Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 23, Summer 1998, 349–362

α-PSEUDOCOMPACTNESS IN C_P-SPACES

Angel Tamariz-Mascarúa

Abstract

We prove that $C_p(X)$ is σ-α-pseudocompact if and only if X is pseudocompact and α-b-discrete, and $C_p(X, [0,1])$ is α-pseudocompact if and only if X is α-b-discrete. We also give an example of an infinite α-pseudocompact α-b-discrete space.

1. Introduction

For a Tychonoff space X the space $C_p(X)$ of the real-valued functions defined on X with the pointwise convergence topology contains a copy of the real line as a closed subset. Thus $C_p(X)$ is not compact for any X. Hence, the following general question arises for a compact-like property $\mathcal{P}$: under which conditions on X is $C_p(X)$ the union of a countable collection of subspaces satisfying $\mathcal{P}$? With respect to this problem, for $\mathcal{P}$ = pseudocompactness, V.V. Tkachuk proved in [9] the following result.

Theorem 1.1. *$C_p(X)$ is σ-pseudocompact if and only if X is pseudocompact and b-discrete*

On the other hand, it was proved in [6] that if $C_p(X)$ is σ-countably compact, then X must be finite. This fact explains why the construction of infinite pseudocompact b-discrete spaces is not trivial (see [5], [2, Example 6.1], [1, I.2.5]).

In Section 2 of this article we generalize Theorem 1.1 by proving that $C_p(X)$ is σ-α-pseudocompact if and only if X is pseudocompact and α-b-discrete. (This result was mentioned in [7]

Mathematics Subject Classification: 54A25, 54C35, 54D99
Key words: α-pseudocompact space; C_α-compact set; spaces of continuous functions; α-b-discrete space

without any proof.) In Section 3 we give an example of an infinite pseudocompact α-b-discrete space.

In order to prove Theorems 2.7 and 2.8, we follow a similar strategy to that given to prove Propositions 3.5 and 3.9 in [9]. The example in Section 3 is obtained by modifying Example I.2.5 in [1]. Proofs hold by applying some results obtained in [3].

We assume that all spaces are Tychonoff spaces. If X is a space and $A \subset X$, then $\mathrm{cl}_X(A)$ (or simply $\mathrm{cl}(A)$) denotes the closure of A in X. The Greek letters ξ, λ, γ stand for infinite ordinal numbers, and the Greek letters α, κ stand for infinite cardinals. For a set X, $|X|$ denotes the cardinality of X. Besides, $[X]^{<\alpha}$ stands for the family of subsets of X of cardinality $< \alpha$. For ordinal numbers ξ and γ with $\xi < \gamma$, (ξ, γ) and $[0, \gamma)$ are the sets $\{\lambda : \xi < \lambda < \gamma\}$ and $\{\lambda : \lambda < \gamma\}$, respectively. If α is a cardinal number, then α also stands for the discrete space of cardinality α. As usual, $\mathbf{R}$ denotes the set of real numbers with its Euclidean topology. For a space X, $\beta(X)$ is its Stone-Čech compactification.

The following concepts and some of its properties were analyzed in [3].

Definition 1.2. *1. A subset B of X is said to be C_α-compact in X if $f[B]$ is a compact subset of $\mathbf{R}^\alpha$ for every continuous function $f : X \to \mathbf{R}^\alpha$.*

2. If X is C_α-compact in itself, then we say that X is α-pseudocompact.

3. A space $X \subset Y$ is σ-C_α-compact in Y if there is a cover $\{X_n : n < \omega\}$ of X where X_n is C_α-compact in Y. The expression X is σ-C_α-compact will mean that X is σ-C_α-compact in X.

If $\alpha < \gamma$, then every C_γ-compact subset of X is C_α-compact. A set $Y \subset X$ is a *G_δ-set* in X if there is a sequence $(U_n)_{n<\omega}$

of nonempty open sets in X such that $Y = \bigcap_{n<\omega} U_n$. A subset Y of X is G_δ*-dense* in X if each nonempty G_δ-set in X has a nonempty intersection with Y. Observe that a space X is pseudocompact iff X is $\aleph_0$-pseudocompact. For each $\alpha < \gamma$, there exists a space X which is α-pseudocompact and is not γ-pseudocompact. In fact, the space of ordinal numbers $[0, \alpha^+)$ endowed with the order topology is α-pseudocompact but is not γ-pseudocompact (see [3]).

If X and Y are two spaces, we denote by $C(X, Y)$ the set of continuous functions defined on X and with values in Y. If $Y = \mathbf{R}$, then we write $C(X)$ instead of $C(X, \mathbf{R})$. The set of real bounded continuous functions defined on X is denoted by $C^*(X)$. A subspace Y of a space X is C^**-embedded* in X if for every $f \in C^*(Y)$ there is $g \in C^*(X)$ such that $g|_Y = f$; and it is a *zero-set* (resp., *cozero-set*) if there is $f \in C(X)$ such that $Y = f^{-1}\{0\}$ (resp., $Y = f^{-1}(\mathbf{R} \setminus \{0\})$). $\mathcal{Z}(X)$ is the collection of zero-sets of X. We write $C_p(X, Y)$, $C_p(X)$ and $C_p^*(X)$ in order to symbolize the sets $C(X, Y)$, $C(X)$ and $C^*(X)$ considered with the pointwise convergence topology. Recall that two disjoint subsets A and B of a space X are *completely separated* if there exists $f \in C(X, [0, 1])$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. For a product $\prod_{j \in J} X_j$ and for $K \subset J$, π_K denotes the projection from $\prod_{j \in J} X_j$ to $\prod_{j \in K} X_j$.

As usual, if $\mathcal{P}$ is a topological property, then a space X is σ-$\mathcal{P}$ if X is the countable union of subspaces having $\mathcal{P}$.

Definition 1.3. *Let α be a cardinal number,*

1. *a space X is α-discrete if every subset of X of cardinality $\leq \alpha$ is discrete, or equivalently is closed in X,*

2. *X is α-b-discrete if every subset Y of X of cardinality $\leq \alpha$ is discrete and C^*-embedded in X,*

3. *a space X is b-discrete if X is ω-b-discrete,*

4. *a subset* Y *of a product* $X = \prod_{j \in J} X_j$ *is said to be* α*-dense in* X *if for every* $K \subset J$ *of cardinality* $\leq \alpha$ *we have* $\pi_K(Y) = \prod_{k \in K} X_k$.

Observe that if $\gamma < \alpha$ and Y is α-dense in X, then Y is γ-dense and dense in X.

The following two results proved in [3] will play an important role for our purposes.

Proposition 1.4. *For a subset* B *of* X*, the following are equivalent:*

1. B *is* C_α*-compact in* X*;*
2. *if* $\{Z_\xi : \xi < \alpha\} \subset \mathcal{Z}(X)$ *and* $B \cap \bigcap_{\xi \in I} Z_\xi \neq \emptyset$ *for every finite subset* I *of* α*, then* $B \cap \bigcap_{\xi < \alpha} Z_\xi \neq \emptyset$.

It is worth mentioning that conditions (1) and (2) in the proposition just formulated are equivalent to B being G_α-dense in $\beta(X)$.

Proposition 1.5. *Let* α *be a cardinal number and let* $X = \prod_{i \in I} X_i$ *be a product of compact spaces of weight not greater than* α*, with* $\alpha \leq |I|$*. Then, for a dense subset* Y *of* X *the following are equivalent.*

1. Y *is* α*-pseudocompact.*
2. Y *is* C_α*-compact in* X*.*
3. Y *is* α*-dense in* X*.*

The following basic results about σ-C_α-compact sets can be easily proven and will be useful.

Proposition 1.6. *Let* $X = \bigcup_{n < \omega} X_n$ *be a space.*

1. *If* $f : X \to Y$ *is a continuous and onto function and* X_n *is* C_α*-compact (resp.,* α*-pseudocompact) in* X *for every* $n < \omega$*, then* Y *is* σ-C_α*-compact (resp.,* σ-α*-pseudocompact).*

2. *If X_n is σ-C_α-compact (resp., σ-α-pseudocompact) in $Y_n \subset Y$ for each $n < \omega$, then X is σ-C_α-compact (resp., σ-α-pseudocompact) in $\bigcup_{n<\omega} Y_n$.*

2. α-Pseudocompactness in $C_p(X)$

To be able to prove the main theorems of this section, we need to establish some previous results.

Proposition 2.1. *A space X is α-b-discrete if and only if $C_p(X,[0,1])$ is α-dense in $[0,1]^X$.*

Proof. Assume that X is α-b-discrete and let K be a subset of X of cardinality $\leq \alpha$. Let h be an element in $[0,1]^K$. Since K is discrete, h is continuous; so there exists $\tilde{h} \in C_p(X,[0,1])$ which extends h because K is C^*-embedded in X. Therefore, $C_p(X,[0,1])$ is α-dense in $[0,1]^X$.

Now, suppose that $C_p(X,[0,1])$ is α-dense in $[0,1]^X$ and let K be a subset of X of cardinality $\leq \alpha$. By hypothesis, every $h \in [0,1]^K$ can be continuously extended to X, so K is discrete and C^*-embedded in X. □

We will use the following α-version of Proposition 3.8 in [9]. Its proof is similar to that given when $\alpha = \omega$.

Lemma 2.2. *For any space X the following conditions are equivalent.*

1. *The space X is α-b-discrete.*
2. *X is α-discrete and $cl_{\beta(X)}A \cong \beta(A)$ for each $A \subset X$ of cardinality $\leq \alpha$.*
3. *X is α-discrete and $cl_{\beta(X)}A \cap cl_{\beta(X)}B = \emptyset$ for every disjoint $A, B \subset X$ of cardinality $\leq \alpha$.*

In order to prove the following two results, we will use Proposition 1.4

Lemma 2.3. *Let $A = \{x_\lambda : \lambda < \alpha\}$ be a subset of X and let z_0 be an element in $cl_X A$. For each C_α-compact subset Y of $Z = \{f \in C_p(X,[0,1]) : f(z_0) = 0\}$, and each $\epsilon \in (0,1)$, there exists $G = \{\xi_1, ..., \xi_k\} \in [\alpha]^{<\omega}$ such that, if $f \in C_p(X,[0,1])$ and $f(x_{\xi_i}) \geq \epsilon \ \forall\ 1 \leq i \leq k$, then $f \notin Y$.*

Proof. For each $n > \epsilon^{-1}$ and each $F = \{\lambda_1, ..., \lambda_n\} \in [\alpha]^{<\omega}$, let $M_F = \{f \in Z : f(x_{\lambda_i}) \in [\epsilon - \frac{1}{n}, 1] \ \forall\ 1 \leq i \leq n\}$. It happens that each M_F is a nonempty zero-set in Z, and if $F_1, ..., F_s \in [\alpha]^{<\omega}$ with $|F_i| > \epsilon^{-1} \ \forall\ 1 \leq i \leq s$, then $M_{F_1} \cap \ldots \cap M_{F_s} = M_{\cup\{F_i : 1 \leq i \leq s\}}$. Let $\mathcal{M} = \{M_F : F \in [\alpha]^{<\omega}$ and $|F| > \epsilon^{-1}\}$. Observe that $|\mathcal{M}| \leq \alpha$. Now, it is easy to see that if $f \in \bigcap \mathcal{M}$, then $f(x_\lambda) \geq \epsilon$ for all $\lambda < \alpha$. But, $f(z_0) = 0$. This means that f is not a continuous function, but this is a contradiction. So $\bigcap \mathcal{M} = \emptyset$. Because of Proposition 1.4, we can find $F_1, ..., F_s \in [\alpha]^{<\omega}$ such that $Y \cap M_{F_1} \cap \ldots \cap M_{F_s} = \emptyset$. Let $G = F_1 \cup \ldots \cup F_s = \{\xi_1, ..., \xi_k\}$. Thus $Y \cap M_G = \emptyset$. So, if $f(x_{\xi_i}) \geq \epsilon$ for all $1 \leq i \leq k$ and $f \in C_p(X,[0,1])$, then $f \notin Y$. □

Lemma 2.4. *Let $A = \{a_\lambda : \lambda < \alpha\}$ and $B = \{b_\lambda : \lambda < \alpha\}$ be two disjoint subsets of X such that $cl_{\beta(X)} A \cap cl_{\beta(X)} B \neq \emptyset$. Let Y be a C_α-compact subspace of $C_p(X,[-1,1])$ and let $\epsilon \in (0,1)$. Then there exist $K = \{\lambda_1, ..., \lambda_n\} \in [\alpha]^{<\omega}$ and $H = \{\xi_1, ..., \xi_m\} \in [\alpha]^{<\omega}$ such that, for any $f \in C_p(X,[-1,1])$ with $f(a_{\lambda_i}) \geq \epsilon$ and $f(b_{\xi_j}) \leq -\epsilon$ for every $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ we have $f \notin Y$.*

Proof. For each $n > \epsilon^{-1}$ and for each $F = \{\lambda_1, ..., \lambda_n\}, G = \{\xi_1, ..., \xi_n\} \in [\alpha]^{<\omega}$ we take $M_{(F,G)} = \{f \in C_p(X,[-1,1]) : f(a_{\lambda_i}) \geq \epsilon - \frac{1}{n}$ and $f(b_{\xi_j}) \leq -\epsilon + \frac{1}{n} \ \forall\ 1 \leq i \leq n, 1 \leq j \leq n\}$. Let $\mathcal{M} = \{M_{(F,G)} : F, G \in [\alpha]^{<\omega}$ and $|F|, |G| > \epsilon^{-1}\}$. The collection $\mathcal{M}$ is closed under finite intersections because $M_{(F_1,G_1)} \cap \ldots \cap M_{(F_n,G_n)} = M_{(\cup_{1 \leq i \leq n} F_i, \cup_{1 \leq i \leq n} G_i)}$. Moreover $\bigcap \mathcal{M} = \emptyset$. In fact, assume that $f \in \bigcap \mathcal{M}$ and let a_λ, b_ξ be arbitrary elements in A and B, respectively. Let $n < \omega$ be such that $n > \epsilon^{-1}$. We can take different elements $\lambda_1, ..., \lambda_n \in \alpha \setminus \{\lambda\}$ and different elements $\xi_1, ..., \xi_n \in \alpha \setminus \{\xi\}$. We have that $f \in M_{(F,G)}$ where $F = \{\lambda, \lambda_1, ..., \lambda_n\}$ and $G = \{\xi, \xi_1, ..., \xi_n\}$. Thus, $f(a_\lambda) \geq \epsilon - \frac{1}{n}$ and $f(b_\xi) \leq -\epsilon + \frac{1}{n}$. Since this can be done for every $n > \epsilon^{-1}$, then $f(a_\lambda) \geq \epsilon$ and $f(b_\xi) \leq -\epsilon$. Let $\hat{f} : \beta(X) \to [-1, 1]$ be the continuous extension of f. So $\hat{f}(a_\lambda) \geq \epsilon$ for all $\lambda < \alpha$ and $\hat{f}(b_\xi) \leq -\epsilon$ for all $\xi < \alpha$. But this is not possible because there is $r \in cl_{\beta(X)}A \cap cl_{\beta(X)}B$. Therefore, $\bigcap \mathcal{M} = \emptyset$.

Each of the elements in $\mathcal{M}$ is a nonempty zero-set in $C_p(X, [-1, 1])$ and the cardinality of $\mathcal{M}$ is $\leq \alpha$, so, by Proposition 1.4 we conclude that there exist $n > \epsilon^{-1}$, $K = \{\lambda_1, ..., \lambda_n\}$ and $H = \{\xi_1, ..., \xi_n\}$ such that $Y \cap M_{(K,H)} = \emptyset$. The sets K and H are as promised. □

Proposition 2.5. *If $C_p(X, [0, 1])$ is σ-C_α-compact, then X is α-discrete.*

Proof. Let $C_p(X, [0, 1]) = \bigcup\{P_n : n < \omega\}$ where, for each $n < \omega$, P_n is C_α-compact in $C_p(X, [0, 1])$. Assume that X is not α-discrete and let A be a non-closed subset of X of cardinality $\leq \alpha$; say $A = \{x_\lambda : \lambda < \alpha\}$. Then, there exists $z_0 \in (cl_X A) \setminus A$. Besides, there exists a retraction R from $C_p(X, [0, 1])$ onto $Z = \{f \in C_p(X, [0, 1]) : f(z_0) = 0\}$ ($R(f) = f - f(z_0)$). So Z is equal to $\bigcup_{0<n<\omega} Z_n$ where each Z_n is C_α-compact in Z (Proposition 1.6). We are going to obtain a contradiction after assuming that $z_0 \in (cl_X A) \setminus A$. By Lemma 2.3, for each $0 < n < \omega$ there is $G_n = \{\lambda_1^n, ..., \lambda_{k(n)}^n\} \in [\alpha]^{<\omega}$ such that if $f \in Z$ and $f(x_{\lambda_i}) \geq 2^{-n}$ for all $1 \leq i \leq k(n)$, then $f \notin Z_n$. Consider the sets $\hat{G}_n = G_1 \cup \ldots \cup G_n$ $(0 < n < \omega)$. Then $\hat{G}_n \subset \hat{G}_{n+1}$ for all $0 < n < \omega$, and if $f \in Z$ and $f(x_\lambda) \geq 2^{-n}$ for all $\lambda \in \hat{G}_n$, then $f \notin Z_n$.

Since X is a Tychonoff space, we can take, for each $0 < n < \omega$, a function $f_n \in Z$ such that $f_n(x_\lambda) = 1$ for all $\lambda \in \hat{G}_n$. Let $f = \Sigma_{n=1}^{\infty} 2^{-n} f_n$. We have that $f \in Z$ and if $n > 0$ and $\lambda \in \hat{G}_n$, then $f(x_\lambda) \geq 2^{-n} f_n(x_\lambda) = 2^{-n}$. Thus, $f \notin Z_n$ for all $0 < n < \omega$.

But this is a contradiction because $Z = \bigcup_{0<n<\omega} Z_n$. Therefore, A must be closed in X. □

The function $r : C_p(X) \to C_p(X, [0,1])$ defined as $r(f) = \xi \circ f$ is a retraction of $C_p(X)$ onto $C_p(X, [0,1])$, where $\xi : \mathbf{R} \to \mathbf{R}$ is defined as follows: $\xi(x) = x$ for $x \in [0,1]$, $\xi(x) = 1$ for $x > 1$, and $\xi(x) = 0$ if $x < 0$. So, $r|_{C_p^*(X)}$ is a retraction of $C_p^*(X)$ onto $C_p(X, [0,1])$. Besides, $C_p^*(X) = \bigcup_{n<\omega} C_p(X, [-n,n])$. Thus, by using Proposition 1.6 we obtain:

Proposition 2.6. *$C_p(X, [0,1])$ is σ-C_α-compact (resp., σ-α-pseudocompact) if and only if $C_p^*(X)$ is σ-C_α-compact (resp., σ-α-pseudocompact).*

Now, we are able to prove the main results of this article.

Theorem 2.7. *Let X be a space and α a cardinal number. Then the following are equivalent:*

1. *X is α-b-discrete.*
2. *$C_p(X, [0,1])$ is α-pseudocompact.*
3. *$C_p(X, [0,1])$ is C_α-compact in $[0,1]^X$.*
4. *$C_p(X, [0,1])$ is σ-α-pseudocompact.*
5. *$C_p(X, [0,1])$ is σ-C_α-compact.*
6. *$C_p^*(X)$ is σ-C_α-compact.*
7. *$C_p^*(X)$ is σ-α-pseudocompact.*

Proof. The equivalencies (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are a consequence of Propositions 1.5 and 2.1, the implications (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are evident, and Proposition 2.6 gives us (4) $\Leftrightarrow$ (7) and (5) $\Leftrightarrow$ (6). So, we have only to prove (5) $\Rightarrow$ (1).

For convenience, we are going to consider the space $C_p(X, [-1,1])$ instead of $C_p(X, [0,1])$. Because of Proposition

2.5 and Lemma 2.2, it is enough to prove that if A and B are two disjoint subsets of X of cardinality $\leq \alpha$, then $cl_{\beta(X)}A \cap cl_{\beta(X)}B = \emptyset$. Assume the contrary. Let A and B be disjoint subsets of X of cardinality $\leq \alpha$, and let r be an element belonging to $cl_{\beta(X)}A \cap cl_{\beta(X)}B$. Enumerate A and B as $\{a_\lambda : \lambda < \alpha\}$ and $\{b_\lambda : \lambda < \alpha\}$, respectively.

Assume that $C_p(X, [-1, 1]) = \bigcup\{Z_n : 0 < n < \omega\}$ where Z_n is C_α-compact in $C_p(X, [-1, 1])$ for each $0 < n < \omega$. Due to Lemma 2.4, we know that for each $0 < n < \omega$ there exist K_n and H_n in $[\alpha]^{<\omega}$ such that if $f \in C_p(X, [-1, 1])$ and $f(a_{\lambda(i)}) \geq 2^{-n}$ for every $\lambda(i) \in K_n$, and $f(b_{\xi(j)}) \leq -2^{-n}$ for every $\xi(j) \in H_n$, then $f \notin Z_n$. Without loss of generality we can assume that $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$, $H_1 \subset H_2 \subset \ldots \subset H_n \subset \ldots$, and there exists a sequence $(k_n)_{0<n<\omega}$ of natural numbers such that $K_n = \{\lambda(0), ..., \lambda(k_n)\}$, and $H_n = \{\xi(0), ..., \xi(k_n)\}$ for every $0 < n < \omega$, where $\lambda(n) \neq \lambda(m)$ and $\xi(n) \neq \xi(m)$ if $n \neq m$.

We know that X is α-discrete (Proposition 2.5). Thus, there exist disjoint open families $\mathcal{U} = \{U_n : n < \omega\}$ and $\mathcal{V} = \{V_n : n < \omega\}$ such that

(a) $(\bigcup \mathcal{U}) \cap (\bigcup \mathcal{V}) = \emptyset$; and

(b) $a_{\lambda(n)} \in U_n$ and $b_{\xi(n)} \in V_n$ for every $n < \omega$.

Now, since X is a Tychonoff space, there exist two collections $\mathcal{F} = \{f_n \in C_p(X, [-1, 1]) : n < \omega\}$ and $\mathcal{G} = \{g_n \in C_p(X, [-1, 1]) : n < \omega\}$ such that, for every $n < \omega$,

(i) $f_n \geq 0$ and $g_n \leq 0$;

(ii) $f_n^{-1}([-1, 1] \setminus \{0\}) \subset U_n$ and $g_n^{-1}([-1, 1] \setminus \{0\}) \subset V_n$; and

(iii) $f_n(a_{\lambda(n)}) = 1$ and $g_n(b_{\xi(n)}) = -1$.

We define, for each $0 < n < \omega$, the function $d_n = 2^{-n} \cdot (\Sigma_{t=0}^{k_n}(f_t + g_t))$. Take $h = \Sigma_{n=1}^{\infty} d_n$. The function h belongs to $C_p(X, [-1, 1])$, and $h(a_{\lambda(i)}) \geq 2^{-n}\ \forall\ 1 \leq i \leq k_n$, and $h(b_{\xi(i)}) \leq -2^{-n}$ $\forall$ $1 \leq i \leq k_n$, for each $0 < n < \omega$. But this means that $h \notin Z_n$ for all $0 < n < \omega$, which is not possible because $C_p(X, [-1, 1]) = \bigcup\{Z_n : 0 < n < \omega\}$. This contradiction leads us to conclude that $cl_{\beta(X)}A \cap cl_{\beta(X)}B = \emptyset$. Therefore, X is α-b-discrete. □

Theorem 2.8. *Let X be a space and α be a cardinal number. Then, the following assertions are equivalent:*

1. *X is pseudocompact and α-b-discrete.*
2. *$C_p(X)$ is σ-α-pseudocompact.*
3. *$C_p(X)$ is σ-C_α-compact.*

Proof. If $C_p(X)$ is σ-C_α-compact, then $C_p(X, [0, 1])$ also has this property because it is a retract of $C_p(X)$. Then X is α-b-discrete and $C_p(X)$ is σ-pseudocompact (Theorem 2.7). Therefore, X is also pseudocompact (Theorem 1.1).

If X is pseudocompact, then $C_p(X) = C_p^*(X)$. Since X is α-b-discrete, then $C_p^*(X) = C_p(X)$ is σ-α-pseudocompact (Theorem 2.7). □

3. An Infinite α-Pseudocompact α-b-Discrete Space

In [1, Example I.2.5] the efforts done in [5] are synthesized, and an example is given of an infinite pseudocompact b-discrete space Z. By reason of Proposition 1.5, a slight modification of Z is enough to obtain an infinite α-pseudocompact α-b-discrete space for each infinite cardinal α. For the sake of completeness we present here the details of this construction. The interval $[0, 1]$ C | R will be denoted by I.

Let α be an uncountable cardinal number, and let M be the set $[0, 2^\alpha)$ of ordinals smaller that 2^α. Let $S = \{x \in I^M : |\{\lambda \in M : \pi_\lambda(x) \neq 0\}| \leq \alpha\} \subset I^M$ be the Σ_α-product based at the point which has all its coordinates equal to zero. Then

$|S| = 2^\alpha = |M|$. Let $\{s_\lambda : \lambda \in M\}$ be an enumeration of the elements of S such that $|\{\lambda \in M : s = s_\lambda\}| = 2^\alpha$ for all $s \in S$. Let $\mathcal{E} = \{A \subset M : |A| \leq \alpha\}$. The cardinality of $\mathcal{E}$ is equal to 2^α, so we can choose an enumeration $\{A_\lambda : \lambda \in M\}$ of the elements of $\mathcal{E}$ such that $|\{\lambda : A_\lambda = A\}| = 2^\alpha$ for each $A \in \mathcal{E}$.

Remark 3.1. *Let A, B be subsets of M of cardinality $\kappa \leq \alpha$, and let $f \in S$. Then, there exist $\xi, \gamma \in M$ greater than* sup$B$ *such that $A_\xi = A$ and $s_\gamma = f$.*

Proof. Indeed, since $|B| = \kappa \leq \alpha$ and $\alpha < cof^{(2^\alpha)}$, sup$B = \gamma < 2^\alpha$. Because of the way we enumerate S and $\mathcal{E}$, there are $\xi, \lambda \in (\gamma, 2^\alpha)$ such that $A_\xi = A$ and $s_\lambda = f$. □

For each $\lambda \in M$ we define a point $x_\lambda \in I^M$ by:

$$\pi_\gamma(x_\lambda) = \begin{cases} \pi_\gamma(s_\lambda) & \text{if } \gamma \leq \lambda; \\ 1 & \text{if } \gamma > \lambda \text{ and } \lambda \in A_\gamma; \\ 0 & \text{if } \gamma > \lambda \text{ and } \lambda \notin A_\gamma. \end{cases}$$

We are going to prove that the subspace $X = \{x_\lambda : \lambda \in M\}$ of I^M is the one we looked for.

Claim 3.2. *X is dense in I^M.*

Proof. Let $\{m_1, ..., m_k\}$ be a finite subset of M and $A_1, ..., A_k$ be open subsets of I. Consider the basic open subset $U = \{f \in I^M : f(m_i) \in A_i \text{ for } i \in \{1, ..., k\}\}$. Take $g \in I^M$ such that

$$g(m_i) = \begin{cases} a_i \in A_i & \text{if } i \in \{1, ...k\}; \\ 0 & \text{if } i \notin \{1, ...k\}. \end{cases}$$

The function g is an element in $S \cap U$. Because of Remark 3.1, there is $\xi \in M$ which is greater than m_i for every i, such that $g = s_\xi$. Now, it can be proved that $x_\xi \in X$ belongs to U. □

Claim 3.3. *Let κ be a cardinal $\leq \alpha$. Then X is κ-pseudocompact.*

Proof. By virtue of Proposition 1.5 and Claim 3.2, in order to prove Claim 3.3, we need to show that for any $B \subset M$ of cardinality $\leq \kappa$, $\pi_B(X) = I^B$ holds.

Let $g \in I^B$ be arbitrary. Take $f \in S$ defined by

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \notin B; \\ g(\lambda) & \text{if } \lambda \in B. \end{cases}$$

Let $\gamma = \sup B$. There is $\xi \in (\gamma, 2^\alpha)$ such that $f = s_\xi$ (Remark 3.1). It is not difficult to see that $\pi_B(x_\xi) = g$. Therefore, $\pi_B(X) = I^B$. □

Claim 3.4. *Let B be a subset of M of cardinality $\kappa \leq \alpha$. Then $cl_{I^M}(\{x_\lambda : \lambda \in B\})$ is homeomorphic to $\beta(\kappa)$.*

Proof. It suffices to prove that for all disjoint $M_1, M_2 \subset M$ of cardinality $\leq \kappa$ we have $\text{cl}_{I^M}(\{x_\lambda : \lambda \in M_1\}) \cap \text{cl}_{I^M}(\{x_\lambda : \lambda \in M_2\}) = \emptyset$ (see [4, 6.5]).

Let $\xi \in M$ be such that $\xi > \sup(M_1 \cup M_2)$ and $A_\xi = M_1$ (Remark 3.1). Then $\pi_\xi(x_\lambda) = 1$ if $\lambda \in M_1$, and $\pi_\xi(x_\lambda) = 0$ if $\lambda \in M_2$. Thus the sets $\{x_\lambda : \lambda \in M_1\}$ and $\{x_\lambda : \lambda \in M_2\}$ are completely separated in I^M. □

As a consequence of this last claim we have the following result.

Claim 3.5. *Every subset of X of cardinality $\leq \alpha$ is closed in X.*

Proof. Let $B \subset M$ with $|B| = \kappa \leq \alpha$, and $\gamma \in M \setminus B$. Due to Claim 3.4, $\text{cl}_{I^M}\{x_\lambda : \lambda \in B\} \cap \{x_\gamma\} = \emptyset$. Thus $\text{cl}_X\{x_\lambda : \lambda \in B\} \cap \{x_\gamma\} = \emptyset$. Therefore, $\{x_\lambda : \lambda \in B\}$ is closed in X. □

Claim 3.6. *Every subset of X of cardinality $\leq \alpha$ is C^*-embedded in X.*

Proof. Let $B \subset M$ with $|B| = \kappa \leq \alpha$. Let $f \in C_p(\{x_\lambda : \lambda \in B\}, I) = I^B$. Since $P = \text{cl}_{I^M}(\{x_\lambda : \lambda \in B\})$ is homeomorphic to $\beta(\kappa)$ (Claim 3.4), there is an $h_0 \in C_p(P, I)$ such that $h_0|_{\{x_\lambda : \lambda \in B\}} = f$. Clearly, there is an $h_1 \in C_p(I^M, I)$ such that $h_1|_P = h_0$. Then $h = h_1|_X$ is the required function on X. □

Recall that a space Y is *left-separated* if there is a well-ordered $\prec$ on Y such that the set $\{y \in Y : y \prec x\}$ is closed in Y for every $x \in Y$.

Claims 3.3, 3.5 and 3.6 say that X is an infinite α-pseudocompact α-b-discrete space. Moreover, X is left separated (Claim 3.5) and connected (Claim 3.3, Proposition 1.5 and Lemma in [8]).

References

[1] A.V. Arkhangel'skii, *Topological Function Spaces*, Kluwer Academic Publishers, 1992.

[2] A.V. Arkhangel'skii and V.V. Tkachuk, *Function spaces and topological invariants*, Moskovsk. Gos. Univ., 1985. (In Russian.)

[3] S. García-Ferreira, M. Sanchis and A. Tamariz-Mascarúa, *On C_α-compact subsets*, Top. Appl., **77** (1997) 139-160.

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, 1960.

[5] D.B. Shakhmatov, *A pseudocompact Tychonoff space all countable subsets of which are closed and C^*-embedded*, Top. Appl. **22** (1986) 139-144.

[6] D.B. Shakhmatov and V.V. Tkachuk, *When is the space $C_p(X)$ σ-countably compact?*, Vestnik Mosk. Univ. Matematika, **41** (1986) 70-72.

[7] A. Tamariz-Mascarúa, *Espacios C_p de funciones Continuas*, Aportaciones Matemáticas, Serie Comunicaciones **18**, Sociedad Matemática Mexicana (1996) 265-285.

[8] M.G. Tkachenko, *Examples of left-separated spaces and topological groups*, Acta Math. Sci. Hungar. **38** (1981) 257-261.

[9] V.V. Tkachuk, *The spaces $C_p(X)$: decomposition into a countable union of bounded subspaces and completeness properties*, Top. and Appl. **22** (1986) 241-253.

Departamento de Matemáticas, Facultad de Ciencias, U.N.A.M., Ciudad Universitaria, México 04510.

E-mail address: atamariz@servidor.unam.mx