# Topology Proceedings

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

**Topology Proceedings**
**Volume 23, Summer 1998, 323–348**

# ON EILENBERG–MOORE ALGEBRAS INDUCED BY CHAINS

Sergio Salbany

Dedicated to Nico Pumplün on the occasion of his 65th anniversary, as an expression of deepest appreciation and sincere good wishes.

**Abstract**

Let $S$ be a fixed topological space. The contravariant Hom functor given by $C(X) = Hom_{\underline{Top}}(X, S)$ has an adjoint specified, on sets, by $P(A) = S^A$ and the composite, $M = C \circ P$, is a Monad on the category of sets. In this paper we characterize the category of Eilenberg–Moore Algebras associated with $M$ in the special case where $S$ is a linearly ordered space in its specialization order. The characterization is presented in terms of the notion of a dual frame which admits a $C(S)$–action.

## 1. Introduction. The Monad $M_S$

Let $S = (S, T)$ be a given topological space. Let $M_S$ be the induced monad on the category of sets specified on objects by

$$M_S(A) = Hom_{\underline{Top}}(S^A, S) = C(S^A)$$

*Mathematics Subject Classification*: Primary: 18C15, 54B30; Secondary: 06D20, 08A99, 54F05

*Key words*: $C(S)$–action, complete dual frame, $C(S)$–consistent complete dual frame, Eilenberg–Moore Algebra

The problem of identifying explicitly the Eilenberg–Moore category of $M_S$–algebras has not been solved for a general topological space $S$, as far as we know, though many interesting special cases have been treated in the literature: for example, R.-E. Hoffmann discusses all cases for which the cardinality of $S$ does not exceed 2 [1]. F. Linton had earlier dealt with the case where $S$ is the Sierpiński dyad. In [2] and [4] the case of the unit interval with its usual topology is discussed. Related and extensive work has been done in other categories, though we shall only single out the "Locally Convex Algebras" of D. Pumplün and H. Röhrl [5]; and that of J.W. Pelletier and J. Rosický [4], where further references are also given. Other examples are provided in P. Johnstone's book *"Stone Spaces"* [3] to which we refer for the general theory.

Our intention has been to obtain a unified description of the Eilenberg–Moore category of $M_S$–algebras for a general $S$. This, however, is not the appropriate place to report on that work; instead we shall consider a further special case: $S$ is a finite linearly ordered set when given the specialization order: $x \leq y$ if and only if $y \in \mathrm{cl} x$, equivalently, when $S$ is a finite $T_0$ space whose topology is linearly ordered by set inclusion.

The reason for studying this special case should be clear if we recall that the category of $M_S$–algebras is the category of Frames and Frame Homomorphisms when $S$ is the Sierpiński dyad. The study also leads to an interesting description of the algebras as "$C(S)$–module"–type structures.

Finally, the case where $S$ is an infinite linearly ordered set seems to involve an additional notion of limit. We have not yet succeeded in describing the resulting category of Eilenberg–Moore algebras in a way analogous to the finite case, not even when $S$ is compact.

We shall record, for ease of reference, the following well known and readily verifiable facts concerning the monad $M_S$ on $\underline{Ens}$:

(i) The unit $\eta$ is specified by:

$$\eta_A : A \to C(S^A, S), \qquad \eta_A(a) = \pi_a \text{ for all } a \in \mathrm{S}.$$

(ii) The multiplication $\mu$ is specified by:

$\mu_A : M_S^2 A \to M_S A$ is given by composition on the right with the evaluation map $e_{S^A} : S^A \to S^{C(S^A,S)}$, where $\pi_f \circ e_{S^A} = f$ for all $f$ in $C(S^A, S)$.

Thus $\mu_A(\alpha) = \alpha \circ e_{S^A}$, for all $\alpha$ in $M_S^2(A)$.

(iii) For $f : A \to B$, we have $\hat{f} : S^B \to S^A$ given by $\pi_a \circ \hat{f} = \pi_{f(a)}$.

Then $Mf : C(S^B, S) \to C(S^A, S)$ is given by composition with $\hat{f}$ on the right: $Mf(\beta) = \beta \circ \hat{f}$, for all $\beta$ in $C(S^B, S)$.

## 2. The Chain *S*

We shall be concerned with finite chains $S$ with a smallest element 0, a largest element 1; equipped with the $u$–topology with basic open sets of the form $[0, a)$, $a \in S$. Denote by $S_u$ the resulting topological space. The $l$–topology on $S$ has basic open sets of the form $(a, 1]$, $a \in S$. The $u \vee l$–topology on $S$ is the discrete topology $d$ on $S$. Denote by $S_l$ and $S_d$ the corresponding topological spaces.

Observe that $S$ has the property that if $x$, $y$, $a$, $b$ are in $S$ and $x < y$ , $a < b$, then there is $\varphi \in C(S, S)$ such that $\varphi(x) = a, \varphi(y) = b$, when $S$ is $S_u$, $S_l$ or $S_d$.

Finally, let us record some facts that will be used later without further explanation: In all cases, $S$ is a complete ordered set; every $\varphi \in C(S_u, S_u)$ is a monotone function and preserves finite infima and finite suprema in $S$; moreover, when $C(S_u)$ is given the pointwise induced partial order: $f \leq g$ if, for all $x \in S$, $f(x) \leq g(x)$, then it is closed under finite suprema and arbitrary infima, where these constructs are specified pointwise. Of course, $C(S_u, S_u)$ is also closed under arbitrary suprema, but these are not given by pointwise evaluation.

Throughout this note $S$ will always refer to a finite chain with the $u$–topology, unless specified otherwise.

## 3. A Theorem of Stone–Weierstrass Type for $C(S^A)$

It will be essential to express the functions in $C(S^A)$ in terms of the projection mappings. A simple description is available in terms of $\bigwedge$ and $\bigvee$ when $S$ is a finite chain.

**Theorem 3.1.** *Every continuous function $f : S_u^A \to S_u$ is the pointwise infimum of finite suprema of functions of the form $\varphi \circ \pi_a$, where $\varphi$ is in $C(S_u)$ and $\pi_a : S^A \to S$ is a projection.*

*Proof.* We first establish a two point approximation property. Given $x$, $y$ in $S^A$, we have $f(y) < f(x)$ or $f(x) < f(y)$ or $f(x) = f(y)$. Assume $f(y) < f(x)$. Since $f$ is, necessarily, a monotone non–decreasing function, we cannot have $x \le y$. Hence there is $a \in A$ such that $\pi_a(x) > \pi_a(y)$. Since $f(x) > f(y)$ there will exist a continuous function $\varphi_a : S_u \to S_u$ such that $\varphi_a(\pi_a(x)) = f(x)$, $\varphi_a(\pi_a(y)) = f(y)$.

Similarly, if $f(x) < f(y)$, then there is $b \in A$ and $\varphi_b \in C(S_u)$ such that $\varphi_b \circ \pi_b(x) = f(x)$, $\varphi_b(\pi_b(y)) = f(y)$.

When $f(x) = f(y) = c$, then $\underline{c} \circ \pi_a$ is the required function, where $\underline{c}$ denotes the constant map to $c$, $\underline{c} : S_u \to S_u$, where $a$ is any element of $A$.

We denote the function constructed above by $f_{xy}$. Thus, $f_{xy} : S_u^A \to S_u$, $f_{xy}(x) = f(x)$, $f_{xy}(y) = f(y)$ ("two point approximation property").

Let us observe that $f_{xy}$ is also a continuous function from $S_l^A$ to $S_l$, since $\pi_a : S_l^A \to S_l$ and $\varphi : S_l \to S_l$ are continuous. Hence $f_{xy} : S_{u\vee l}^A \to S_{u \vee l}$ is continuous.

Now, fix $x$. For each $y$ there is a $\Pi l$–neighbourhood of $y$, $V_y$, whose image under $f_{xy}$ is contained in $[f(y), 1]$, by $(\Pi\, l - l)$–continuity of $f_{xy}$. By continuity of $f$, there is a $\Pi u$–neighbourhood, $W_y$, of $y$, whose image under $f$ is contained in $[0, f(y)]$. Thus $V_y \cap W_y = U_y$ is a $\Pi l \bigvee \Pi u = \Pi(l \vee u)$–neighbourhood of $y$. Since $l \vee u$ is the discrete topology on $S$ and $S$ is a finite set, the product space $\Pi_{a\in A}(S, u\vee l)_a$ is a compact. Hence, there are finitely many $U_y$'s which cover $S^A$, say $U_{y1}, U_{y2}, \ldots, U_{y_n}$.

Let $f_x = f_{xy_1}\vee f_{xy_2}\vee\ldots\vee f_{xy_n}$. Observe that $f_x : S_u^A \to S_u$ is continuous and that $f_x(x) = f(x)$, since $f_{xy_j}(x) = f(x)$, $\quad 1 \leq j \leq n$.

We now show that $f \leq f_x$ for all $x$. Consider an arbitrary $z$ in $S_A$. By above, $z \in U_{y_j}$ for some $j$. Now, since $z \in V_{y_j}$ we have $f_{xy_j}(z) \geq f(y_j)$; since $z \in W_{y_j}$, we have $f(z) \leq f(y_j)$. Hence $f(z) \leq f_{xy_j}(z) \leq f_x(z)$. Now $z$ is arbitrary, hence $f \leq f_x$ for all $x$. Letting $h = \bigwedge\limits_{x\in S^A} f_x$, we have $h \in C(S_u^A, S_u)$ and $f \leq h$. Observe that, for a given $t$ in $S^A$, $h(t) = \bigwedge\limits_{x\in S^A} f_x(t) \leq f_t(t) \leq f(t)$. Thus $h \leq f$. Hence $h = f$, as required. □

**Definition 3.2.** Let $\mathcal{S}$ denote the set of functions $g : S_u^A \to S_u$ such that $g = \bigvee\limits_{i=1}^{n} \varphi_i \circ \pi_{a_i}$, where $\varphi_i \in C(S_u)$ and $a_i \in A$ for $1, 2, \ldots, n$.

Every $f$ in $C(S^A, S)$ can be expressed as the infimum of all members of $\mathcal{S}$ that dominate it.

**Proposition 3.3.** *Let $f : S_u^A \to S_u$ be a continuous function, then $f = \bigwedge\{g | g \in \mathcal{S},\ f \leq g\}$.*

*Proof.* Let $\mathcal{S}_f$ consist of all $g$ in $\mathcal{S}$ such that $f \leq g$. Then, clearly, $f \leq \bigwedge\{g \mid g \in \mathcal{S}_f\}$ . On the other hand, $f_x$, defined in the proof of 3.1, is in $\mathcal{S}_f$. Hence $\bigwedge\{g \mid g \in \mathcal{S}_f\} \leq f_x$. This inequality holds for all $x$ in $S^A$, hence $\bigwedge\{g | g \in \mathcal{S}_f\} \leq \bigwedge\limits_{x\in S^A} f_x = f$. The proof is complete. □

The set $\mathcal{S}$ has an important compactness property:

**Theorem 3.4.** *Let $f \in \mathcal{S}$, and suppose $\{g_i | \ i \in I\}$ is a family of functions in $\mathcal{S}$ such that $\bigwedge\limits_{i\in I} g_i \leq f$. Then, there exists $g_{i_1}, g_{i_2}, \ldots, g_{i_n}$ such that $\bigwedge\limits_{r=1}^{n} g_{i_r} \leq f$.*

*Proof.* Let $g = \bigwedge_{i\in I} g_i$. For each $i$, there are $\varphi_{ij} \in C(S_u)$, $a_{ij} \in A$, $1 \leq j \leq n(i)$, such that $g_i = \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{a_{ij}}$. Let $\varphi_p, a_p$, $1 \leq p \leq m$, be such that $f = \bigvee_{p=1}^{m} \varphi_p \circ \pi_{a_p}$. Consider $x$ in $S^A$. We have $g(x) = g_i(x)$ for some $i = i(x)$. Similarly, there is $p = p(x)$ such that $f(x) = \varphi_p \circ \pi_{a_p}(x)$. Now, as observed in 3.1, both $f$ and $g_i$ are continuous as functions from $S_d^A$ to $S_d$, where $S_d$ denotes $S$ with the discrete topology. Hence, there is a $\Pi d$–neighbourhood of $x$, $W_x$, say, such that for all $z$ in $W_x$ we have $g_i(z) = g_i(x)$ and $\varphi_p(\pi_{a_p}(z)) = \varphi_p(\pi_{a_p}(x))$, where $i = i(x)$ and $p = p(x)$, determined above. Because of compactness, $S_d^A$ is covered by finitely many $W_x$'s, say $W_{x_1}, W_{x_2}, \ldots, W_{x_N}$. Let $w$ be any element of $S^A$. There is $W_{x_r}$ such that $w \in W_{x_r}$. For $i_r = i(x_r)$ and $p_r = p(x_r)$, we then have:

(1) $g_{i_r}(w) = g_{i_r}(x_r) = g(x_r)$.
(2) $\varphi_{p_r} \circ \pi_{a_{p_r}}(x_r) = f(x_r)$.

Hence:

$$\begin{aligned}\bigwedge_{t=1}^{N} g_{i_t}(w) \leq g_{i_r}(w) &= g_{i_r}(x_r)\\ &= g(x_r) \leq f(x_r)\\ &= \varphi_{p_r} \circ \pi_{a_{p_r}}(w) \leq f(w).\end{aligned}$$

Since $(\bigwedge_{t=1}^{N} g_{i_t})(w) \leq f(w)$ for all $w$ in $S^A$, we have $\bigwedge_{t=1}^{N} g_{i_t} \leq f$, as required. □

As a Corollary, we obtain an apparently stronger version of the Theorem. The formulation below arose from a suggestion by the anonymous referee.

**Corollary 3.5.** If the infimum of a set $F$ of functions in $\mathcal{S}$ is in $\mathcal{S}$, then it is also the infimum of a finite subset of $F$.

It should be noted, however, that the property does not hold generally, as shown by the following example. Thus emphasizing the special nature of $\mathcal{S}$.

**Example 3.6.** Let $\pi_n : 2_u^w \to 2_u$ denote the $n^{\text{th}}$ projection map. Then $\bigwedge\limits_{n=1}^{\infty} \pi_n$ is never equal to the infimum of a finite set of $\pi_n$'s.

## 4. The $M_S$–Algebras

The set $C(S_u^A, S_u)$, with the order induced by $S_u$, is a partially ordered set which is closed under arbitrary infima and finite suprema, defined pointwise. It also admits an action by elements of $C(S_u)$ defined by composition on the left: $\varphi * f = \varphi \circ f$, where $f \in C(S_u^A, S_u)$, $\varphi \in C(S_u)$. We shall show that these operations can be transferred to an $M_S$–algebra $A$ by means of the structure map $h : C(S_u^A, S_u) \to A$.

Firstly, the definition of an action by $C(S_u)$ on a lattice that admits arbitrary infima and finite suprema.

**Definition 4.1.** Let $(A, \leq)$ be a complete lattice. $A$ is said to admit a $C(S_u)$–action if there is a map $C(S_u) \times A \to A$, where $(\varphi, a) \mapsto \varphi * a$, satisfying the following:

(1) $I_S * a = a$ for all $a \in A$, where $I_S$ is the identity map in $C(S_u)$.

(2) $(\varphi \circ \psi) * a = \varphi * (\psi * a)$, where $\circ$ denotes composition in $C(S_u)$.

(3) $\left(\bigwedge\limits_{i \in I} \varphi_i\right) * a = \bigwedge\limits_{i \in I} (\varphi_i * a)$

(3′) $\varphi * \left(\bigwedge\limits_{i \in I} a_i\right) = \bigwedge\limits_{i \in I} (\varphi * a_i)$, $I$ arbitrary.

(4) $\left(\bigvee\limits_{i \in I} \varphi_i\right) * a = \bigvee\limits_{i \in I} (\varphi_i * a)$ (4)′ $\varphi * \left(\bigvee\limits_{i \in I} a_r\right) = \bigvee\limits_{i \in I} (\varphi * a_r)$, $I$ finite.

(5) Let $\varphi_1,\ \varphi_2,\ldots,\varphi_n,\ \psi_1,\ldots,\psi_n \in C\left(S_u\right)$ be such that for all $s_i \in S$ we have $\bigwedge \langle\varphi_i\rangle \leq \bigvee \langle\psi_i\rangle$, i.e.: $\varphi_1(s_1) \wedge \varphi_2\left(s_2\right) \wedge \ldots \wedge \varphi_n\left(s_n\right) \leq \psi_1\left(s_1\right) \vee \ldots \vee \psi_n\left(s_n\right)$, then for any choice $a_1,\ldots,a_n$, we have:

$$\bigwedge_{i=1}^{n} \langle\varphi_1,\ldots\varphi_n\rangle * \langle a_1,\ldots a_n\rangle \leq \bigvee_{i=1}^{n} \langle\psi_1,\psi_2,\ldots,\psi_n\rangle * \langle a_1,\ldots a_n\rangle,$$

i.e.:

$$\varphi_1 * a_1 \wedge \varphi_2 * a_2 \wedge \ldots \wedge \varphi_n * a_n \leq \psi_1 * a_1 \vee \psi_2 * a_2 \vee \ldots \vee \psi_n * a_n.$$

**Examples 4.2.**

**1.** If $(X,T)$ is a topological space, then $C\left(X,S_u\right)$ is a complete lattice, where arbitrary infima and finite suprema are specified pointwise by:

$$f = \bigwedge_{i\in I} f_i \Longleftrightarrow f(x) = \bigwedge_{i\in I} f_i\left(x\right), \quad \text{for all } x\in X.$$

$$f = \bigvee_{j=1}^{n} f_j \Longleftrightarrow f(x) = \bigvee_{j=1}^{n} \left(f_j\left(x\right)\right), \quad \text{for all } x\in X.$$

In this example, $(A,\leq) = C\left(X,S_u\right)$ and the action is specified by composition: for $\varphi\in C\left(S_u\right)$ and $f\in C\left(X,S_u\right)$, $\varphi * f = \varphi\circ f$. The verification of the requisite properties is straightforward, taking into account that $S$ is a finite chain and $\varphi\in C\left(S_u\right)$.

**2.** $(S,\leq)$ itself admits a $C\left(S_u\right)$ action given by evaluation: $\varphi * s = \varphi(s)$ for all $\varphi\in C\left(S_u\right)$, $s\in S$. This example is a particular case of the first one when $X$ is a singleton set.

## 5. The Transfer of Structure from $C\left(S_u^A, S_u\right)$ to $A$ for an Algebra $(A, h)$

We shall assume that $(A, h)$ is an $M_S$–algebra with structure map $h$, so that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal, "1_A"'] & MA \arrow[d, "h"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} M^2A \arrow[r, "Mh"] \arrow[d, "\mu_A"'] & MA \arrow[d, "h"] \\ MA \arrow[r, "h"'] & A \end{tikzcd}$$

We shall start by defining $\bigwedge\limits_{i\in I} a_i$, for an arbitrary index set, and $\bigvee\limits_{i=1}^{n} a_i$ for a finite index set, all elements belonging to $A$; as well as the $C(S_u)$ action on $A$.

**Definitions 5.1.**

**(1)** Let $I$ be a nonempty set of indices, $a_i \in A$ for $i \in I$. Define $\bigwedge\limits_{i\in I} a_i$ to be $h\left(\bigwedge\limits_{i\in I} \pi_{a_i}\right)$.

**(2)** For a finite index set $J$, $a_j \in A$ for $j \in J$, define $\bigvee\limits_{j\in J} a_j$ to be $h\left(\bigvee\limits_{j\in J} \pi_{a_j}\right)$.

**(3)** For $a \in A$, $\varphi \in C(S_u)$ define $\varphi * a$ to be $h(\varphi \circ \pi_a)$.

**Theorem 5.2.** *Under the operations $\bigwedge$ and $\bigvee$, $A$ is a complete* *lattice, and* $h$ *preserves arbitrary infima and finite suprema:*

$$h\left(\bigwedge_{i\in I} f_i\right) = \bigwedge_{i\in I} h\left(f_i\right)$$
$$h\left(\bigvee_{j=1}^{n} f_j\right) = \bigvee_{j=1}^{n} h\left(f_j\right)$$

*Moreover, the dual frame identity holds:* $a \vee \bigwedge\limits_{i\in I} a_i = \bigwedge\limits_{i\in I}\left(a\vee a_i\right).$

**Lemma 5.3.** $Mh$ *and* $\mu_A$ *preserve arbitrary infima and finite suprema of functions, both operations being defined pointwise.*

*Proof.* Both maps are defined as composition on the right with some function. The definition of pointwise order on the function spaces yields the conclusion. □

**Proposition 5.4.** $h$ *preserves arbitrary infima.*

*Proof.* Let $\{f_i \mid i\in I\}$ be a set of functions in $C\left(S_u^A, S_u\right)$. Then $f = \bigwedge\limits_{i\in I} f_i$ is also in $C\left(S_u^A, S_u\right)$. Now $f_i \in C\left(S_u^A, S_u\right)$, so $\pi_{f_i} \in C\left(S_u^{C\left(S_u^A, S_u\right)}, S_u\right) = M_S^2 A.$

Observe that $\bigwedge\limits_{i\in I} \pi_{f_i}$ is in $M_S^2\, A$. Now:

$$\begin{aligned}\mu_A\left(\bigwedge_{i\in I}\pi_{f_i}\right) &= \bigwedge_{i\in I}\mu_A\left(\pi_{f_i}\right) \text{ (by the lemma)}\\ &= \bigwedge_{i\in I} f_i \text{ (by the nature of } \mu_A\text{).}\end{aligned}$$

Hence: $h\left(\mu_A\left(\bigwedge\limits_{i\in I}\pi_{f_i}\right)\right) = h\left(\bigwedge\limits_{i\in I} f_i\right).$

We know that $h \circ \mu_A = h \circ M\,h$, so we now calculate $h \circ M\,h$ :

$$\begin{aligned} M\,h\left(\bigwedge_{i\in I}\pi_{f_i}\right) &= \bigwedge_{i\in I} M\,h\left(\pi_{f_i}\right) \text{ (by the lemma)} \\ &= \bigwedge_{i\in I} \pi_{f_i}\circ \hat{h} \text{ (by the definition of } M\,h) \\ &= \bigwedge_{i\in I} \pi_{h(f(i))} \text{ (by definition of } \hat{h}\,). \end{aligned}$$

Hence: $h\left(M\,h\left(\bigwedge_{i\in I}\pi_{f_i}\right)\right) = h\left(\bigwedge_{i\in I}\pi_{h(f_i)}\right) = \bigwedge_{i\in I} h\left(f_i\right)$ (by definition of $\bigwedge$ in $A$). Thus $h\left(\bigwedge_{i\in I} f_i\right) = \bigwedge_{i\in I} h\left(f_i\right)$, as required. □

The proof of the following proposition is analogous.

**Proposition 5.5.** *$h$ preserves finite suprema*

Similarly, we have

**Proposition 5.6.** *The lattice identities hold in $A$ with respect to the operators $\bigwedge$ (over arbitrary sets), $\vee$ (over finite sets ), as well as the dual frame distributive law.*

*Proof.* We shall only verify one identity in order to illustrate the method: To show that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, observe that $a \wedge (b \vee c) = h\left(\pi_a \wedge \left(\pi_b \vee \pi_c\right)\right)$, since $h$ preserves the operators. Now $\pi_a \wedge (\pi_b \vee \pi_c) = (\pi_a \wedge \pi_b) \vee (\pi_a \wedge \pi_c)$, hence $h\left(\pi_a \wedge \left(\pi_b \vee \pi_c\right)\right) = h\left(\left(\pi_a \wedge \pi_b\right) \vee \left(\pi_a \wedge \pi_c\right)\right) = h\left(\pi_a \wedge \pi_b\right) \vee h\left(\pi_a \wedge \pi_c\right) = (a \wedge b) \vee (a \wedge c)$, as required. □

**Note :**It is perhaps worth observing that $h$ is the order inducing map; by contrast $\mu_A$ has little to do with order, indeed if $\mu_A(a) \leq \mu_A\left(b\right)$, then $\pi_a \leq \pi_b$, hence $\pi_a = \pi_b$, hence $a = b$.

We now consider the action of $C\left(S_u\right)$ on $(A, \leq)$.

**Proposition 5.7.** *With* $\varphi * a = h(\varphi \circ \pi_a)$, *we have:*

*(1)* $1_S * a = a$, *where* $1_S$ *is the identity map on* $S_u$.

*(2)* $(\varphi \circ \psi) * a = \varphi * (\psi * a)$.

*(3)* *h preserves the action:* $h(\varphi \circ f) = \varphi * h(f)$.

*Proof.* (1) $1_S * a = h(1_S \circ \pi_a)$ (by definition of $*$ on $A$) $= h(\pi_a) = a$.

(2) This is a remarkable identity with a simple proof that is, perhaps, not too obvious. We prefer to deduce it from (3):

$$(\varphi \circ \psi) * a = h((\varphi \circ \psi) \circ \pi_a) = h(\varphi \circ (\psi \circ \pi_a)) = \varphi * h(\psi \circ \pi_a) = \varphi * (\psi * a).$$

(3) To prove this, consider $\varphi \circ \pi_f$ in $M_S^2 A$, where $f \in M_S A$. Now $h(\mu_A(\varphi \circ \pi_f)) = h(M\,h(\varphi \circ \pi_f))$, also:

(i) $h(\mu_A(\varphi \circ \pi_f)) = h(\varphi \circ \pi_f \circ e_{S^A}) = h(\varphi \circ f)$.

(ii) h($M\,h(\varphi \circ \pi_f)) = h\left(\varphi \circ \pi_f \circ \hat{h}\right) = h\left(\varphi \circ \pi_{h(f)}\right) = \varphi * h(f)$.

Hence, $h(\varphi \circ f) = \varphi * h(f)$. ∎

For completeness we state the remaining identities relating to $*$:

**Proposition 5.8.**

***(i)*** $\varphi * \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} \varphi * a_i$.

***(ii)*** $\varphi * \left(\bigvee_{j=1}^{n} a_j\right) = \bigvee_{j=1}^{n} \varphi * a_j$.

*Proof.* (i) $\varphi * \left(\bigwedge_{i\in I} a_i\right) = h\left(\varphi \circ \bigwedge_{i\in I} \pi_{a_i}\right) = h\left(\bigwedge_{i\in I} \varphi \circ \pi_{a_i}\right)$
(since $\varphi$ preserves arbitrary infima in the finite chain $S$)
$= \bigwedge_{i\in I} h\left(\varphi \circ \pi_{a_i}\right) = \bigwedge_{i\in I} \left(\varphi * a_i\right).$

The proof of (ii) is similar. □

Finally, the compatibility of $*$ with respect to the $\bigwedge \bigvee$ inequality:

**Proposition 5.9.** *Assume that* $\varphi_i,\ \psi_i \in C\left(S_u, S_u\right)$ *are such that, for all* $s_i$ *in* $S$ :

$$\bigwedge_{i=1}^{n} \varphi_i\left(s_i\right) \leq \bigvee_{i=1}^{n} \psi_i\left(s_i\right) \tag{+}$$

*Then, for* $a_1, \ldots, a_n \in A$, *we have:*

$$\bigwedge_{i=1}^{n} \varphi_i * a_i \leq \bigvee_{i=1}^{n} \psi_i * a_i.$$

*Proof.* Let $a_1, \ldots, a_n$ be given. Observe that the condition expressed in (+) is equivalent to the following:

$$\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i} \leq \bigvee_{i=1}^{n} \psi_i \circ \pi_{a_i}\,.$$

By monotonicity of $h$ and the fact that $h$ preserves finite suprema and finite infima, we have;

$$\bigwedge_{i=1}^{n} \varphi_i * a_i = h\left(\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) \leq h\left(\bigvee_{i=1}^{n} \psi_i \circ \pi_{a_i}\right) = \bigvee_{i=1}^{n} \psi_i * a_i,$$

as required. □

We have now established that an $M_S$ algebra $(A, h)$ can be given the structure of a complete lattice compatible with a $C(S_u)$–action on $A$, and satisfies the dual frame distributive law.

## 6. The Constants in $M_S$–Algebras

It is natural to enquire if algebras contain copies of $S$. The natural representative of the element $c$ of $S$ is the constant function $\underline{c} : S_u^A \to S_u$, so we define $\underline{c}$ in $A$ to be $h(\underline{c})$. To what extent are the $h(\underline{c})$'s distinct? We shall show that if $c \neq d$ are elements of $S$ and $h(\underline{c}) = h(\underline{d})$, then $A$ is a singleton set, a trivial $M_S$–algebra: Suppose $h(\underline{c}) = h(\underline{d})$, $c \neq d$. We may assume, since $S$ is a chain, that $c < d$. Let $\varphi \in C(S_u)$ be a monotone map such that $\varphi(c) = 0$ and $\varphi(d) = 1$. Then $\varphi \circ \underline{c} = \underline{0}$ and $\varphi \circ \underline{d} = \underline{1}$, hence

$$h(\varphi \circ \underline{c}) = h(0) = 0_A, \quad h(\varphi \circ \underline{d}) = h(\underline{1}) = 1_A.$$

But

$$h(\varphi \circ \underline{c}) = \varphi * h(\underline{c}) = \varphi * h(\underline{d}) = h(\varphi \circ d).$$

Hence $0_A = 1_A$.

Thus, we have proved the following proposition.

**Proposition 6.1.** *Every nontrivial $M_S$ algebra contains a copy of $S$.*

**Corollary 6.2.** *The two element chain is not a $M_3$–algebra, where 3 is the three element chain with the $u$–topology.*

## 7. The Algebra Maps

Let $(A, h_A)$, $(B, h_B)$ be $M_S$–algebras. Let $f : (A, h_A) \to (B, h_B)$ be an algebra map, then the following diagram commutes.

$$\begin{tikzcd} MA \arrow[r, "Mf"] \arrow[d, "h_A"'] & MB \arrow[d, "h_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

**Proposition 7.1.** *$f$ preserves arbitrary infima, finite suprema, the $C(S_u)$–action and constants:*

**(i)** $f\left(\bigwedge\limits_{i\in I} a_i\right) = \bigwedge\limits_{i\in I} f(a_i)\,.$

**(ii)** $f\left(\bigvee\limits_{j=1}^{n} a_j\right) = \bigvee\limits_{j=1}^{n} f(a_j)\,.$

**(iii)** $f(\varphi * a) = \varphi * f(a)\,,$ for all $\varphi \in C(S_u)\,,\ a \in A.$

**(iv)** $f(\underline{c}) = \underline{c}$, for all $c \in S$.

*Proof.* (i) Let $a = \bigwedge\limits_{i\in I} a_i$. Then $f(a) = f\left(h_A\left(\bigwedge\limits_{i\in I}\pi_{a_i}\right)\right) = h_B\left(M\,f\left(\bigwedge\limits_{i\in I}\pi_{a_i}\right)\right).$

(ii) $= h_B\left(\bigwedge\limits_{i\in I} M\,f(\pi_{a_i})\right)$ (by 5.3) $= h_B\left(\bigwedge\limits_{i\in I}\pi_{f(a_i)}\right) = \bigwedge\limits_{i\in I} h_B\left(\pi_{f(a_i)}\right) = \bigwedge\limits_{i\in I} f(a_i)\,.$

(iii) $f(\varphi * a) = f(h_A(\varphi \circ \pi_a)) = h_B(M\,f(\varphi \circ \pi_a)) = h_B\left(\varphi \circ \pi_a \circ \hat{f}\right) = h_B\left(\varphi \circ \pi_{f(a)}\right) = \varphi * h_B\left(\pi_{f(a)}\right) = \varphi * f(a)$.

(iv) $f(\underline{c}) = f(h_A(\underline{c})) = h_B(M\,f(\underline{c})) = h_B\left(\underline{c} \circ \hat{f}\right) = h_B(\underline{c}) = \underline{c}$.

□

## 8. The Eilenberg–Moore Category of $M_{S_u}$–Algebras

We have established that on every non trivial $M_{S_u}$–algebra $(A, h)$ there can be defined a complete lattice structure, and a $C(S_u)$–action. The following properties hold:

**(i)** $a \vee \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} (a \vee a_i)$.

**(ii)** $\underline{c} * a = c$, for all $c \in S$, where $\underline{c} : A \to S$ denotes the constant function mapping $A$ onto $\{c\}$.

**(1)** $1_S * a = a$.

**(2)** $(\varphi \circ \psi) * a = \varphi * (\psi * a)$, where $\circ$ denotes composition in $C(S_u)$.

**(3)** $\left(\bigwedge_{i \in I} \varphi_i\right) * a = \bigwedge_{i \in I} (\varphi_i * a)$ **(3)′** $\varphi * \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} \varphi * a_i$.

**(4)** $\left(\bigvee_{i=1}^{n} \varphi_i\right) * a = \bigvee_{i=1}^{n} (\varphi_i * a)$ **(4)′** $\varphi * \left(\bigvee_{i=1}^{n} a_i\right) = \bigvee_{i=1}^{n} \varphi * a_i$.

**(5)** Axiom of $*$–consistency: If $\varphi_i,\ \psi_i \in C(S_u)$, $1 \le i \le n$, are such that, for all possible choices of $s_i \in S$, we have:

$$\bigwedge_{i=1}^{n} \varphi_i(s_i) \le \bigvee_{i=1}^{n} \psi_i(s_i),$$

then, for all choices $a_1, \ldots, a_n$ in $A$, we have

$$\bigwedge_{i=1}^{n} \varphi_i * a_i \le \bigvee_{i=1}^{n} \psi_i * a_i.$$

We shall refer to such a lattice as a $\boldsymbol{C(S)}$**–consistent dual frame.**

That (3), (3)′, (4), (4)′ hold is a straightforward consequence of the definition of $*$ and of the fact that a structure map preserves arbitrary infima and finite suprema.

We can now formulate the characterization of $M_S$–algebras, up to isomorphism.

**Theorem 8.1.** *The Eilenberg–Moore Category of $M_{S_u}$–algebras has, as objects, the $C(S)$–consistent dual frames and, as morphisms, the maps which preserve $C(S)$–action, arbitrary infima, finite suprema and constants.*

The explicit nature of the definitions involved renders it sufficient to verify that, indeed:

- **(i)** On every $C(S)$–consistent dual frame $A$ there can be defined a (structure) map $h_A : M_S\, A \to A$ such that $(A, h_A)$ is an $M_S$–algebra.
- **(ii)** Every map $f : A \to B$ between any two such $C(S)$–consistent dual frames that preserves $C(S)$–action as well as arbitrary infima, finite suprema and constants is an $M_S$–algebra map from $(A, h_A)$ to $(B, h_B)$; and the converse is also true.

Let $A$ be a $C(S)$–consistent dual frame. We define $h_A : M_S A \to A$.

**Definition 8.2.** *Let $f \in C\left(S^A, S\right)$. Define*

$$h_A(f) = \bigwedge\{\varphi_1 * a_i \vee \quad \ldots \vee \varphi_n * a_n \mid \\ f \leq \varphi_1 \circ \pi_{a_1} \vee \varphi_2 \circ \pi_{a_2} \vee \ldots \vee \varphi_n \circ \pi_{a_n}\}.$$

By Theorem 3.1, it is clear that $h_A$ is well defined. It is also clear that $h_A$ is monotone in the sense that $h_A(f) \leq h_A(g)$ if $f \leq g$.

**Proposition 8.3.** $h_A(\varphi \circ \pi_a) = \varphi * a$.

*Proof.* Since $g = \varphi \circ \pi_a$ is such that $g \leq \varphi \circ \pi_a$, we have $h_A(\varphi \circ \pi_a) \leq \varphi * a$.

To prove the reverse inquality we consider two cases:

(i) Assume that $g = \bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i} \geq \varphi \circ \pi_a$. If no $a_i$ is equal to $a$, then $\varphi(1) \leq \varphi_1(0) \vee \varphi_2(0) \vee \ldots \vee \varphi_n(0) = \varphi_r(0)$, for some $r$ (since $S$ is linearly ordered). Then $\varphi * a \leq \underline{\varphi(1)} * a \leq \underline{\varphi_r(0)} * a = \underline{\varphi_r(0)} * a_r \leq \varphi_r * a_r$, so that $\varphi * a \leq \bigvee_{i=1}^{n} \varphi_i * a_i$.

(ii) If $a$ is one of the $a_r$'s, say $a_1$, then we have $\varphi(s) \leq \varphi_1(s) \vee \varphi_2(s_2) \vee \ldots \vee \varphi_n(s_n)$, for all choices of $s, s_2, \ldots, s_n$ in $S$. Hence:

$$\varphi(s) \leq \varphi_1(s) \vee \varphi_2(0) \vee \ldots \vee \varphi_n(0) = \varphi_1(s) \vee c = (\varphi_1 \vee \underline{c})(s),$$

where $c = \bigvee_{r=2}^{n} \varphi_r(0)$. Hence $\varphi * a \leq (\varphi_1 \vee \underline{c}) * a$, so that $\varphi * a \leq \varphi_1 * a \vee \underline{c} * a = \varphi_1 * a \vee c$. But $c = \varphi_r(0)$, for some $r$, so that $c = \underline{\varphi_r(0)} * a_r \leq \varphi_r * a_r$. Moreover, $a = a_1$, hence $\varphi * a \leq \varphi_1 * a_1 \vee \varphi_r * a_r \leq \bigvee_{i=1}^{n} \varphi_i * a_i$. In conclusion: $\varphi * a \leq h_A(\varphi \circ \pi_a)$. Hence $\varphi * a = h_A(\varphi \circ \pi_a)$. □

**Corollary 8.4.** $h_A(\pi_a) = a$.

This follows from the fact that $1_S * a = a$.

**Proposition 8.5.** $h_A\left(\bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) = \bigvee_{i=1}^{n} \varphi_i * a_i$.

*Proof.* It is clear that $h_A\left(\bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) \leq \bigvee_{i=1}^{n} \varphi_i * a_i$, by definition of $h_A$. Also, for each $j$, $\varphi_j \circ \pi_{a_j} \leq \bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}$, hence, by monotonicity of $h_A$, we have $h_A(\varphi_j \circ \pi_{a_j}) \leq h_A\left(\bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right)$, so that $\varphi_j * a_j \leq h\left(\bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right)$, since $h_A\left(\varphi_j \circ \pi_{a_j}\right) = \varphi_j * a_j$. Hence $\bigvee_{j=1}^{n} \varphi_j * a_j \leq h_A\left(\bigvee_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right)$. The proof is complete. □

**Corollary 8.6.** Let $g_i = \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{a_{ij}}$, $i = 1, 2$. Then $h_A(g_1) \vee h_A(g_2) = h_A(g_1 \vee g_2)$.

*Proof.* From Proposition 8.5, we have, for $n = 1$, $h_A(\varphi \circ \pi_a) = \varphi * a$. We can write $g_i$ as $\bigvee_{j=1}^{n} \varphi_{ij} \circ a_{ij}$, where $n = n(1) \vee n(2)$, by taking some $\varphi_{ij}$ to be $\underline{0}$, if necessary. Hence $g_1 \vee g_2 = \bigvee_{\substack{1 \leq j \leq n \\ 1 \leq i \leq 2}} \varphi_{ij} \circ \pi_{a_{ij}}$. By Proposition 8.5, we have $h_A(g_1 \vee g_2) = \bigvee_{\substack{1 \leq j \leq n \\ 1 \leq i \leq 2}} \varphi_{ij} * a_{ij}$. Also, $h_A(g_1) \vee h_A(g_2) = \bigvee_{i=1}^{2} \left[\bigvee_{j=1}^{n} \varphi_{ij} * a_{ij}\right]$. Hence $h_A(g_1 \vee g_2) = h_A(g_1) \vee h_A(g_2)$. □

**Proposition 8.7.** $h_A\left(\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) = \bigwedge_{i=1}^{n} \varphi_i * a_i$.

*Proof.* Since $\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i} \leq \varphi_j \circ \pi_{a_j}$ for $1 \leq j \leq n$, we have $h_A\left(\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) \leq \varphi * a_j$, for $1 \leq j \leq n$. Hence $h_A\left(\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i}\right) \leq \bigwedge_{i=1}^{n} (\varphi_i * a_i)$. To prove the reverse inequality, let $s = \bigvee_{k=1}^{m} \psi_k \circ \pi_{b_k}$ be such that $\bigwedge_{i=1}^{n} \varphi_i \circ \pi_{a_i} \leq s$. We consider two cases:

(1) If no $a_i$ is a $b_k$, then $\bigwedge_{i=1}^{n} \varphi_i(1) = \varphi_r(1)$, for some $r$. Hence
$\varphi_r(1) \leq \bigvee_{k=1}^{m} \psi_k(0) = \psi_t(0)$, for some $t$. Hence
$\varphi_r * a_r \leq \underline{\varphi_r(1)} * a_r = \varphi_r(1) \leq \psi_t(0) = \underline{\psi_t(0)} * b_t \leq \psi_t * b_t$,

so that $\bigwedge\limits_{r=1}^{n} \varphi_r * a_r \leq \bigvee\limits_{k=1}^{m} \psi_k * b_k$.

(2) if $a_1, \ldots, a_p$ are the same as, respectively, $b_1, \ldots, b_p$, then we have $\bigwedge\limits_{i=1}^{n} \varphi_i * a_i \leq \bigwedge\limits_{i=1}^{p} \varphi_i * a_i \leq \bigvee\limits_{i=1}^{p} \psi_i * b_i$ (by the axiom of $*$–consistency).

Hence $\bigwedge\limits_{i=1}^{n} \varphi_i * a_i \leq \bigvee\limits_{i=1}^{m} \psi_i * b_i$.

Thus $\bigwedge\limits_{i=1}^{n} \varphi_i * a_i \leq \bigvee\limits_{i=1}^{m} \psi_i * b_i$, in both cases. Consequently $\bigwedge\limits_{i=1}^{n} \varphi_i * a_i \leq h_A \left( \bigwedge\limits_{i=1}^{n} \varphi_i \circ \pi_{a_i} \right)$, by definition of $h_A$. The proof is complete. □

**Proposition 8.8.** $h_A \left( \bigwedge\limits_{i \in I} g_i \right) = \bigwedge\limits_{i \in I} h_A (g_i)$, where $g_i \in C \left( S_u^A, S_u \right), i \in I$.

*Proof.* Let $g = \bigwedge\limits_{i \in I} g_i$. By monotonicity of $h_A$, we have $h_A (g) \leq \bigwedge\limits_{i \in I} h_A (g_i)$. To establish the reverse inequality, let $f = \bigvee\limits_{r=1}^{N} \varphi_r \circ \pi_{a_r}$ be such that $g \leq f$.

By Theorem 3.4, we have $g_{i_1}, \ldots, g_{i_m}$ such that $\bigwedge\limits_{s=1}^{m} g_{i_s} \leq f$. Then

$$h_A \left( \bigwedge_{s=1}^{m} g_{i_s} \right) \leq h_A (f) = h_A \left( \bigvee_{r=1}^{N} \varphi_r \circ \pi_{a_r} \right) = \bigvee_{r=1}^{N} \varphi_r * a_r$$

by Corollary 8.6. Hence $\bigwedge\limits_{s=1}^{m} h_A (g_{i_s}) \leq \bigvee\limits_{r=1}^{N} \varphi_r * a_r$, since $h_A \left( \bigwedge\limits_{s=1}^{m} g_{i_s} \right) = \bigwedge\limits_{s=1}^{m} h_A (g_{i_s})$, by Proposition 8.7. Thus

$\bigwedge_{i\in I} h_A(g_i) \leq \bigvee_{r=1}^{N} \varphi_r * a_r$, for all $f = \bigvee_{r=1}^{N} \varphi_r \circ \pi_{a_r}$ such that $g \leq f$. Hence $\bigwedge_{i\in I} h_A(g_i) \leq h_A(g)$, as required. □

**Proposition 8.9.**. $h_A(\varphi \circ f) = \varphi * h_A(f)$

*Proof.* We have $f = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi a_{ij}$ by Theorem 3.1. Hence $\varphi \circ f = \varphi \circ \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{a_{ij}} = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi \circ \varphi_{ij} \circ \pi_{a_{ij}}$, since $S$ is a finite chain and $\varphi$, being monotone, preserves arbitrary infima and finite suprema. Hence $h_A(\varphi \circ f) = \bigwedge_{i\in I} h_A\left(\bigvee_{j=1}^{n(i)} \varphi \circ \varphi_{ij} \circ \pi_{a_{ij}}\right)$, by Proposition 8.7. Now

$$h_A\left(\bigvee_{j=1}^{n(i)} \varphi \circ \varphi_{ij} \circ \pi_{ij}\right) = \bigvee_{j=1}^{n(i)} h_A\left(\varphi \circ \varphi_{ij} \circ \pi_{a_{ij}}\right),$$

by Corollary 8.6. Also $h_A\left(\varphi \circ \varphi_{ij} \circ \pi_{a_{ij}}\right) = (\varphi \circ \varphi_{ij}) * a_{ij}$, by Proposition 8.3. Now $(\varphi \circ \varphi_{ij}) * a_{ij} = \varphi * (\varphi_{ij} * a_{ij})$, hence $h_A(\varphi \circ f) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi * \varphi_{ij} * a_{ij}$. Putting $\varphi = 1_s$, we get $h_A(f) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * a_{ij}$. Hence

$$\varphi * h_A(f) = \varphi * \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * a_{ij} = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi * \varphi_{ij} * a_{ij} = h_A(\varphi \circ f),$$

as required. □

**Proposition 8.10.** $h_A(f_1 \vee f_2) = h_A(f_1) \vee h_A(f_2)$

*Proof.* By Theorem 3.1, we have $f_1 = \bigwedge_{i\in I} g_{1i}$, $f_2 = \bigwedge_{j\in J} g_{2j}$, where each $g_{kr}$ is in $\mathcal{S}$. By the dual frame law, we have $f_1 \vee f_2 =$

$\bigwedge_{i\in I} g_{1i} \vee \bigwedge_{i\in I} g_{2j} = \bigwedge_{ij} (g_{1i} \vee g_{2j})$. Hence $h_A(f_1 \vee f_2) = \bigwedge_{ij} h_A(g_{1i} \vee g_{2j})$, by Proposition 8.8. Also, $h_A(g_{1i} \vee g_{2j}) = h_A(g_{1i}) \vee h_A(g_{2j})$, by Corollary 8.6. Hence

$$\begin{aligned} h_A(f_1 \vee f_2) &= \bigwedge_{ij} (h_A(g_{1i}) \vee h_A(g_{2j})) \\ &= \bigwedge_{i\in I} h_A(g_{1i}) \vee \bigwedge_{j\in I} h_A(g_{2j}) \text{ by the dual frame law,} \\ &= h_A(f_1) \vee h_A(f_2). \end{aligned}$$

□

**Proposition 8.11.** $h_A \circ Mh_A = h_A \circ \mu_A$.

*Proof.* Let $\alpha \in M^2 A$. Then $\alpha = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{fij}$, where $\varphi_{ij} \in C(S_u)$ and $f_{ij} \in C\left(S_u^A, S_u\right)$. Thus

$$\begin{aligned} h_A(Mh_A(\alpha)) &= h_A\left(\bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{fij} \circ \hat{h}_A\right) \\ &= h_A\left(\bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{h_A(f_{ij})}\right) \\ &= \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * h_A(f_{ij}). \end{aligned}$$

Also, $h_A(\mu_A(\alpha)) = h_A\left(\bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ f_{ij}\right)$ ($\mu_A$ preserves composition and $\mu_A(\pi_f) = f$ ) $= \bigwedge_{i\in I} h_A\left(\bigvee_{j=1}^{n} \varphi_{ij} \circ f_{ij}\right) = \bigwedge_{i\in I} \bigvee_{j=1}^{n} \varphi_{ij} * h_A(f_{ij})$, since $h_A$ preserves finite suprema and $C(S_u)$–action. The proof is complete. □

We now complete the picture by describing the $M_S$–algebra morphisms. Let $A$ and $B$ be two nontrivial $C(S)$–consistent dual frames and let $h_A$, $h_B$ be the corresponding structure maps defined above.

**Proposition 8.12.** *If $f : A \to B$ preserves $C(S)$–action, arbitrary infima and finite suprema, then $f : (A, h_A) \to (B, h_B)$ is an algebra map.*

*Proof.* We verify that $h_B \circ Mf = f \circ h_A$. Let $\alpha \in MA$, then $\alpha = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \alpha_{ij} \circ \pi_{aij}$, hence $Mf(\alpha) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{a_{ij}} \circ \hat{f} = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} \circ \pi_{f(a_{ij})}$, so that $h_B(Mf(\alpha)) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * f(a_{ij})$. Now $h_A(\varphi) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * a_{ij}$. Hence $f(h_A(\alpha)) = \bigwedge_{i\in I} \bigvee_{j=1}^{n(i)} \varphi_{ij} * f(a_{ij})$, by our assumption on $f$. $\square$

Conversely,

**Proposition 8.13.** *Let $f : (A, h_A) \to (B, h_B)$ be an $M_S$–* algebra map, then *$f : A \to B$ preserves arbitrary infima, finite suprema and $C(S)$–action.*

*Proof.* 1. $f$ preserves arbitrary infima: Consider $a = \bigwedge_{i\in I} a_i$ in $A$. By Corollary 8.4 and Proposition 8.8, we have $a = h_A\left(\bigwedge_{i\in I} \pi_{a_i}\right)$, hence

$$\begin{aligned} f(a) &= f \circ h_A\left(\bigwedge_{i\in I} \pi_{a_i}\right) = h_B\left(Mf\left(\bigwedge_{i\in I} \pi_{a_i}\right)\right) \\ &= h_B\left(\bigwedge_{i\in I} \pi_{a_i} \circ \hat{f}\right) = h_B\left(\bigwedge_{i\in I} \pi_{f(a_i)}\right) \\ &= \bigwedge_{i\in I} f(a_i)\,. \end{aligned}$$

2. $f$ preserves finite suprema: Consider $a_1 \vee a_2$. Now $h_A(\pi_{a_1} \vee \pi_{a_2}) = a_1 \vee a_2$, so that

$$\begin{aligned} f(a_1 \vee a_2) &= f \circ h_A(\pi_{a_1} \vee \pi_{a_2}) = h_B \circ Mf(\pi_{a_1} \vee \pi_{a_2}) \\ &= h_B\left((\pi_{a_1} \vee \pi_{a_2}) \circ \hat{f}\right) = h_B\left(\pi_{f(a_1)} \vee \pi_{f(a_2)}\right) \\ &= f(a_1) \vee f(a_2). \end{aligned}$$

3. $f$ preserves $C(S_u)$–action:

$$\begin{aligned} f(\varphi * a) &= f(h_A(\varphi \circ \pi_a)) = h_B(Mf(\varphi \circ \pi_a)) \\ &= h_B\left(\varphi \circ \pi_a \circ \hat{f}\right) = h_B\left(\varphi \circ \pi_{f(a)}\right) = \varphi * f(a). \end{aligned}$$

□

## 9. Summary

The condition involving "replacement" in the description of the algebras is a consequence of a more general property. It is easy to show that this property is shared by all $M_S$–algebras, thus allowing the following description of the nontrivial objects of the Eilenberg–Moore category of $M_S$–algebras:

Complete lattices $A$ which contain a copy of $S$ and admit an $n$–ary action $* : C(S^n) \times A^n \to A$, for every integer $n, n \geq 1$, such that for all $\Phi_i \in C(S^n)$, $\underline{a} = \langle a_1, a_2, \ldots, a_n \rangle \in A^n$, $c \in S$, we have:

**(I)** $\left(\bigvee_{i=1}^{n} \Phi_i\right) * \underline{a} = \bigvee_{i=1}^{n} \Phi_i * \underline{a}$ **(I)′** $\Phi * \left(\bigvee_{i=1}^{n} \underline{a}_i\right) = \bigvee_{i=1}^{n} \Phi * \underline{a}_i$

**(II)** $\left(\bigwedge_{i=1}^{n} \Phi_i\right) * \underline{a} = \bigwedge_{i=1}^{n} \Phi_i * \underline{a}$ **(II)′** $\Phi * \left(\bigwedge_{i=1}^{n} \underline{a}_i\right) = \bigwedge_{i=1}^{n} \Phi * \underline{a}_i$

**(III)** $1_{S^n} * \underline{a} = \underline{a}$

**(IV)** $\underline{c} * \underline{a} = c$

## 10. The Case of the Sierpiński Dyad $S = (2, u)$

The Eilenberg–Moore category of $M_S$–algebras, as mentioned in the introduction, is the category of Frames and Frame homomorphism, with no reference to a $C(S)$–action. The results in this paper refer explicitly to the $C(S)$–action, but this is easily seen to be trivial: $C(2_u, 2_u)$ consists of three elements $\underline{0}$, $1_2$, $\underline{1}$ and we have $\underline{0} * a = 0$, $\underline{1} * a = 1$, $1_2 * a = a$, for all $a$ in an algebra $A$, so that, when $S = 2_u$, reference to the $C(S)$–action is superfluous.

## 11. Generalizations

It would be most interesting to be able to describe the $M_S$–algebras when $S$ is, for instance, $\omega = \{0 < 1 < 2 < \cdots < n < \ldots\}$ with the $u$–topology, and with the $l$–topology; as well as when $S = (\omega^+, u)$, $S = (\omega^+, l)$, etc. As far as we can ascertain, a purely algebraic/discrete discription as the one given in this paper would not be possible, and an $M_S$–algebra would need to satsify, in addition, a topological condition compatible with the algebraic ones given in section 9, above.

## 12. Acknowledgment

The author would like to thank the anonymous referee for the comments and suggestions that were made. The responsibility for the final form of the paper rests with the author, but the paper has only gained in clarity and precision as a result of the referre's careful work and sensitive comments.

## References

[1] Hoffmann R.E., *Monads induced by topological spaces and their Eilenberg–Moore categories*, Seminarberichte Fachbereich Mathematik und Informatik, Fern Universität Hagen, 19 (1984), 207–216.

[2] Isbell J., *Generating the algebraic theory of* $C(X)$, Alg. Universalis **15** (1982),153–155.

[3] Johnstone P., *Stone Spaces,* Cambridge University Press, 1983.

[4] Pelletier J.W. and J. Rosický, *Generating the equational theory of* $C^*$*–algebras and related categories,* Category Theory in Relation to Analysis, Algebra and Combinatorics, World Scientific, Singapore, (1989), 163-180.

[5] Pumplün D. and Röhrl H., *Banach spaces and totally convex spaces I,* Comm. in Algebra **12** (1984), 953–1019.

[6] Semadeni Z., *On Isbell's approach to the unit ball of* $C(X)$ *as an abstract algebra*, Seminarberichte Fachbereich Mathematik und Informatik, Fern Universität Hagen, **24**, (1986), 75–101.

Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa

*E-mail address*: `salbasdo@alpha.unisa.ac.za`