu_{\mathbf{U}}$ [4].

Let $\mathbf{U}$ be a quasi-uniformity on a frame L and let $u \in \mathbf{U}$. Then u^* is defined by $u^*(x) = \bigvee\{c \in S_u : c \wedge x \neq 0\} = S_u x$. The uniformity $\mathbf{U}^* = \{u^* : u \in \mathbf{U}\}$ is the coarsest frame uniformity containing $\mathbf{U}$. Equivalently, [4] $\{S_u : u \in \mathbf{U}\}$ is a base for $\mathbf{U}^*$ when considered as a covering uniformity. We shall use the fact that for any $x \in L$, and any $u \in \mathbf{U}$, x is u^*-small if and only if x is u-small [7, Proposition 2.1(1)]. If $\mathbf{U}$ is a quasi-uniformity on L and $Fr(\mathbf{U}^*) = L$, then $(L, \mathbf{U})$ is said to be a quasi-uniform frame [5]. Let $(L, \mathbf{U})$ be a quasi-uniform frame and let $a, b \in L$. We write $a \lhd b$ if $u^*(a) \leq b$ for some $u^* \in \mathbf{U}^*$.

Let L and M be frames and let $\mathbf{U}$ and $\mathbf{V}$ be (quasi-) uniformities on L and M respectively. Let $f : L \to M$ be a frame homomorphism, and for $u \in \mathbf{U}$, let $u_f = \bigvee\{f(x) \sharp f(y) : x \sharp y \leq u\}$. Then f is a (quasi-) uniform frame homomorphism if and only if

$u_f \in \mathbf{V}$ whenever $u \in \mathbf{U}$. Let $(L, \mathbf{U})$ and $(M, \mathbf{V})$ be (quasi-) uniform frames, and let f be a (quasi-) uniform frame homorphism of L onto M. Then f is a (quasi-) uniform surjection provided that $\{u_f : u \in \mathbf{U}\}$ is a base for $\mathbf{V}$. There is an equivalent formulation of the concept of a quasi-uniform frame homomorphism that will be used in §4: for each $u \in \mathbf{U}$ there exists $v \in \mathbf{V}$ such that $v \circ f \leq f \circ u$.

The following proposition is proved in [10] and will be used in describing the completion of a quasi-uniform frame in §3. Let L be a frame and let $u : L \to L$. Then $u^- = \bigvee\{a \sharp b : a \sharp b \leq u\}$.

Proposition 2.2. *Let $(L, \mathbf{U})$ be a quasi-uniform frame. If $u\,,\ v \in \mathbf{U}$ and $v^3 \leq u$ then $v \leq u^- \leq u$. Consequently, $\{u^- : u \in \mathbf{U}\}$ is a base for $\mathbf{U}$.*

Proposition 2.3. [10, Proposition 2.6] *Let $g : L \to M$ be a dense frame homomorphism of L onto M. Let $\mathbf{U}$ be a quasi-uniformity on M. Then $(u_{g_*})_g = u^-$ for $u \in \mathbf{U}$.*

Proposition 2.4. [10, Proposition 2.3] *Let $f : (M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ be a quasi-uniform surjection. Then $f^* : (M, \mathbf{V}^*) \to (\mathbf{L}, \mathbf{U}^*)$ is a uniform surjection where $f^*(x) = f(x)$ for $x \in M$.*

The reader is invited to compare the following proposition with Proposition 2.2 of [10].

Proposition 2.5. *Let $(L, \mathbf{U})$ and $(M, \mathbf{V})$ be uniform frames and let $f : (L, \mathbf{U}) \to (\mathbf{M}, \mathbf{V})$ be a uniform surjection. Then $f : (L, \mu_{\mathbf{U}}) \to (M, \mu_{\mathbf{V}})$ is a uniform surjection, where $\mu_{\mathbf{U}}(\mu_{\mathbf{V}})$ is the covering uniformity associated with $\mathbf{U}(\mathbf{V})$.*

Definition. A quasi-uniform frame $(L, \mathbf{U})$ is *complete* if every dense quasi-uniform surjection $(M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ is an isomorphism.

This is equivalent to the statement that $(L, \mathbf{U}^*)$ is a complete uniform frame [10, Proposition 3.3].

Definition. Let $(M, \mathbf{V})$ and $(L, \mathbf{U})$ be quasi-uniform frames with $(M, \mathbf{V})$ complete. Then $(M, \mathbf{V})$ is a completion of $(L, \mathbf{U})$ provided that there is a dense quasi-uniform surjection $(M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$.

In [10, Theorem 3.4] it was proved that each quasi-uniform frame has a unique completion.

3. The Completion of a Quasi-Uniform Frame

Let $(L, \mathbf{U})$ be a quasi-uniform frame. The frame CL described below will carry the completion of $(L, \mathbf{U})$. The description is due to B. Banaschewski and A. Pultr [3, §4]. Recall that a nonempty subset A of L is called a downset provided that if $x \leq y$ and $y \in A$ then $x \in A$. Let DL be the frame of all downsets of L ordered by inclusion. For any $a \in L$, let $k(a) = \{x \in L : x \triangleleft a\}$.

Banaschewski and Pultr utilized the quotient of DL determined by the prenucleus $l_0 : DL \to DL$, where

$$l_0(A) = \{a \in L : k(a) \subseteq A\} \bigcup \{a \in L : a \wedge S_u \subseteq A \text{ for some } u \in \mathbf{U}\}.$$

This quotient is denoted by CL. (Recall that $A \in CL$ provided that $A = l_0(A)$). B. Banaschewski and A. Pultr [2] show that CL, with set inclusion as the partial order, is a frame.

In CL, meet is intersection and for $C_\alpha \in CL$,

$$\bigvee_\alpha C_\alpha = l\Big(\bigcup_\alpha C_\alpha\Big),$$

where l is the nucleus on DL defined by $l(A) = \bigwedge\{B \in DL : l_0(B) = B,\ A \subset B\}$ for $A \in DL$ [3].

Let $CL \to L$ be the join map. Then the right adjoint $r : L \to CL$ of the join is given by $r(a) = \downarrow a$, where $\downarrow a = \{x \in L : x \leq a\}$ for $a \in L$. We shall make use of the following formulas which are proved in [3]. For any $a \in L$ and any $V \in CL$,

$$r(a) = \bigvee\{r(x) : x \triangleleft a\}$$

and

$$V = \bigvee \{r(a) : a \in V\}.$$

Lemma 3.1. *Let $(L, \mathbf{U})$ be a quasi-uniform frame, and let r be the right adjoint of the frame homomorphism $CL \to L$ given by the join map. Then $\{u_r : u \in \mathbf{U}\}$ is a base for a quasi-uniformity $C\mathbf{U}$ on CL and $(CL, C\mathbf{U})$ is a quasi-uniform frame.*

Proof. Clearly each u_r for $u \in \mathbf{U}$ is nondecreasing. To show that u_r is a Δ-map we must show that the u_r-smalls cover CL. We show that $\bigvee S_{u_r} = \downarrow e$. By definition, $\bigvee S_{u_r} = l\Big(\bigcup\{c : c \in S_{u_r}\}\Big)$. It follows from the definition of u_r that if x is u-small, then $r(x)$ is u_r-small. Since $r(x) = \downarrow x$, it suffices to show that $\bigvee_{x \in S_u} \downarrow x = \downarrow e$. Note that $e \wedge S_u = S_u \subseteq \bigcup_{x \in S_u} \downarrow x$. By the definition of l_0, we have $e \in l_0\Big(\bigcup_{x \in S_u} \downarrow x\Big)$. But $l_0\Big(\bigcup_{x \in S_u} \downarrow x\Big) \subseteq l\Big(\bigcup_{x \in S_u} \downarrow x\Big)$. Consequently $\downarrow e = l\Big(\bigcup_{x \in S_u} (\downarrow x)\Big) = \bigvee_{x \in S_u} \downarrow x$. This shows that each u_r is a Δ-map. Let $v \in \mathbf{U}$ and choose $u \in \mathbf{U}$ such that $u^2 \leq v$. It can be verified that $u_r^2 \leq v_r$. Also, if u, v, $w \in \mathbf{U}$ are such that $u \leq v \wedge w$, then $u_r \leq v_r \wedge w_r$. This proves that $\{u_r : u \in \mathbf{U}\}$ is a base for a frame quasi-uniformity which we denote by $C\mathbf{U}$.

For $V, W \in CL$, we write $W \triangleleft V$ whenever $S_{u_r} W \subseteq V$ for some $u \in \mathbf{U}$. Let $V \in CL$. We show that $V = \bigvee\{W \in CL : W \triangleleft V\}$. We have previously commented that $V = \bigvee\{r(a) : a \in V)\}$. Consequently, $V = \bigvee\{r(x) : x \triangleleft a, a \in V\} \leq \bigvee\{r(x) : r(x) \triangleleft r(a) \leq V, a \in V\}$. Since each $r(x) \in CL$, we have that $V \leq \bigvee\{W \in CL : W \triangleleft V\} \leq V$. Therefore $(CL, C\mathbf{U})$ is a quasi-uniform frame. □

The following lemma is based upon the proof of [3, Proposition 8].

Lemma 3.2. *Let* $(M, \mathbf{V})$ *and* $(L, \mathbf{U})$ *be quasi-uniform frames. Let* $h : (M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ *be a dense quasi-uniform surjection. Then there exists a dense quasi-uniform surjection* $g : (CL, C\mathbf{U}) \to (M, \mathbf{V})$ *such that* $hg : CL \to L$ *is the join map.*

Proof. Let $h : (M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ be a dense quasi-uniform surjection. Define $f : DL \to M$ by $f(A) = \bigvee h_*(A)$. In the proof of Proposition 8 [3], it is established that: (1) f is a frame homomorphism, (2) f determines a frame homomorphism $g : CL \to M$ such that $f = gl$, (3) $hg : CL \to L$ is the join map, (4) g is dense and onto, (5) $h_*(x) = f(\downarrow x)$, and (6) $l(\downarrow x) = \downarrow x$.

We now show that g is a quasi-uniform frame homomorphism. Let $u_r \in C\mathbf{U}$, where $u \in \mathbf{U}$. Since h is a quasi-uniform surjection, we may assume that $u = v_h$ for some $v \in \mathbf{V}$. We show that $v^- \leq (u_r)_g$. Suppose that $a \sharp b \leq v$. Then $h(a) \sharp h(b) \leq u$ and consequently $rh(a) \sharp rh(b) \leq u_r$. From this we have that $grh(a) \sharp grh(b) \leq (u_r)_g$. In the proof of Proposition 8 [3] it is established that $gr = h_*$. Let $x \in M$. Then $x \leq h_* h(x) = grh(x)$. Consequently

$$\begin{aligned} v^- = \bigvee\{a \sharp b : a \sharp b \leq v\} &\leq \bigvee\{grh(a) \sharp grh(b) : a \sharp b \leq v\} \\ &\leq (u_r)_g. \end{aligned}$$

We now show that $g : (CL, C\mathbf{U}) \to (\mathbf{M}, \mathbf{V})$ is a quasi-uniform surjection. By [10, Proposition 3.1] it suffices to show that $\{v_{g_*} : v \in \mathbf{V}\}$ is a base for $C\mathbf{U}$. Let $v \in \mathbf{V}$. By [10, Proposition 3.1], we may assume that $v = u_{h_*}$. Then

$$\begin{aligned} v_{g_*} = (u_{h_*})_{g_*} &= \bigvee\{g_*(a) \sharp g_*(b) : a \sharp b \leq u_{h_*}\} \\ &\geq \bigvee\{g_* h_*(x) \sharp g_* h_*(y) : x \sharp y \leq u\} \\ &= \bigvee\{g_* f(\downarrow x) \sharp g_* f(\downarrow y) : x \sharp y \leq u\} \\ &= \bigvee\{g_* gl(\downarrow x) \sharp g_* gl(\downarrow y) : x \sharp y \leq u\} \\ &= \bigvee\{g_* g(\downarrow x) \sharp g_* g(\downarrow y) : x \sharp y \leq u\} \\ &\geq \{\downarrow x \sharp \downarrow y : x \sharp y \leq u\} \\ &= \bigvee\{r(x) \sharp r(y) : x \sharp y \leq u\} = u_r \end{aligned}$$

Therefore $v_{g_*} \in C\mathbf{U}$. It remains to show that $\{v_{g_*} : v \in \mathbf{V}\}$ is a base for $C\mathbf{U}$. Let $A \in CL$, and let $v \in \mathbf{V}$. Then

$$\begin{aligned} v_{g_*}(A) &= \bigvee\{g_*(b) : a \sharp b \leq v,\, A \wedge g_*(a) \neq 0\} \\ &= \bigvee\{g_*(b) : a \sharp b \leq v,\, a \wedge g(A) \neq 0\} \end{aligned}$$

If $a \wedge g(A) \neq 0$ and $a \sharp b \leq v$, then $b \leq v(g(A))$ and hence $g_*(b) \leq g_* v g(A)$. Consequently $v_{g_*}(A) \leq g_* v g(A)$, or $v_{g_*} \leq g_* v g$. The proof will be completed by showing that $\{g_* v g : v \in \mathbf{V}\}$ is a base for $C\mathbf{U}$. Let $u, w \in C\mathbf{U}$ be such that $w^2 \leq u$. In the proof of Theorem 2.7 [10] it is proved that $g_* w_g g \leq u$. Since $w_g \in \mathbf{V}$, this completes the proof. □

It follows from the definition of a complete quasi-uniform frame that the function g in the statement of Lemma 3.2 is an isomorphism whenever M is complete. It also follows from Lemma 3.2 that if $(L, \mathbf{U})$ has a completion, then the completion is unique up to isomorphism.

Theorem 3.3. *Let $(L, \mathbf{U})$ be a quasi-uniform frame. Then $(CL, C\mathbf{U})$ is the completion of $(L, \mathbf{U})$.*

Proof. Let $(M, \mathbf{V})$ be a quasi-uniform frame, and let $f : (M, \mathbf{V}) \to (\mathbf{CL}, \mathbf{CU})$ be a dense quasi-uniform surjection. Then $\bigvee \circ f : (M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ is a dense surjection. By Lemma 3.2 there exists a dense surjection $g : (CL, C\mathbf{U}) \to (\mathbf{M}, \mathbf{V})$ such that $\bigvee \circ f \circ g = \bigvee$. Dense maps are monic for maps between regular frames. Consequently $f \circ g = id_{CL}$. Also $(\bigvee \circ f) \circ g \circ f = \bigvee \circ f$. But $\bigvee \circ f$ is dense and hence monic. Therefore $g \circ f = id_M$. We have that f is an isomorphism and consequently $(CL, C\mathbf{U})$ is complete. It follows from Lemma 3.2 that $(CL, C\mathbf{U})$ is a completion of $(L, \mathbf{U})$. By the remarks following Lemma 3.2 we have that each quasi-uniform frame has a unique completion. □

Corollary 3.4. *Each uniform frame has a unique completion.*

Proof Let $(L, \mathbf{U})$ be a uniform frame. Let $(CL, C\mathbf{U})$ be the quasi-uniform completion of $(L, \mathbf{U})$. Then $(CL, (C\mathbf{U})^*)$ is a complete uniform frame [10, Proposition 3.3]. Let $\phi_L : (CL, C\mathbf{U}) \to (\mathbf{L}, \mathbf{U})$ be a dense quasi-uniform surjection. Then, by [10, Proposition 2.3], $\phi_L^* : (CL, (C\mathbf{U})^*) \to (\mathbf{L}, \mathbf{U})$ is a dense uniform surjection. It follows that $(CL, (C\mathbf{U})^*)$ is a completion of $(L, \mathbf{U})$. But the quasi-uniform completion is unique and consequently $(CL, C\mathbf{U})$ is isomorphic to $(CL, (C\mathbf{U})^*)$. It follows that $C\mathbf{U}$ is a frame uniformity. Furthermore, any uniform completion of $(L, \mathbf{U})$ is a quasi-uniform completion of $(L, \mathbf{U})$. Therefore the uniform completion is unique. □

Corollary 3.5. *Let $(L, \mathbf{U})$ be a quasi-uniform frame and let $(CL, C\mathbf{U})$ be the completion of $(L, \mathbf{U})$. Then $(C\mathbf{U})^* = \mathbf{C}(\mathbf{U}^*)$.*

Proof. $(CL, (C\mathbf{U})^*)$ is a complete uniform frame. If $h : (CL, C\mathbf{U}) \to (\mathbf{L}, \mathbf{U})$ is a dense quasi-uniform surjection, then $h : (CL, (C\mathbf{U})^*) \to (\mathbf{L}, \mathbf{U}^*)$ is a dense uniform surjection. By definition, $(CL, C\mathbf{U}^*)$ is the completion of $(L, \mathbf{U}^*)$ which implies that $C\mathbf{U}^* = (\mathbf{CU})^*$. □

Corollary 3.6. *Let $(L, \mathbf{U})$ be a uniform frame. Then the quasi-uniform completion, $(CL, C\mathbf{U})$, is the uniform completion of $(L, \mathbf{U})$.*

4. The Totally Bounded Coreflection

Recall [5] that a quasi-uniform frame $(L, \mathbf{U})$ is totally bounded provided that for each $u \in \mathbf{U}$ there is a finite cover of L by u-small elements. The following theorem is taken from [5, Theorem 2.3].

Theorem 4.1. *Let L be a frame and let $\triangleleft$ be a quasi-proximity inclusion on L. For a, $b \in L$ define*

$$u_{a,b}(x) = \begin{cases} 0 & \text{if } \ x = 0 \\ b & \text{if } \ x \leq a, \quad x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

Then $\{u_{a,b} : a \lhd b\}$ *is a subbase for the unique totally bounded frame quasi-uniformity* $\mathbf{U}_{\lhd}$ *on* L *that determines* $\lhd$*. Furthermore* $\mathbf{U}_{\lhd}$ *is the coarsest frame quasi-uniformity that determines* $\lhd$*.*

Proposition 4.2. *The category of totally bounded quasi-uniform frames forms a coreflective subcategory of the category of quasi-uniform frames.*

Proof. Let $(L, \mathbf{U})$ be a quasi-uniform frame. By [5, Proposition 2.1] there is an associated quasi-proximity inclusion $\lhd$ on L defined by $a \lhd b$ if and only if $u(a) \leq b$ for some $u \in \mathbf{U}$. We show that $(L, \mathbf{U}_{\lhd})$ gives the totally bounded coreflection. Clearly the identity map $(L, \mathbf{U}_{\lhd}) \to (L, \mathbf{U})$ is a quasi-uniform frame homomorphism. Let $f : (M, \mathbf{V}) \to (\mathbf{L}, \mathbf{U})$ be a quasi-uniform frame homomorphism where $\mathbf{V}$ is totally bounded. Consider the diagram

$$\begin{array}{ccc} & & (L, \mathbf{U}_{\lhd}) \\ & h \nearrow & \downarrow i \\ (M, \mathbf{V}) & \xrightarrow{f} & (L, \mathbf{U}) \end{array}$$

where $h(x) = f(x)$ for $x \in M$. Clearly h is a frame homomorphism. So it remains to show that h is a quasi-uniform frame homomorphism. This follows from the fact that given a, $b \in M$ with $v(a) \leq b$ for some $v \in V$, we have that $f(a) \lhd f(b)$ and

$$u_{f(a),f(b)} \circ h \leq h \circ u_{a,b}.$$

Obviously h is unique. This completes the proof. □

Theorem 4.3. *A quasi-uniform frame* $(L, \mathbf{U})$ *is compact if and only if it is totally bounded and complete.*

Proof. Suppose that $(L, \mathbf{U})$ is totally bounded and complete. Then $(L, \mathbf{U}^*)$ is totally bounded, and by [10, Proposition 3.3], $(L, \mathbf{U}^*)$ is complete. Hence $L = CL$, and by [1, Proposition 3], L is compact.

Suppose that $(L, \mathbf{U})$ is a quasi-uniform frame and that L is compact. Then if $u \in \mathbf{U}$, S_u has a finite subcover and therefore

$\mathbf{U}$ is totally bounded. Since L is compact, it is paracompact, and consequently the fine uniformity is complete [2]. Since L is compact, the fine uniformity is the unique admissible uniformity. Since $\mathbf{U}^*$ is admissible, it is the fine uniformity. Hence $(L, \mathbf{U}^*)$ is complete. By [10, Proposition 3.3], $(L, \mathbf{U})$ is complete. □

Corollary 4.4. *Let $(L, \mathbf{U})$ be a totally bounded quasi-uniform frame. Then $(L, \mathbf{U})$ is complete if and only if L is compact.*

Theorem 4.5. *The completion of a totally bounded quasi-uniform frame is compact.*

Proof. Let $(L, \mathbf{U})$ be a totally bounded quasi-uniform frame. Then $\mathbf{U}^*$ is totally bounded. By [1, Proposition 3], $(CL, C\mathbf{U}^*)$ is compact. The result now follows from $(CL, (C\mathbf{U})^*) = (CL, C\mathbf{U}^*)$.

Since the completion of a totally bounded quasi-uniform frame is compact [10, Corollary 4.9], it is natural to extend the notion of the Samuel compactification of a uniform frame to quasi-uniform frames. The Samuel compactification of a quasi-uniform frame $(L, \mathbf{U})$ is defined to be the completion of the totally bounded coreflection of $(L, \mathbf{U})$.

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