Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Volume 23, Summer 1998, 207–220

CHARACTERIZATION OF BASES OF COUNTABLE ORDER AND FACTORIZATION OF MONOTONE DEVELOPABILITY

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Abstract

Heath's Theorem that semi-metrizability + point-countable base $\Rightarrow$ developability has been sharpened in many directions. It has been shown that we can replace the point-countable base with a $\delta\theta$-base or the so-called open (G), and still have developability. On the other hand, we can also simultaneously replace semi-metrizability with the property of being a β-space, and have the weaker monotone developability. In this paper, we introduce the concept of a *borrowed local base* that generalizes simultaneously the $\delta\theta$-base and the open (G), and the concept of an *$\iota\iota$-space* that generalizes the β-space and prove that borrowed local base + $\iota\iota$-space = monotone developability, in the presence of regularity, thereby improving upon all the results cited above. The concept of a *borrowed local base* comes about in a careful study of the concept of the BCO of Arhangel'skiǐ that rewards us here with a canonical form of the BCO. There is also another factorization of monotone developability where concepts of ϵ- and o-spaces replace that of spaces with borrowed local bases.

Mathematics Subject Classification: 54E35, 54E30, 54E25, 54E20, 54E18, 54E99

Key words: Characterizations of Bases of Countable Order and factorizations of monotone developability, laterally well-ordered trees, branches, left-shifted lines, inscriptions, borrowed local bases, o-, ϵ-, ι- and $\iota\iota$-spaces.

In this paper, we are to extend the work of Wicke and Worrell [16], [17], [18], [19] and Chaber, Čoban and Nagami [5], [6] on Arhangel'skiĭ's Bases of Countable Order [1] (For Definition, see 0.3 below). Worrell and Wicke gave a characterization of *spaces with a* BCO in Theorem 2 of [19] (See 0.3 below) that suggested to Chaber, Čoban and Nagami [6] the designation *monotonically developable* spaces in their reference to *spaces with a* BCO. We are to follow their lead and adopt the designation. Chaber, Čoban and Nagami also gave a characterization of *spaces with a* BCO in terms of what they called a *sieve* (Lemma 1.1 of [6]).

We are to give here two characterizations of BCO's (as opposed to the characterization of monotone developability) in the framework of what I call *laterally well-ordered trees* (Theorems 1 and 3, cf. Definition 1.1 and Theorem 5.2 of [16]). Part of the observation in Theorem 1 (Cor. 2) allows the formulation of the idea of spaces with *borrowed local bases*, making it possible to factor the notion of monotone developability (Theorem 5) in a manner complementary to an earlier theorem (Theorem 4). While Theorem 4 was largely established in [14], [15], improving results of Hodel [11] and Wicke and Worrell [17], [18], Theorem 5 represents a simultaneously generalization of Heath [10], Chaber (Cor. 2.16 of [5]) and Gruenhage (Lemma 3.2 of [9]), and Theorem 1.2 of [13].

0. Definitions, Notations, Terminology and Simple Facts

1. Throughout this paper, $(X, \mathcal{T})$ denotes a T_1-space. We confine our discourse to T_1-spaces so that we can always have *proper* open subsets given a non-void open set, unless it is a singleton.

2. Let there be $A : \{(x, U) : x \in U \in \mathcal{T}\} \to \mathcal{T}$. A is said to be a *shrinking of open neighbourhoods* on X, if $x \in A(x, U) \subset U$, whenever $x \in U \in \mathcal{T}$. Given two shrinkings, A and B, of open neighbourhoods on X, if $B(x, U) \subset A(x, U)$, whenever $x \in U \in \mathcal{T}$, we write $B < A$. A property $\mathcal{P}$ on the shrinking

A of open neighbourhoods on X is said to be *monotone* if, A has property $\mathcal{P} \Rightarrow B$ has property $\mathcal{P}$ whenever $B < A$. In the following, we define three monotone [1] properties on the shrinking A of open neighbourhoods on X:

(o) given open neighbourhoods U and V of, respectively, x and y, $x \in A(y, V)$ and $y \in A(x, U) \Rightarrow$ either $A(y, V) \subset U$ or $A(x, U) \subset V$;

(ϵ) given open neighbourhoods U_n, $n \in \omega$, of x, $U_{n+1} \subset A(x, U_n)$ for all $n \in \omega \Rightarrow \bigcap\{U_n : n \in \omega\}$ is not a neighbourhood of x, unless x is isolated, and

(ι) given open neighbourhoods U_n, $n \in \omega$, of, respectively, x_n, $U_{n+1} \subset A(x_n, U_n) \backslash \{x_n\}$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} \neq \emptyset \Rightarrow \langle x_n \rangle$ has a cluster point.

We say X is an o-, an ϵ- and an ι-space, if on X is, respectively, a shrinking of open neighbourhoods with (o), with (ϵ) and with (ι). These properties are quite weak. Indeed, the θ-spaces of Hodel [11], themselves weak (see, for example, [7]), are o-spaces. Spaces of countable pseudocharacter are ϵ-spaces. We can also have a *sequence* of decreasing shrinkings $\langle A_n \rangle$ of open neighbourhoods on X and define on them a *monotone* property:

($\iota\iota$) given open neighbourhoods U_n, $n \in \omega$, of, respectively, x_n, $U_{n+1} \subset A_n(x_n, U_n) \backslash \{x_n\}$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} \neq \emptyset \Rightarrow \langle x_n \rangle$ has a cluster point,

and speak of $\iota\iota$-spaces. We can see that β-spaces of Hodel [11] and monotonic β-spaces of Chaber [5], themselves weak (see, for example, remarks after 7.7 of [8]) are $\iota\iota$-spaces [2].

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[1] The usage of the term *monotone* here is different from that in [5] and [6].

[2] In [14], we made the assertion that β-spaces are ι-spaces. That is not quite right.

For, given any open neighbourhood U of x, we can always let $A_n(x,U) \equiv U \cap g(n,x)$ (to use Definition 7.7 of β-spaces in [8]), for example. If we have *regularity* on X, we can of course assume that $x \in A(x,U) \subset ClA(x,U) \subset U$, and likewise $x \in A_n(x,U) \subset ClA_n(x,U) \subset U$ whenever $x \in U \in \mathcal{T}$.

3. A base $\mathcal{V}$ on X is a Base of Countable Order (BCO) if every sequence $\langle V_i \rangle$ of elements of $\mathcal{V}$, *strictly* decreasing according to set inclusion, constitutes a local base at every point $\xi \in \bigcap\{V_i : i \in \omega\}$. There are two characterizations of spaces with BCO's, the one used as definition in the Handbook, Definition 6.1 of [8] (Theorem 2 of [19]), and the one formulated in terms of the celebrated *sieve* of Chaber, Čoban and Nagami [6]. Theorem 2 of [19] says that X has a BCO if, and only if, there is on it a sequence $\langle \mathcal{B}_n \rangle$ of bases so that every decreasing sequence $\langle B_n \rangle$ of open sets constitutes a local base at every point $\xi \in \bigcap\{B_i : i \in \omega\}$, provided $B_n \in \mathcal{B}_n$ for every $n \in \omega$. We think of the *sieve* as a tree of open sets of height ω and speak of a BCO tree (see Theorem 6.3 of [8]).

4. A *laterally well-ordered tree* on X is a *tree* of open subsets of height ω on which

   i) *immediate successors*[3] of any element are *subsets* of that element,

   ii) every *level*[3] $\mathcal{U}_i$ is a *cover* of X, and

   iii) there is on every level $\mathcal{U}_i$ a *well-order* $<_i$ so defined that, if we write, for every $U \in \mathcal{U}_i$, $\tilde{U}$ for $U \setminus \bigcup\{V : V <_i U\}$, we have

      a) $\tilde{U} = \bigcup\{\tilde{W} : W$ is an *immediate successor*[3] of $U\} \neq \emptyset$, and

[3] Concepts of *immediate successors*, *levels* and *branches* are standard concepts on trees in the literature, quite independent of the lateral well-order being introduced, and are therefore not explained here.

b) $W <_{i+1} W'$, if $U <_i U'$, W and W' are respectively *immediate successors*[3] of U and U'.

When we speak of the collection of open subsets $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$, we often understand it to be the *laterally well-ordered tree*, without mentioning explicitly the structures among the elements, and vice versa. These laterally well-ordered trees are very common objects. Indeed, we have

Theorem 0. *Given any base $\mathcal{V}$ of the topology on X, there is a* laterally well-ordered tree $\mathcal{U} = \bigcup\{\mathcal{U}_i : i \in \omega\} \subset \mathcal{V}$, *every branch of which that involves no singleton open sets is a* strictly *decreasing sequence of open sets.*

Proof. Out of the members of $\mathcal{V}$, we can construct a collection $\mathcal{U}_0$ *well-ordered* by $<_0$ so that

i) $\bigcup\mathcal{U}_0 = X$ and

ii) $\tilde{U} \equiv U \setminus \bigcup\{V \in \mathcal{U}_0 : V <_0 U\} \neq \emptyset, \ \forall U \in \mathcal{U}_0$.

For every $\Xi \in \mathcal{U}_0$, out of the members of $\mathcal{V}$ that are subsets of Ξ, *proper* ones unless Ξ is a singleton set, we can construct a collection $\mathcal{U}_1(\Xi)$ *well-ordered* by $<_\Xi$ so that

i) $\bigcup\mathcal{U}_1(\Xi) \supset \tilde{\Xi}$ and

ii) $\tilde{U} \equiv U \setminus \bigcup\{V \in \mathcal{U}_1(\Xi) : V <_\Xi U\} \setminus \bigcup\{V \in \mathcal{U}_0 : V <_0 \Xi\} \neq \emptyset$, $\forall U \in \mathcal{U}_1(\Xi)$,
and let members of $\mathcal{U}_1(\Xi)$ be the *immediate successors* [3] of Ξ.

Let $\mathcal{U}_1 \equiv \bigcup\{\mathcal{U}_1(\Xi) : \Xi \in \mathcal{U}_0\}$ and define a *well-order*, $<_1$, on $\mathcal{U}_1$, *extending* the well-orders on the individual $\mathcal{U}_1(\Xi)$'s, so that, if $U \in \mathcal{U}_1(\Xi), U' \in \mathcal{U}_1(\Xi')$ and $\Xi <_0 \Xi'$, we have $U <_1 U'$, *ad infinitum.* We thus have for every $n \in \omega$ a *well-ordered* cover $\mathcal{U}_n$ of X, and a *laterally well-ordered tree* in $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$. □

A sequence $\langle U_n \rangle$ of elements of $\mathcal{U}$ so that, for all $n \in \omega$, $U_n \in \mathcal{U}_n$ and $U_{n+1} \leq_{n+1}$ *some* immediate successor of U_n, is to be called a *left-shifted line*. *Branches*[3] are of course *left-shifted lines*. Clearly, $(*)$ given $U, V \in \mathcal{U}_n$, for any $n \in \omega, U \subsetneq V$, as subsets on $X \Rightarrow U <_n V$, as elements on $\mathcal{U}_n$.

5. Given $x \in X$. If $x \in \widetilde{B}_i \subset B_i \in \mathcal{U}_i$ for every $i \in \omega$, then clearly $\langle B_i \rangle$ is a *branch* and is designated $\mathcal{U}^x$.

6. Given a sequence $\langle U_i \rangle$ of subsets and a sequence $\langle x_i \rangle$ of points on X. If $x_i \in U_i$, for every $i \in \omega$, we say $\langle x_i \rangle$ is *inscribed* in $\langle U_i \rangle$. Given a *laterally well-ordered tree* $\mathcal{U}$ on X, we say X has *borrowed local bases* with respect to $\mathcal{U}$ if there is, for every $x \in X$, a countable collection $\{B_i(x) : i \in \omega\}$ of open subsets such that $\{B_i(x_j) : \xi \in B_i(x_j), i, j \in \omega\}$ is a local subbase at ξ, whenever $\langle x_i \rangle$ is inscribed in $\langle \widetilde{U}_i \rangle$ for some *left-shifted line* $\langle U_i \rangle$ on $\mathcal{U}$ so that $x_m \in U_n$ for all $n < m$, and clusters to $\xi \in \bigcap\{U_i : i \in \omega\}$.

7. A space $(X, \mathcal{T})$ is said to have (G) if, for every $x \in X$, there is a sequence $\langle W_n(x) \rangle$ of subsets, each containing x, so that, if $\xi \in U \in \mathcal{T}$, there is an open neighbourhood $V(\xi, U)$ of ξ so that, for every $y \in V(\xi, U)$, $\xi \in W_m(y) \subset U$ for some $m \in \omega$ (dependent on y). If $\langle W_n(x) \rangle$ is *decreasing*, X is said to have *decreasing* (G). If $W_n(x)$ is an *open* neighbourhood of x, X is said to have *open* (G) [4]. The notion of a decreasing (G) is equivalent to that of a *point-network* of Balogh [3]. Clearly, spaces $(X, \mathcal{T})$ with decreasing (G) are ι-spaces (0.2 above). For, if, for any $x \in U \in \mathcal{T}$, we let $A(x, U) = V(x, U)$, we see that, given the hypothesis of the condition of (ι) with $\xi \in \bigcap\{U_n : n \in \omega\}$ and therefore $m_n \in \omega$ such that $x_n \in W_{m_n}(\xi) \subset U_n$, for every $n \in \omega$, the sequence $\langle m_n \rangle$ has to be strictly increasing and $x_n \to \xi$.

1. Results

Theorem 1. *If* $\mathcal{U}$ *is a laterally well-ordered tree* such that every *branch* $\mathcal{B}$ is a local base at (every) $\xi \in \bigcap \mathcal{B}$, then $\mathcal{U}$ *is* a BCO on X. Conversely, every BCO on X contains in it such a $\mathcal{U}$.

Proof. We are to prove that given a sequence $\langle U_i \rangle$ of elements of $\mathcal{U}$, strictly decreasing in the sense of set inclusion, there is such a *branch* $\mathcal{B}$ on $\mathcal{U}$ that every member of $\mathcal{B}$ can count among its successors on $\mathcal{U}$ at least one member of $\langle U_i \rangle$. For every $n \in \omega$, because $\mathcal{U}_n$ is well-ordered, in view of $(*)$ of 0.4 above, we have $| \mathcal{U}_n \bigcap \{U_i : i \in \omega\} | < \omega$ and therefore $\{U_i : i \in \omega\} \cap \bigcup \{\mathcal{U}_m : n < m\} \neq \emptyset$. Consequently, for every $n \in \omega$, we have a *first* member, B_n, on $\mathcal{U}_n$, to count among its successors on $\mathcal{U}$ elements of $\langle U_i \rangle$. Given $n < m$. If A is the predecessor of B_m on $\mathcal{U}_n$, then we see that we cannot have $A < B_n$ (otherwise B_n is not the *first* on $\mathcal{U}_n$ to count among its successors on $\mathcal{U}$ elements of $\langle U_i \rangle$). We cannot have $B_n < A$ either (otherwise B_m is not the *first* on $\mathcal{U}_m$ to count among its successors on $\mathcal{U}$ members of $\langle U_i \rangle$). $\therefore A = B_n$, i.e., $\langle B_n \rangle$ is a *branch.*

If $x \in \bigcap \{U_i : i \in \omega\}$, then clearly $x \in \bigcap \mathcal{B}$. If $\mathcal{B}$ is a local base at x, then $\{U_i : i \in \omega\}$ is also a local base at x, i.e. $\mathcal{U}$ *is* a BCO.

The converse follows from Theorem 0 where $\mathcal{V}$ is now a BCO. □

Remarks. 1. In the characterization of monotone developability of a space X in terms of the existence of *sieves*, we have essentially a tree of open sets of height ω, such that X is covered by the family of elements on the first level, each elemental open set on the tree is *covered* by its successors and every branch $\mathcal{B}$ constitutes a local base at every $\xi \in \bigcap \mathcal{B}$ (6.3 of [8]). Clearly, if we well order the levels of this tree in the manner suggested in the proof of 6.3 of [8] and take care to *jettison* the elements that fail to satisfy ii) in the proof of Theorem 0, as every element is required to do on a *laterally well-ordered tree*, effectively lopping off some of the branches, we will be rewarded with a *laterally well-ordered tree*. This laterally well-ordered tree *being* a BCO, according to our theorem, the tree with all the branches intact *contains* a BCO. Our theorem therefore implies Chaber-Čoban-Nagami and our proof is good for their theorem. The argument presented in our proof is much more elementary than the method of Wicke and Worrell on which Chaber, Čoban and Nagami relied, and we have effectively a simpler proof of Chaber-Čoban-Nagami. There is no need to resort to *decreasing sequences of bases*, for one thing. Note also that the *laterally well-ordered tree* $\mathcal{U}$, the branches $\mathcal{B}$ of which constitute local bases at $\xi \in \bigcap \mathcal{B}$, *is* itself a BCO and we have a special kind of BCO's that are present within every BCO, that is, we have a *canonical* form of a BCO. Furthermore, in such a canonical $\mathcal{U}$, there is clearly (†) a decreasing sequence $\langle \mathcal{W}_n \rangle$ of bases so that $x \in W_n \in \mathcal{W}_n$, $W_{n+1} \subset W_n, n \in \omega \Rightarrow \{W_n : n : \omega\}$ is a base at x. (For, we can always let $\mathcal{W}_n \equiv \bigcup\{\mathcal{U}_i : i \geq n\}$ for every $n \in \omega$, noting that a decreasing $\langle W_n \rangle$ *contains* either a *strictly* decreasing subsequence or a *smallest* member.) On the other hand, if we have (†), we can construct a canonical $\mathcal{U}$ out of $\bigcup\{\mathcal{W}_n : n \in \omega\}$ as we did in the proof of Theorem 0, only taking care this time that $\mathcal{U}_n \subset \mathcal{W}_n$, for every $n \in \omega$. It follows then the characterization of *spaces with a* BCO of Worrell and Wicke cited in the introductory paragraphs is also an easy corollary of our Theorem 1 above.

2. If the sequence $\langle U_i \rangle$ is a *left-shifted line*, instead of a strictly decreasing sequence, on the laterally well-ordered tree $\mathcal{U}$, we can still arrive at the conclusion that $x \in \bigcap\{U_i : i \in \omega\}$, under the assumption that every branch of $\mathcal{U}$ is a local base at (every) $\xi \in \bigcap \mathcal{B}, \Longrightarrow \{U_i : i \in \omega\}$ is a local base at x.

3. Our characterization of a BCO here should remind one of Definition 1.1 in [16] of *primitive bases*, and be compared with Theorem 5.2 of the same paper.

4. If we bring in the concept of a local *pseudo*-base at a point

as opposed to a local base (so that we can talk about *pseudo*-character of the point instead of its character) and the concept therewith of *pseudo*-bases of countable order (*P*BCO), we have a similar theorem in those terms.

Corollary 2. On monotonically developable spaces X, there are always *borrowed local bases* with respect to some *laterally well-ordered tree* $\mathcal{U}$ (every branch $\mathcal{B}$ of which is a local base at (every) $\xi \in \bigcap \mathcal{B}$).

Proof. Every monotonically developable space X has on it a *laterally well-ordered tree* $\mathcal{U}$ (every branch $\mathcal{B}$ of which is a local base at (every) $\xi \in \bigcap \mathcal{B}$). If we let $\{B_i(x) : i \in \omega\} = \mathcal{U}^x$ so that $B_i(x)$ is the first on $\mathcal{U}_i$ to contain x, we see that, given any *left-shifted line* $\langle U_i \rangle$ on $\mathcal{U}$ and any $\langle x_i \rangle$ *inscribed* in $\langle \widetilde{U}_i \rangle$, the family $\{U_i : i \in \omega\}$ is a local base at any $\xi \in \bigcap\{U_i : i \in \omega\}$ (by item 2 of Remarks on Theorem 1) and *a fortiori* the *bigger* family $\{B_i(x_j) : \xi \in B_i(x_j), i, j \in \omega\}$ is a local base at $\xi \in \bigcap\{U_i : i \in \omega\}$ (whether $\langle x_i \rangle$ clusters to ξ or not). □

Remarks. A point-countable open base or more generally the so called open (G) (especially as it is represented in Lemma 1.2 of [9]) and the H-spaces in [13] on X clearly bring about *borrowed local bases* with respect to *any* laterally well-ordered tree. One can also see that the $\delta\theta$-base of Aull [2] brings about *borrowed local bases* with respect to some laterally well-ordered tree wrested from the countable closed cover $\{F_n : n \in \omega\}$ in the proof of 8.2 of [8] in the manner indicated in the following. Let $\mathcal{C}_0 = (\backslash F_0, X)$ be an *ordered open cover* of X. Suppose we have already *ordered open covers* $\mathcal{C}_0, \mathcal{C}_1, \ldots \mathcal{C}_n$ of X, and $C \in \mathcal{C}_n$, we have two immediate sucessors $C \cap \backslash F_{n+1}$ and C and order $\mathcal{C}_{n+1}$ so that $C \cap \backslash F_{n+1}$ preceeds C and, of $C, C' \in \mathcal{C}_n$, the immediate successors of C preceed those of C' provided C preceeds C' on $\mathcal{C}_n$. If we take care to jettison, during our construction, any element C for which $\tilde{C} = \emptyset$, we see that $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in \omega\}$ is a *laterally well-ordered tree.*

Theorem 3. A necessary (and sufficient) condition for a *laterally well-ordered tree* $\mathcal{U}$ on X to be a *canonical* BCO (see first item of Remarks on Theorem 1) is that every branch $\mathcal{B}$ on $\mathcal{U}$ of non-void intersection be $\mathcal{U}^x$ for some $x \in X$ and, for every $x \in X$, $\mathcal{U}^x$ be a local base at x.

Proof. If $\mathcal{U}$ on X is a *canonical* BCO, any branch $\mathcal{B} \equiv \langle B_i \rangle$ on $\mathcal{U}$ is a local base at $x \in \bigcap \mathcal{B}$ (if there is such an x). Necessarily the point $x \in \bigcap\{\widetilde{B}_i : i \in \omega\}$, otherwise there is an $i \in \omega$ such that $x \in B_i \backslash \widetilde{B}_i$, and $\bigcup\{U \in \mathcal{U}_i : U < B_i\}$ is an open neighbourhood of x that contains no members of $\mathcal{B}$ (even though $\mathcal{B}$ is a base at x), clearly a contradiction. We therefore have $x \in \bigcap\{\widetilde{B}_i : i \in \omega\}$ and $\mathcal{B} = \mathcal{U}^x$. □

Remarks. Our characterization of a BCO here is a very surprising one in view of Definition 1.1 in [16] of primitive bases and Theorem 5.2 of the same paper.

Theorem 4. Monotonically developable spaces are o-, ϵ- and ι-spaces. T_3-spaces that are simultaneously o-, ϵ- and $\iota\iota$-spaces are monotonically developable.

Proof. It was pointed out in Fletcher and Lindgren [7] that spaces with *primitive bases* are θ-spaces and θ-spaces are o-spaces [14], [15]. Monotonically developable spaces are of course first countable and therefore ϵ-spaces [14], [15].

To prove that monotonically developable spaces $(X, \mathcal{T})$ are ι-spaces, let $\mathcal{V}$ be a BCO on X. If, given $x \in U \in \mathcal{T}$, we let $A(x, U) \in \mathcal{V}$, we see that, given the hypothesis of the condition of (ι) with $\xi \in \bigcap\{A(x_n, U_n) : n \in \omega\}$, the sequence, being *strictly decreasing*, is a local base at ξ making sure that $x_i \to \xi$ and X an ι-space.

The second statement is largely Theorem 2.1 of [15] and Remarks on it there. The construction in the proof of Theorem 2.1 of [15] can be slightly refined to take into account of the weakening of the (ι) property to $(\iota\iota)$ in our hypothesis. □

Theorem 5. A regular X is monotonically developable if, and only if, it is an $\iota\iota$-space and has *borrowed local bases* with respect to some *laterally well-ordered* tree $\mathcal{U}$.

Proof. That our conditions are necessary for monotone developability follows from Cor. 2 and Theorem 4 above. To prove they are sufficient, we proceed to construct a tree $\mathcal{V}$ of open neighbourhoods (of *specific* points) of height ω, each element of which, as an open neighbourhood of a specific point, is the union of the family of its immediate successors (as open subsets), and each branch $\mathcal{B}$ of which constitutes a base at any $\xi \in \bigcap \mathcal{B}$. More specifically, noting that X is first countable and therefore an ϵ-space, we let the first level $\mathcal{V}_o$ be $\{A(x,U) \bigcap A_o(x,U) : x \in \widetilde{U}, U \in \mathcal{U}_o\}$, A and A_0 being the A and A_0 of (ϵ) and $(\iota\iota)$ in 0.2. Suppose we have defined $\mathcal{V}_o, \mathcal{V}_1, \ldots, \mathcal{V}_n$ of our tree of open neighbourhoods and we are to define the next level $\mathcal{V}_{n+1}$. We take a member of $\mathcal{V}_n$, say, an open neighbourhood V_n of x_n with predecessors, the open neighbourhoods V_{n-1} of x_{n-1} on $\mathcal{V}_{n-1}$, V_{n-2} of x_{n-2} on $\mathcal{V}_{n-2}$, ..., V_o of x_o on $\mathcal{V}_o$. For each $x \in V_n$, we provide an open neighbourhood of x as follows. We let

i) $U \in \mathcal{U}_{n+1}$ such that $x \in \widetilde{U}$, and

ii) $B = \bigcap \{B_i(x_j) : x \in B_i(x_j), i, j \leq n\}$.

If $x \neq x_n$, we let

iiia) $W \equiv [V_n \backslash \{x_n\}] \cap U \cap B$ and provide x with the open neighbourhood $A_\nu(x, W)$, where $\nu = \mid \{x_o, x_1, \ldots x_n\} \mid$.

If $x = x_n$, we let

iiib) $W \equiv V_n \cap U \cap B$, and provide x with the open neighbourhood $A(x, W)$.

Clearly, the tree so constructed is a BCO tree (0.3), *provided* we can show that, for each branch $\mathcal{B}$, $\xi \in \bigcap \mathcal{B} \Rightarrow \mathcal{B}$ is a local base at ξ. Given the branch $\mathcal{B}$, explicitly, $V_o, V_1, V_2, \ldots$, open neighbourhoods of $x_o, x_1, x_2, \ldots$ on levels $\mathcal{V}_o, \mathcal{V}_1, \mathcal{V}_2, \ldots$ respectively.

The case of its alternative being almost trivial, we assume that the sequence $\langle x_i \rangle$ consists of *infinitely* many *distinct* points, and property $(\iota\iota)$ ensures that $\langle x_i \rangle$ clusters to some $\eta \in \bigcap\{V_i : i \in \omega\}$. Clearly, if $x_n \in \widetilde{U}_n \subset U_n \in \mathcal{U}_n$ for every $n \in \omega$, we have a left-shifted line in $\langle U_n \rangle$ so that $x_m \in U_n$ for all $n < m$ and $\eta \in \bigcap\{U_n : n \in \omega\}$, $\langle x_n \rangle$ *inscribed* in $\langle \widetilde{U}_n \rangle$, and therefore a local subbase at η of the form $\{B_i(x_j) : \eta \in B_i(x_j), i, j \in \omega\}$. Given an open neighbourhood Ξ of η, we have $\mu \in \omega$ such that $\eta \in \Upsilon \equiv \bigcap\{B_i(x_j) : \eta \in B_i(x_j), i, j \leq \mu\} \subset \Xi$. But then, Υ being a neighbourhood of η, there is a $\nu > \mu$ such $x_\nu \in \Upsilon$ and such that $\eta \in V_\nu \subset \Xi$, making $\{V_i : i \in \omega\}$ a local base at η. X being T_1, $\xi \notin \bigcap\{V_i : i \in \omega\}$ unless $\xi = \eta$. We therefore have on X a BCO tree and the monotone developability of X. □

Remarks. 1. This result generalizes at once Lemma 3.2 of [9] and Theorem 8.2 of [8] (due to Chaber). It has the additional advantage of being a *necessary* and sufficient condition, and provides a second factorization of monotone developability after Theorem 4. These two results, Theorems 4 and 5, provide two solutions to the question of Morton Brown that asks for properties that ensure developability in the face of semimetrizability. There is also an answer to Brown's question in [12] where the property semi-metrizability is much more fully exploited to produce an answer.

2. Clearly then, Cor. 3.3 in [9] can be generalized to: Let X be a submetacompact $\iota\iota$-space satisfying open (G). Then X is a developable space with a point-countable base. Since spaces with decreasing (G) are *ι-spaces* and therefore *$\iota\iota$-spaces* (0.7 above) and since spaces with open (G) have *borrowed local bases* (Remarks on Corollary 2 above), an immediate corollary to Theorem 5 is that spaces with decreasing open (G) are monotonically developable, and the Theorem of Balogh-Collins-Reed-Roscoe-Rudin (Corollary 2.3 of [3], Theorem 8 of [4]) follows, paracompactness taken care of by decreasing (G) (Lemma 1.3 of [3] and Theorem 4 of [4]).

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