Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 23, Summer 1998, 379–409

METRIZABLE SUBSPACES OF FREE TOPOLOGICAL GROUPS ON METRIZABLE SPACES

Kohzo Yamada *

Dedicated to Professor Alexander V. Arhangel'skiĭ on his sixtieth birthday

Abstract

Let $F(X)$ and $A(X)$ be respectively the free topological group and the free abelian topological group on a Tychonoff space X. For all natural number n we denote by $F_n(X)$ ($A_n(X)$) the subset of $F(X)$ ($A(X)$) consisting of all words of reduced length $\leq n$. For every natural number n, we construct a neighborhood base at the identity in $F_{2n}(X)$ for a pseudocompact space X. In the abelian case, it was already obtained for a Tychonoff space in [22], [23]. Using the neighborhood bases we prove that for each natural number $n \geq 2$, $uu(X)$ is equal to the character of $A_n(X)$ for a Tychonoff space X and to the character of $F_n(X)$ for a pseudocompact space X, where $uu(X)$ is a cardinal function on X called the universal uniform weight, which is defined in this paper. As applications of these facts, we characterize the spaces X such that $F_n(X)$ and $A_n(X)$ are first countable or metrizable for each $n \in \mathbb{N}$, respectively. In addition, we also prove that if $F_n(X)$ ($A_n(X)$) is first countable, then it is metrizable for each $n \geq 2$. We also

* This paper was written while the author was visiting Auburn University.

consider when the natural mappings from $(X \oplus X^{-1} \oplus \{e\})^n$ to $F_n(X)$ and from $(X \oplus -X \oplus \{0\})^n$ to $A_n(X)$ are closed.

1. Introduction

Let $F(X)$ and $A(X)$ be respectively the *free topological group* and the *free abelian topological group* on a Tychonoff space X in the sense of Markov [12]. As an abstract group, $F(X)$ is free on X and it carries the finest group topology that induces the original topology of X; every continuous map from X to an arbitrary topological group lifts in a unique fashion to a continuous homomorphism from $F(X)$. Similarly, as an abstract group, $A(X)$ is the free abelian group on X, having the finest group topology that induces the original topology of X, so that every continuous map from X to an arbitrary abelian topological group extends to a unique continuous homomorphism from $A(X)$.

For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose length is less than or equal to n. It is known that X itself and each $F_n(X)$ are closed in $F(X)$. The subspace $A_n(X)$ is defined similarly and each $A_n(X)$ is closed in $A(X)$. Let e be the identity of $F(X)$ and 0 be that of $A(X)$. For each $n \in \mathbb{N}$ and an element $(x_1, x_2, \ldots, x_n)$ of $(X \oplus X^{-1} \oplus \{e\})^n$ we call $x_1 x_2 \cdots x_n$ a *form*. In the abelian case, $x_1 + x_2 + \cdots + x_n$ is also called a *form* for $(x_1, x_2, \ldots, x_n) \in (X \oplus -X \oplus \{0\})^n$. We remark that a form may contain some reduced letter. Then the reduced form of $x_1 x_2 \cdots x_n$ is a word of $F(X)$ and that of $x_1 + x_2 + \cdots + x_n$ is a word of $A(X)$. For each $n \in \mathbb{N}$ we denote the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ onto $F_n(X)$ by i_n

Key words: Free topological group, free abelian topological group, character, universal uniformity, universal uniform weight, first countable space, metrizable space, closed mapping

Mathematics Subject Classification: Primary 54H11, 22A05; Secondary 54A25, 54E15, 54E35, 54C10

and we also use the same symbol i_n in the abelian case, that is, i_n means the natural mapping from $(X \oplus -X \oplus \{0\})^n$ onto $A_n(X)$. Clearly the mapping i_n is continuous for each $n \in \mathbb{N}$.

Graev [7] proved that if a space X is non-discrete, then neither $F(X)$ nor $A(X)$ is first countable. On the other hand, since the mappings i_n are continuous, if a space X is compact metrizable, then both $F_n(X)$ and $A_n(X)$ are compact metrizable and hence first countable for each $n \in \mathbb{N}$. Then the following natural question can be considered:

> *Is each $F_n(X)$ ($A_n(X)$) metrizable or first countable for such spaces X as the real line $\mathbb{R}$, the space $\mathbb{Q}$ of rational numbers, $\mathbb{R} \setminus \mathbb{Q}$ or the hedgehog $J(\kappa)$ of spininess κ ?*

Some answers to the above question were obtained earlier. For example, Fay, Ordman and Thomas [6] showed that $F_8(\mathbb{Q})$ is not a k-space and hence it is not first countable. The author [22] obtained that neither $A_3(\mathbb{Q})$ nor $A_3(\mathbb{R} \setminus \mathbb{Q})$ is a k-space and hence they are not first countable.

In the next section, we first give neighborhoods of the identity in $F_{2n}(X)$ and $A_{2n}(X)$ for each $n \in \mathbb{N}$, respectively. In the abelian case, we introduce the neighborhood base at 0 in $A_{2n}(X)$ for a Tychonoff space X, which was obtained by the author in [22], [23]. In the non-abelian case, applying the Graev's continuous pseudometric on $F(X)$ defined in [7] and the neighborhood base at e in $F(X)$ for a Tychonoff space X constructed by Tkačenko [16] we construct a neighborhood base at e in $F_{2n}(X)$ for a pseudocompact space X. When $n = 1$, our neighborhood base is of the same form as Pestov's one constructed in [14] and it is easy to see that pseudocompactness of a space X is not necessary.

In §3, we define the cardinal function $uu(X)$ on a Tychonoff space X called the *universal uniform weight*. Then, using the above neighborhood bases at the identity, we prove that

$\chi(A_n(X)) = \chi(A_2(X)) = uu(X)$ for a Tychonoff space X and each natural number $n \geq 2$, and $\chi(F_n(X)) = \chi(F_2(X)) = uu(X)$ for a pseudocompact space X and each natural number $n \geq 2$. As applications of these facts, in §4 we characterize metrizable spaces X such that $F_n(X)$ and $A_n(X)$ are metrizable for each $n \in \mathbb{N}$, respectively. We prove that for a metrizable space X, the following are equivalent:

(i) $A_n(X)$ is metrizable for each $n \in \mathbb{N}$;

(ii) $A_n(X)$ is first countable for each $n \in \mathbb{N}$;

(iii) $A_2(X)$ is metrizable;

(iv) $A_2(X)$ is first countable;

(v) the set of all non-isolated points of X is compact.

In the non-abelian case the following are equivalent:

(i) $F_n(X)$ is metrizable for each $n \in \mathbb{N}$;

(ii) $F_n(X)$ is first countable for each $n \in \mathbb{N}$;

(iii) $F_4(X)$ is metrizable;

(iv) $F_4(X)$ is first countable;

(v) X is compact or discrete.

Furthermore, the following are also equivalent:

(i) $F_3(X)$ is metrizable;

(ii) $F_3(X)$ is first countable;

(iii) $F_2(X)$ is metrizable;

(vi) $F_2(X)$ is first countable;

(v) the set of all non-isolated point of X is compact.

It is well known that every first countable topological group is metrizable. Though each $F_n(X)$ ($A_n(X)$) is not a topological group, it follows from the above results that if $F_n(X)$ ($A_n(X)$) is first countable, then it is metrizable for each $n \geq 2$.

In the same section, we also consider the closedness of the natural mappings i_n. Tkačenko [18] raised the following problem:

> *Characterize spaces X for which the natural mapping i_n is quotient (closed, z-closed, R-quotient, etc.), $n \in \mathbb{N}$.*

He showed in [17] that if X^{2n} is normal and countably compact, then the mapping i_n is closed. Pestov [14] proved that the mapping i_2 is quotient if and only if every neighborhood of the diagonal in X^2 is an element of the universal uniformity on X. We prove for a metrizable space X that if i_3 is closed, then X is compact or discrete. For $n = 2$, we improve Pestov's result by showing that i_2 is closed under the same condition as in the above result.

All topological spaces are assumed to be Tychonoff. By $\mathbb{N}$ we denote the set of all positive natural numbers. Our terminology and notations follow [5]. We refer to [9] for elementary properties of topological groups and to [3] and [7] for the main properties of free topological groups.

2. Neighborhood Base at the Identity in $F_n(X)$ and $A_n(X)$

In this section, we consider a neighborhood base at the identity. We first introduce the neighborhood base at 0 in $A(X)$ constructed by Tkačenko [15], Pestov [14] and the neighborhood base at 0 in $A_{2n}(X)$ constructed by the author [22], [23].

Let $\mathcal{U}_X$ be the universal uniformity on a space X. For each

$P = \{U_1, U_2, \dots\} \in (\mathcal{U}_X)^\omega$, let

$$V(P) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U_i \text{ for } i = 1, \dots, k, k \in \mathbb{N}\},$$

and $\mathcal{V} = \{V(P) : P \in (\mathcal{U}_X)^\omega\}$. Furthermore, fix any $n \in \mathbb{N}$. For each $U \in \mathcal{U}_X$ let

$$V_n(U) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U \text{ for } i = 1, \dots, k, k \leq n\},$$

and $\mathcal{V}_n = \{V_n(U) : U \in \mathcal{U}_X\}$. Then the following are known.

Theorem 2.1. *Let X be a space. Then:*

(1) ([14]) $\mathcal{V}$ is a neighborhood base at 0 in $A(X)$, and

(2) ([22], [23]) $\mathcal{V}_n$ is a neighborhood base at 0 in $A_{2n}(X)$ for each $n \in \mathbb{N}$.

In the non-abelian case, some neighborhood bases at e in $F(X)$ were constructed by Tkačenko [16], Pestov [14] and Uspenskiĭ [21], and every element of their neighborhood bases has a complicated form while $V(P)$ and $V_n(U)$ are simpler. Here we introduce Tkačenko's neighborhood base, which is used for proving Theorem 2.6 below.

Let X be a space and $\overline{X} = X \oplus X^{-1} \oplus \{e\}$. For each $n \in \mathbb{N}$, we define a mapping j_n from $\overline{X}^n \times \overline{X}^n$ to $F_{2n}(X)$ by $j_n((\boldsymbol{x}, \boldsymbol{y})) = i_n(\boldsymbol{x}) i_n(\boldsymbol{y})^{-1}$ for every $(\boldsymbol{x}, \boldsymbol{y}) \in \overline{X}^n \times \overline{X}^n$. Let $\mathcal{U}_n$ be the universal uniformity on $\overline{X}^n$ for each $n \in \mathbb{N}$. For each $R = \{U_n : n \in \mathbb{N}\} \in \prod_{i=1}^\infty \mathcal{U}_n$, we put

$$W'_n(R) = \bigcup \{j_{\pi(1)}(U_{\pi(1)}) \cdots j_{\pi(n)}(U_{\pi(n)}) : \pi \in S_n\}$$

and

$$W(R) = \bigcup_{n=1}^\infty W'_n(R),$$

where S_n is the permutation group on $\{1, 2, \dots, n\}$. Then Tkačenko proved the following.

Theorem 2.2 ([16]). *$\mathcal{W} = \{W(R) : R \in \prod_{i=1}^{\infty} \mathcal{U}_n\}$ is a neighborhood base at e in $F(X)$.*

From the definition, we have that $|\mathcal{V}_n| = |\mathcal{U}_X| \leq |\mathcal{U}_X|^{\aleph_0} = |\mathcal{V}|$ for each $n \in \mathbb{N}$. In particular, if $\mathcal{U}_X$ is countably infinite, so is $\mathcal{V}_n$ for each $n \in \mathbb{N}$, even though the cardinality of $\mathcal{V}$ is $\mathfrak{c}$. We shall construct a family $\mathcal{W}_n$ consisting of neighborhoods of e in $F_{2n}(X)$ and satisfying the same property. First, we prove that it is a neighborhood base at e in $F_{2n}(X)$ assuming that $\mathcal{U}_n = (\mathcal{U}_{\overline{X}})^n$ for each $n \in \mathbb{N}$, where $(\mathcal{U}_{\overline{X}})^n$ is the uniformity on $\overline{X}^n$ generated by the sets $U^n = \{((x_1, \dots, x_n), (y_1, \dots, y_n)) \in \overline{X}^n \times \overline{X}^n : (x_i, y_i) \in U \text{ for } i = 1, 2, \dots, n\}$ with $U \in \mathcal{U}_{\overline{X}}$ (we use the uniformity in the proof of Lemma 2.4). Then we show that it is a neighborhood base at e in $F_{2n}(X)$ for a pseudocompact space X.

In general, the universal uniformity $\mathcal{U}_n$ on $\overline{X}^n$ is finer than $(\mathcal{U}_{\overline{X}})^n$. It was proved in [10] and [8] that for a space X, $\mathcal{U}_2 = (\mathcal{U}_X)^2$ if and only if either there is a cardinal $\mathfrak{m}$ for which X^2 is pseudo-$\mathfrak{m}$-compact and $P(\mathfrak{m})$, or X is discrete (a space X is $P(\mathfrak{m})$ if each family of fewer than $\mathfrak{m}$ open sets has open intersection).

Definition. *Fix an arbitrary $n \in \mathbb{N}$. For a subset U of $\overline{X}^2$ which includes the diagonal of $\overline{X}^2$, let $W_n(U)$ be a subset of $F_{2n}(X)$ which consists of the identity e and all words g satisfying the following conditions;*

(1) *g can be represented as the reduced form $g = x_1 x_2 \cdots x_{2k}$, where $x_i \in \overline{X}$ for $i = 1, 2, \dots, k$ and $1 \leq k \leq n$,*

(2) *there is a partition $\{1, 2, \dots, 2k\} = \{i_1, i_2, \dots, i_k\} \cup \{j_1, j_2, \dots, j_k\}$,*

(3) *$i_1 < i_2 < \cdots < i_k$ and $i_s < j_s$ for $s = 1, 2, \dots, k$,*

(4) *$(x_{i_s}, x_{j_s}^{-1}) \in U$ for $s = 1, 2, \dots, k$ and*

(5) *$i_s < i_t < j_s \Longleftrightarrow i_s < j_t < j_s$ for $s, t = 1, 2, \dots, k$.*

We shall show that $\mathcal{W}_n = \{W_n(U) : U \in \mathcal{U}_{\overline{X}}\}$ is the required family.

Theorem 2.3. *Let X be a space. Then for every $U \in \mathcal{U}_{\overline{X}}$ and every $n \in \mathbb{N}$, $W_n(U)$ is a neighborhood of e in $F_{2n}(X)$.*

Proof. Given $U \in \mathcal{U}_{\overline{X}}$, one can find a continuous pseudometric d on X such that

$$\{(x, y) \in X \times X : d(x, y) < 1\} \subseteq U$$

and

$$\{(x^{-1}, y^{-1}) \in X^{-1} \times X^{-1} : d(x, y) < 1\} \subseteq U.$$

Without loss of generality we can assume that $d \leq 1$. Extend d to a continuous pseudometric $\overline{d}$ on $\overline{X}$ defining, $\overline{d}(x^{-1}, y^{-1}) = d(x, y)$, $\overline{d}(e, x) = \overline{d}(e, x^{-1}) = 1$ and $\overline{d}(x^{-1}, y) = \overline{d}(x, y^{-1}) = 2$ for all $x, y \in X$. Then there exists an invariant continuous pseudometric $\hat{d}$ on $F(X)$ such that $\hat{d}(x, y) = \overline{d}(x, y)$ for all $x, y \in \overline{X}$ (see Theorem 1 of [7]).

Now we put

$$O_d = \{g \in F(X) : \hat{d}(g, e) < 1\}$$

and

$$U_d = \{(x, y) \in \overline{X}^2 : \overline{d}(x, y) < 1\}.$$

From the continuity of $\hat{d}$ it follows that O_d is an open neighborhood of e in $F(X)$. To finish the proof it suffices to verify the inclusion

$$O_d \cap F_{2n}(X) \subseteq W_n(U_d).$$

Suppose that $g \in O_d \cap F_{2n}(X)$, $g \neq e$, and let $g = x_1 x_2 \cdots x_m$ be the reduced form of g (hence $m \leq 2n$). Our definitions of $\overline{d}$ and $\hat{d}$ imply that every element of O_d has even length, so $m = 2k$ for some $k \leq n$ and $x_i \neq e$ for each $i \leq 2k$. From Graev's construction of $\hat{d}$ (see the proof of Theorem 1 in [7]) it follows that there exists a partition $\{1, 2, \dots, 2n\} = \{i_1, i_2, \dots, i_k\} \cup \{j_1, j_2, \dots, j_k\}$ satisfying (3) and (5) of the definition of $W_n(U)$ and that $\hat{d}(g, e) = \sum_{p=1}^{k} \overline{d}(x_{i_p}, y_{j_p}^{-1})$. Since $\hat{d}(g, e) < 1$, we conclude that $\overline{d}(x_{i_p}, y_{j_p}^{-1}) < 1$ and hence $(x_{i_p}, y_{i_p}) \in U_d \subseteq U$ for each $p \leq k$. This immediately implies that $g \in W_n(U)$, thus finishing the proof. □

We introduce the notion of a *thin* set defined in [16]. A subset X of a topological group G is *thin* in G if for every neighborhood U of the identity e in G there is a neighborhood V of e such that $xVx^{-1} \subseteq U$ for each $x \in X$.

Let $\mathcal{W}$ be the family of neighborhoods of the identity e of $F(X)$ constructed with the use of the uniformities $(\mathcal{U}_{\overline{X}})^i$ for each $i \in \mathbb{N}$ by Tkachenko's method (see the construction described before Theorem 2.2). In other words, a typical element V of $\mathcal{W}$ has the form $V = W(R)$, where $R \in \prod_{i=1}^{\infty} (\mathcal{U}_{\overline{X}})^i$. It is easy to see that $\mathcal{W}$ is a neighborhood base at e for a Hausdorff group topology τ on $F(X)$ (see the argument in the proof of Theorem 1.1 of [16]). Denote by $G(X)$ the group $F(X)$ endowed with the group topology τ.

Lemma 2.4. *The set $\overline{X}$ is thin in $G(X)$. Hence, for each $n \in \mathbb{N}$ the set $G_n(X)$ is thin in $G(X)$.*

Proof. Consider a basic neighborhood $W(R)$ of the identity in $G(X)$, where $R = (V_1, V_2, \dots) \in \prod_{i=1}^{\infty} (\mathcal{U}_{\overline{X}})^i$. For every $i \in \mathbb{N}$, there exists $V_i' \in (\mathcal{U}_{\overline{X}})^i$ such that $(z, x_1, \dots, x_i, z, y_1, \dots, y_i) \in V_{i+1}$ whenever $z \in \overline{X}$ and $(x_1, \dots, x_i, y_1, \dots, y_i) \in V_i'$. Put $R' = (V_1', V_2', \dots)$. A straightforward verification shows that the set $O = W(R')$ satisfies $xOx^{-1} \subseteq W(R)$ for each $x \in \overline{X}$. Thus, $\overline{X}$ is thin in $G(X)$. Applying the fact n times, we can prove that $G_n(X)$ is thin in $G(X)$. □

Now we need one more lemma. Suppose that $U \in \mathcal{U}_{\overline{X}}$. Let us call an element $g \in F(X)$ *U-canonical* if it has the form $g = hxy^{-1}h^{-1}$, where $(x, y) \in U$ and $h \in F(X)$.

Lemma 2.5. *Let $U \in \mathcal{U}_{\overline{X}}$ and $n \in \mathbb{N}$. Then every element $h \in W_n(U)$ can be represented as a product $h = g_1 g_2 \cdots g_k$ of U-canonical elements $g_1, g_2, \ldots, g_k$, where $k \leq n$ and the length of g_i is less or equal to $2n$ (we write this $\ell(g_i) \leq 2n$) for each $i \leq k$.*

Proof. Let $h = x_1 x_2 \cdots x_{2k} \in W_n(U)$, where $k \leq n$. We prove the lemma by induction on k. It is clear if $k = 1$, so let $k \geq 2$ and assume that the lemma holds for each $i < k$. By the properties (4) and (5) of the definition of $W_n(U)$, there is $i \leq k$ such that $(x_1, x_{2i}^{-1}) \in U$. If $i < k$, then let $h_1 = x_1 \cdots x_{2i}$ and $h_2 = x_{2i+1} \cdots x_{2k}$. Clearly, $h_1 \in W_i(U)$ and $h_2 \in W_{k-i}(U)$. Hence, by the inductive assumption, there are U-canonical elements $u_1, \ldots, u_s$ and $v_1, \ldots, v_t$ such that

$$h_1 = u_1 \cdots u_s, s \leq i \text{ and } \ell(u_j) \leq 2i \text{ for each } j \leq s$$

and

$$h_2 = v_1 \cdots v_t, t \leq k - i \text{ and } \ell(v_j) \leq 2(k - i) \text{ for each } j \leq t.$$

Since $h = u_1 \cdots u_s v_1 \cdots v_t$, $s + t \leq k$, $\ell(u_j) \leq 2n$ for each $j \leq s$ and $\ell(v_j) \leq 2n$ for each $j \leq t$, h is represented as the required product.

On the other hand, if $i = k$, then let $h' = x_2 \cdots x_{2k-1}$. Since $h' \in W_{k-1}(U)$, by the inductive assumption, there are U-canonical elements $g'_1, \ldots, g'_m$ such that $h' = g'_1 \cdots g'_m$, $m \leq k-1$ and $\ell(g'_i) \leq 2(k-1)$. Define g_i $(i = 1, \ldots, m, m+1)$ by

$$g_i = x_1 g'_i x_1^{-1} \text{ for } i = 1, \ldots, m \text{ and } g_{m+1} = x_1 x_{2k}.$$

Then h can be represented as the product

$$\begin{aligned} h &= x_1 h' x_{2k} \\ &= x_1 g'_1 x_1^{-1} x_1 g'_2 x_1^{-1} x_1 \cdots x_1^{-1} x_1 g'_m x_1^{-1} x_1 x_{2k} \\ &= g_1 g_2 \cdots g_{m+1}. \end{aligned}$$

Since every g_i is U-canonical, $m + 1 \leq k$ and $\ell(g_i) \leq 2k \leq 2n$ for each $i \leq m + 1$, h is represented as the required product. □

Applying the above lemmas, we shall show that if a space X is pseudocompact, then for every $n \in \mathbb{N}$ the family $\mathcal{W}_n$ is a neighborhood base at the identity in $F_{2n}(X)$.

Theorem 2.6. *If* $\mathcal{U}_i = (\mathcal{U}_{\overline{X}})^i$ *for each* $i \in \mathbb{N}$*, then for each* $n \in \mathbb{N}$ *and* $R \in \prod_{i=1}^{\infty} \mathcal{U}_i$ *there is* $U \in \mathcal{U}_{\overline{X}}$ *such that* $W_n(U) \subseteq W(R) \cap F_{2n}(X)$*.*

Proof. Fix $n \in \mathbb{N}$ and let $R \in \prod_{i=1}^{\infty} \mathcal{U}_i$. Since $\mathcal{U}_i = (\mathcal{U}_{\overline{X}})^i$ for each $i \in \mathbb{N}$, $W(R)$ is a neighborhood of e in $G(X)$. Let W be a neighborhood of e in $G(X)$ such that $W^n \subseteq W(R)$. Then we apply Lemma 2.4 to choose a neighborhood O of e in $G(X)$ such that $gOg^{-1} \subseteq W$ for each $g \in G_n(X)$.

Let $U \in \mathcal{U}_{\overline{X}}$ such that $U \subseteq \{(x, y) \in \overline{X}^2 : xy^{-1} \in O\}$. We claim that $W_n(U) \subseteq W(R) \cap F_{2n}(X)$. Indeed, by Lemma 2.5, every element $h \in W_n(U)$ can be written as a product $h = g_1 \cdots g_k$ of U-canonical elements $g_1, \dots, g_k$, where $k \le n$ and $\ell(g_i) \le 2n$ for each $i \le k$. Let $g_i = h_i x_i y_i^{-1} h_i^{-1}$, where $h_i \in G(X)$ and $x_i, y_i \in \overline{X}$, $i = 1, \dots, k$. Clearly, $\ell(h_i) \le n - 1 < n$ and $(x_i, y_i) \in U$, whence $x_i y_i^{-1} \in O$. Therefore, the choice of O implies that $g_i = h_i x_i y_i^{-1} h_i^{-1} \in h_i O h_i^{-1} \subseteq W$ for each $i \le k$. In its turn, this implies that $h = g_1 \cdots g_k \in W^k \subseteq W^n \subseteq W(R)$, and the proof is completed. ∎

We should mention here that the referee of this paper informed that Theorem 2.6 also follows from Theorem 1.8 of [20] and that the next corollary also follows from Pestov's Lemma in [13].

Corollary 2.7. *In Theorem 2.5, the hypothesis* $\mathcal{U}_n = (\mathcal{U}_{\overline{X}})^i$ *for each* $i \in \mathbb{N}$ *can be dropped when* $n = 1$*. In other words, for each* $R \in \mathcal{R}$ *there is* $U \in \mathcal{U}_{\overline{X}}$ *such that* $W_1(U) \subseteq W(R) \cap F_2(X)$*.*

Proof. Let $R = (V_1, V_2, \dots) \in \prod_{i=1}^{\infty} \mathcal{U}_i$ and $U \in \mathcal{U}_{\overline{X}}$ be such that $U \subseteq V_1 \cap (X^2 \cup (X^{-1})^2 \cup \{(e, e)\})$. Then, by (1) – (5) of the definition of $W_1(U)$, each $g \in W_1(U)$ has the reduced form $g = x_1 x_2^{-1}$, where $(x_1, x_2) \in U$ or $g = e$. It follows that $g \in W(R)$, and hence $W_1(U) \subseteq W(R) \cap F_2(X)$. ∎

If X is a subspace of a space Y, then the natural monomorphism $i : F(X) \to F(Y)$ is continuous. The following result is due to Pestov.

Theorem 2.8 ([13]). *For a space X the natural monomorphism $i : F(X) \to F(\beta X)$ is an embedding if and only if X is pseudocompact.*

From the above results we have the following.

Theorem 2.9. *Let X be a pseudocompact space. Then $\mathcal{W}_n$ is a neighborhood base at e in $F_{2n}(X)$.*

Proof. Let X be a pseudocompact space and fix $n \in \mathbb{N}$. Then $F_{2n}(X)$ can be considered as a subspace of $F_{2n}(\beta X)$ by Theorem 2.8. By Theorem 2.3 and Theorem 2.6, $\mathcal{W}'_n = \{W_n(U) : U \in \mathcal{U}_{\overline{\beta X}}\}$ is a neighborhood base at e in $F_{2n}(\beta X)$, and hence so is $\mathcal{W}'_n|_{F_{2n}(X)}$ in $F_{2n}(X)$. To prove that $\mathcal{W}_n$ is a neighborhood base at e in $F_{2n}(X)$ let $U' \in \mathcal{U}_{\overline{\beta X}}$. Then $U' \cap \overline{X}^2 \in \mathcal{U}_{\overline{\beta X}}|_{\overline{X}}$ and $\mathcal{U}_{\overline{\beta X}}|_{\overline{X}}$ is a uniformity on $\overline{X}$ which induces the original topology for $\overline{X}$. Since $\mathcal{U}_{\overline{X}}$ is the universal uniformity on $\overline{X}$ there is $U \in \mathcal{U}_{\overline{X}}$ such that $U \subseteq U' \cap \overline{X}^2$. It is easy to see that $W_n(U) \subseteq W_n(U' \cap \overline{X}^2) \subseteq W_n(U') \cap F_{2n}(X) \in \mathcal{W}'_n|_{F_{2n}(X)}$. Since, by Theorem 2.3, $\mathcal{W}_n$ is a family consisting of neighborhoods of e in $F_{2n}(X)$, this implies that it is a neighborhood base at e in $F_{2n}(X)$. □

As a corollary to Theorem 2.3 and Corollary 2.7, the following result is obtained, which was proved by Pestov [13] with a different proof.

Corollary 2.10 ([13]). *Let X be a space. Then $\mathcal{W}_1$ is a neighborhood base at e in $F_2(X)$.*

3. The Character of $F_n(X)$ and $A_n(X)$

In this section, using the neighborhood bases at the identity in $F_n(X)$ and in $A_n(X)$ constructed in the previous section we calculate the character of $F_n(X)$ and $A_n(X)$. Since both $F(X)$ and $A(X)$ are topological groups and hence homogeneous, the character of $F(X)$ and $A(X)$ depends on the character at the identities, respectively. On the other hand, neither $F_n(X)$ nor $A_n(X)$ is a topological group, in fact they are not homogeneous. This means that it is not sufficient to calculate the character at the identities in order to investigate the character of $F_n(X)$ and $A_n(X)$. However, we have the following useful fact.

Lemma 3.1. *Let X be a space and $m, n \in \mathbb{N}$ with $n \leq m$. If B is a neighborhood of e in $F_{m+n}(X)$ and $g \in F_n(X)$, then $gB \cap F_m(X)$ is a neighborhood of g in $F_m(X)$. The same is true in the abelian case.*

Proof. Let U be a neighborhood of e in $F(X)$ such that $U \cap F_{m+n}(X) \subseteq B$. Since $gU \cap F_m(X)$ is a neighborhood of g in $F_m(X)$ it suffices to prove that $gU \cap F_m(X) \subseteq gB \cap F_m(X)$. Let $h \in gU \cap F_m(X)$. Then there is $u \in U$ such that $h = gu$. Since the length of $h \leq m$ (we write it $\ell(h) \leq m$.) and $\ell(g) \leq n$, $\ell(u) \leq m + n$, that is $u \in F_{m+n}(X)$. It follows that $u \in U \cap F_{m+n}(X) \subseteq B$ and $h = gu \in gB \cap F_m(X)$. Therefore, we have $gU \cap F_m(X) \subseteq gB \cap F_m(X)$. □

Proposition 3.2. *Let $m, n \in \mathbb{N}$, $n \leq m$, and κ be a cardinal.*

(1) *If $\chi(e, F_{m+n}(X)) \leq \kappa$, then $\chi(g, F_m(X)) \leq \kappa$ for each $g \in F_n(X)$, and*

(2) *If $\chi(0, A_{m+n}(X)) \leq \kappa$, then $\chi(g, A_m(X)) \leq \kappa$ for each $g \in A_n(X)$.*

Proof. Since the proofs of (1) and (2) are similar, we only show (1). Let $\mathcal{U}$ be a neighborhood base at e in $F(X)$ and $\mathcal{B}_{m+n}$ be a neighborhood base at e in $F_{m+n}(X)$ such that $|\mathcal{B}_{m+n}| \leq \kappa$. Take an arbitrary $g \in F_n(X)$ and put $\mathcal{B}_m(g) = \{gB \cap F_m(X) : B \in \mathcal{B}_{m+n}\}$. Then every element of $\mathcal{B}_m(g)$ contains g and $|\mathcal{B}_m(g)| \leq \kappa$. It is clear that for each $U \in \mathcal{U}$ there is $B \in \mathcal{B}_{m+n}$ such that $gB \cap F_m(X) \subseteq gU \cap F_m(X)$. On the other hand, Lemma 3.1 shows that every element of $\mathcal{B}_m(g)$ is a neighborhood of g in $F_m(X)$. Hence $\mathcal{B}_m(g)$ is a neighborhood base of g in $F_m(X)$ whose cardinality $\leq \kappa$. This implies that $\chi(g, F_m(X)) \leq \kappa$ for each $g \in F_n(X)$. □

Now we introduce a cardinal function on a space X. Let $\mathcal{U}_X$ the universal uniformity of a space X. We define the cardinal function $uu(X)$ called the *universal uniform weight* of X by

$$uu(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } \mathcal{U}_X\}.$$

From the above results we deduce the following equalities.

Theorem 3.3. *For a space X and a cardinal κ the following are equivalent:*

(1) $\chi(A_n(X)) \leq \kappa$ *for each* $n \in \mathbb{N}$,

(2) $\chi(A_2(X)) \leq \kappa$,

(3) $uu(X) \leq \kappa$.

Proof. By (2) of Theorem 2.1 and (2) of Proposition 3.2, we have (3) ⇒ (1) and (1) ⇒ (2) is clear. So we shall show that (2) ⇒ (3). By (2) of Theorem 2.1, $\mathcal{V}_1 = \{V_1(U) : U \in \mathcal{U}_X\}$ is a neighborhood base at 0 in $A_2(X)$. Since $\chi(A_2(X)) \leq \kappa$ there is $\mathcal{B} \subseteq \mathcal{U}_X$ with $|\mathcal{B}| \leq \kappa$ such that $\{V_1(B) : B \in \mathcal{B}\}$ is also a neighborhood base at 0 in $A_2(X)$. From the definition of $V_1(U)$ it is easy to see that $V_1(U_1) \subseteq V_1(U_2)$ if and only if $U_1 \subseteq U_2$. It follows that $\mathcal{B}$ is a base for $\mathcal{U}_X$, and hence $uu(X) \leq \kappa$. □

Theorem 3.4. *For a pseudocompact space X and a cardinal κ the following are equivalent:*

(1) $\chi(F_n(X)) \leq \kappa$ *for each* $n \in \mathbb{N}$,

(2) $\chi(F_2(X)) \leq \kappa$,

(3) $uu(X) \leq \kappa$.

Proof. Since $uu(\overline{X}) = uu(X)$ the argument in the proof of Theorem 3.3 applies here. □

Remark 3.5. Pseudocompactness of a space X is not necessary to prove the equivalence of (2) and (3) of Theorem 3.4, that is, $\chi(F_2(X)) = uu(X)$. Indeed, by Corollary 2.10, $\chi(e, F_2(X)) \leq uu(X)$. Furthermore, since the open subset $F_2(X) \setminus F_1(X)$ of $F_2(X)$ is homeomorphic to a subspace of $\overline{X}^2$ and the open subset $F_1(X) \setminus \{e\}$ of $F_2(X)$ is homeomorphic to $X \cup X^{-1}$, $\chi(g, F_2(X)) \leq \chi(\overline{X}^2) = \chi(\overline{X}) \leq uu(X)$ for each $g \in F_2(X) \setminus \{e\}$. Hence $\chi(F_2(X)) \leq uu(X)$. The converse inequality can be proved similarly to (2) $\Rightarrow$ (3) in Theorem 3.3.

In other words, Theorems 3.3 and 3.4 imply the following: *Let X be a space and $n \in \mathbb{N}$ with $n \geq 2$. Then*

(1) $\chi(A_n(X)) = \chi(A_2(X)) = uu(X)$, *and*

(2) $\chi(F_n(X)) = \chi(F_2(X)) = uu(X)$ *if X is pseudocompact.*

Clearly $\chi(X) \leq uu(X)$ for every space X. On the other hand, the converse inequality is not always true (take X to be the real line $\mathbb{R}$). This implies that neither $\chi(F_1(X)) = uu(X)$ nor $\chi(A_1(X)) = uu(X)$ holds since $F_1(X) = X \oplus X^{-1} \oplus \{e\}$ and $A_1(X) = X \oplus -X \oplus \{0\}$.

4. First Countability and Metrizability of $A_n(X)$ and $F_n(X)$

In this section we study metrizable spaces X for which $F_n(X)$ and $A_n(X)$ are first countable or metrizable. We begin with definitions and well-known results.

Let X be a space. For a subset E of $F(X)$, the set of all elements of X taking part in the reduced form of words in E is called *carrier* of E and denoted by $\operatorname{car} E$ (see [4]). Recall that a subset Y of a space X is called *bounded* if every real-valued continuous function on X is bounded on Y. Arhangel'skiĭ, Pestov and Okunev (Theorem 1.5 of [4]) proved that if E is a bounded set in $F(X)$, in particular if E is compact, then car E is bounded in X. The same is true for $A(X)$. In the same paper they proved the following important results, which are used below.

Theorem 4.1 ([4]). *For a metrizable space X the following are equivalent:*

(1) *$A(X)$ is a k-space;*

(2) *$A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space;*

(3) *X is locally compact and the set of all non-isolated points in X is separable.*

Theorem 4.2 ([4]). *For a metrizable space X, the following are equivalent:*

(1) *$F(X)$ is a k-space;*

(2) *$F(X)$ is k_ω-space or discrete;*

(3) *X is locally compact separable or discrete.*

Let Y be a closed subset of a metrizable space X. Then it is well known that $F(Y)$ $(A(Y))$ is a closed subgroup of $F(Y)$ (of $A(X)$) (see [21]) and in addition for each $n \in \mathbb{N}$, $F_n(Y)$ $(A_n(Y))$ is also a closed subset of $F_n(X)$ $(A_n(X))$, respectively (see [7]). We first discuss the abelian case.

Proposition 4.3. *Let X be a metrizable space. If $A_2(X)$ is first countable, then the set of all non-isolated points of X is compact. The same is true for $F_2(X)$.*

Proof. Suppose that the set of all non-isolated points of X is not compact. Then there are sequences $\{x_n : n \in \mathbb{N}\}$ and

$\{x_{m,n} : m \in \mathbb{N}\}$, $n \in \mathbb{N}$ such that $\{x_{m,n} : m \in \mathbb{N}\}$ converges to x_n for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $Y_n = \{x_{m,n} : m \in \mathbb{N}\} \cup \{x_n\}$. We can take the above sequences such that the family $\{Y_n : n \in \mathbb{N}\}$ will be closed and discrete in X. Let $C_n = \{x_{m,n} - x_n : m \in \mathbb{N}\} \cup \{0\}$ for each $n \in \mathbb{N}$ and $C = \bigcup_{n=1}^{\infty} C_n$. Then C is a closed subset of $A_2(Y)$ such that each sequence $\{x_{m,n} - x_n : m \in \mathbb{N}\}$ converges to 0 and $C_i \cap C_j = \{0\}$ if $i \neq j$, where $Y = \bigcup_{n=1}^{\infty} Y_n$. Since $A_2(Y)$ is a (closed) subspace of $A_2(X)$, to show that $A_2(X)$ is not first countable, it suffices to prove that $A_2(Y)$ is not first countable. Indeed we shall prove that the subspace C of $A_2(Y)$ is homeomorphic to the sequential fan S_ω. Let E be a subset of C such that $E \cap C_n$ is closed in C_n for each $n \in \mathbb{N}$ and K be a compact set in $A_2(Y)$. Since $\operatorname{car} K$ is bounded in Y there is $m \in \mathbb{N}$ such that $\operatorname{car} K \subseteq \bigcup_{i=1}^{m} Y_m$. Hence $K \subseteq \langle \bigcup_{i=1}^{m} Y_m \rangle$, where $\langle \bigcup_{i=1}^{m} Y_m \rangle$ is the subgroup of $A(Y)$ generated by $\bigcup_{i=1}^{m} Y_m$. It follows that $E \cap K = (E \cap C) \cap (K \cap \langle \bigcup_{i=1}^{m} Y_m \rangle) = E \cap (\bigcup_{i=1}^{m} C_i) \cap K = (\bigcup_{i=1}^{m} (E \cap C_i)) \cap K$. Since each C_i is compact, this means that $E \cap K$ is closed in K. Now, Y is locally compact separable. Hence, by Theorem 4.1, $A(Y)$ is a k-space and so is the closed subset $A_2(Y)$ of $A(Y)$. Since $E \cap K$ is closed for each compact set K in $A_2(Y)$, E is closed in $A_2(Y)$ and hence in C. This implies that C is homeomorphic to S_ω. Consequently, $A_2(X)$ is not first countable.

In the non-abelian case, put $C_n = \{x_{m,n} x_n^{-1} : n \in \mathbb{N}\} \cup \{e\}$. Then, applying Theorem 4.2, the above argument also implies that $F_2(X)$ is not first countable. □

Let X be a metrizable space such that the set C of all non-isolated points of X is compact. For each $k \in \mathbb{N}$, let $\mathcal{G}_k$ be the family of all open balls of radius $1/k$ (with respect to certain metric on X) centered in points of C and put $U_k = \bigcup \{G \times G : G \in \mathcal{G}_k\} \cup \Delta_X$, where Δ_X is the diagonal of $X \times X$. Then $\{U_k : k \in \mathbb{N}\}$ is a countable base for the universal uniformity $\mathcal{U}_X$ of X. Hence, by Theorem 3.3, $A_n(X)$ is first countable for each $n \in \mathbb{N}$. As is shown below (see Proposition 4.8) the natural mapping i_2 is closed if every neighborhood of Δ_X is a member of $\mathcal{U}_X$, in particular if X is paracompact. Hence the Hanai-Morita-Stone theorem implies that $A_2(X)$ is metrizable because the natural mapping i_2 from the metrizable space $\overline{X}^2$ onto the first countable space $A_2(X)$ is closed.

On the other hand, the mappings i_n, $(n \geq 3)$ are not closed unless a metrizable space X is compact or discrete (see Theorem 4.9). This means that we need another method to know whether the first countable spaces $A_n(X)$, $n \geq 3$ are metrizable or not. Fortunately, Gruenhage proved the following result which is presented here with his kind permission.

Theorem 4.4. *Let X be a metrizable space such that the set C of all non-isolated points in X is compact. Then $A_n(X)$ has a σ-disjoint base for each $n \in \mathbb{N}$.*

Proof. For each $k \in \mathbb{N}$, let $\mathcal{G}_k$ be the family of all open balls of radius $1/k$ (with respect to a certain metric on X) centered in points of C and put $U_k = \bigcup\{G \times G : G \in \mathcal{G}_k\} \cup \Delta_X$, where Δ_X is the diagonal of $X \times X$. Then, by Theorem 2.1, $\mathcal{V}_m = \{V_m(U_k) : k \in \mathbb{N}\}$ is a neighborhood base at 0 in $A_{2m}(X)$ for each $m \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. For each $g \in A_n(X)$, put $g = g_{X\setminus C} + g_C$, where $g_{X\setminus C} \in A_n(X \setminus C)$ and $g_C \in A_n(C)$. We first prove the following claim.

Claim. Let $g = g_{X\setminus C} + g_C \in A_n(X)$ and $x_i, x'_i \in C$, $i = 1, 2, \ldots, l$ $(l \leq n)$, where $g_C = \sum_{i=1}^{l} \varepsilon_i x_i$ and $x_i, x'_i \in G_i$ for some $G_i \in \mathcal{G}_k$ for $i = 1, 2, \ldots, l$. Put $g'_C = \sum_{i=1}^{l} \varepsilon_i x'_i$ and $g' = g_{X\setminus C} + g'_C$. Then $g + V_m(U_k) \subseteq g' + V_{m+n}(U_k)$ for each $m \in \mathbb{N}$.

Let $v \in V_m(U_k)$. Then $g + v = g_{X\setminus C} + g_C + v = g_{X\setminus C} + g'_C + g_C - g'_C + v = g' + \sum_{i=1}^{l} \varepsilon_i(x_i - x'_i) + v$. Since $x_i, x'_i \in G_i$, we have

$(x_i, x_i') \in U_k$ and $(x_i', x_i) \in U_k$. Hence $\sum_{i=1}^{l} \varepsilon_i(x_i - x_i') + v \in V_{m+n}(U_k)$. This implies that $g + v \in g' + V_{m+n}(U_k)$.

For every $g \in A_n(X)$ let $k(g) = \min\{m \in \mathbb{N} : x \notin \bigcup \mathcal{G}_m$ for each $x \in \operatorname{car} g_{X\setminus C}\}$. Let D be a countable dense subset of C. For $k, m \in \mathbb{N}$ with $k \geq m$ and $h \in A_n(D)$, let

$$\mathcal{B}_{k,m,h} = \{(g + V_{2n}(U_k)) \cap A_n(X) \,:\, g \in A_n(X), \\ g_C = h \text{ and } k(g) = m\}.$$

Let g, g' be distinct words in $A_n(X)$ with $g_C = g'_C = h$ and $k(g) = k(g') = m \leq k$. Suppose that $(g + V_{2n}(U_k)) \cap (g' + V_{2n}(U_k)) \neq \emptyset$. Then there are $v, v' \in V_{2n}(U_k)$ such that $g + v = g' + v'$. Since $g \neq g'$, we have $v - v' \neq 0$, that is $v - v'$ has the reduced form $v - v' = \sum_{i=1}^{s}(a_i - b_i)$, where $a_i, b_i \in \bigcup \mathcal{G}_k$. Then $g = g' + v' - v = g'_C + g'_{X\setminus C} + \sum_{i=1}^{s}(a_i - b_i) = h + g'_{X\setminus C} + \sum_{i=1}^{s}(a_i - b_i)$. Since $g_C = h$ and $\sum_{i=1}^{s}(a_i - b_i)$ is the reduced form of $v - v'$, the elements a_i, b_i belong to $X \setminus C$. Furthermore, since $k(g') = m \leq k$, that is $x \notin \bigcup \mathcal{G}_k$ for each $x \in \operatorname{car} g_{X\setminus C}$, it is clear that $g'_{X\setminus C} + \sum_{i=1}^{s}(a_i - b_i)$ is the reduced form. This contradicts the fact that $k(g) = m \leq k$. Hence we conclude that $\mathcal{B}_{k,m,h}$ is a pairwise disjoint family of subsets of $A_n(X)$.

To prove that $\mathcal{B} = \bigcup\{\mathcal{B}_{k,m,h} : m \leq k, h \in A_n(D)\}$ is a base for $A_n(X)$, consider an element $g \in A_n(X)$ and a neighborhood W of 0 in $A(X)$. Choose $k \in \mathbb{N}$ such that $k \geq k(g)$ and $V_{3n}(U_k) \subseteq W$. Let $g_C = \sum_{i=1}^{l} \varepsilon_i x_i$. Since D is dense in C there are $d_i \in D, i = 1, 2, \ldots, l$ such that $x_i, d_i \in G_i$ for some $G_i \in \mathcal{G}_k$ for $i = 1, 2, \ldots, l$. Put $h = \sum_{i=1}^{l} \varepsilon_i d_i$ and $g' = g_{X\setminus C} + h$. Then $(g' + V_{2n}(U_k)) \cap A_n(X) \in \mathcal{B}_{k,k(g),h}$. From the above claim and the choice of U_k it follows that

$$g + V_n(U_k) \subseteq g' + V_{2n}(U_k) \subseteq g + V_{3n}(U_k) \subseteq g + W.$$

Hence

$$g \in (g + V_n(U_k)) \cap A_n(X) \subseteq (g' + V_{2n}(U_k)) \cap A_n(X) \\ \subseteq (g + V_{3n}(U_k)) \cap A_n(X) \subseteq (g + W) \cap A_n(X).$$

Since $V_n(U_k)$ is a neighborhood of 0 in $A_{2n}(X)$ and $g \in A_n(X)$ we have, by Lemma 3.1, that $(g + V_n(U_k)) \cap A_n(X)$ is a neighborhood of g in $A_n(X)$ and hence so is $(g' + V_{2n}(U_k)) \cap A_n(X)$. Consequently, $\mathcal{B}$ is a σ-disjoint base for $A_n(X)$. □

It was proved by Arhangel'skiĭ [2] that if a space X is a paracompact σ-space, then so are $F(X)$ and $A(X)$, and hence they are perfectly normal. In addition, it is known that every perfectly normal space with a σ-disjoint base is metrizable. These facts and Theorem 4.4 imply that each $A_n(X)$ is metrizable if X is metrizable and the set of all non-isolated points of X is compact.

Consequently, applying Proposition 4.3 and Theorem 4.4, we obtain the following criterion of metrizability of the spaces $A_n(X)$.

Theorem 4.5. *For a metrizable space X, the following are equivalent:*

(1) $A_n(X)$ *is metrizable for each* $n \in \mathbb{N}$*;*

(2) $A_n(X)$ *is first countable for each* $n \in \mathbb{N}$*;*

(3) $A_2(X)$ *is metrizable;*

(4) $A_2(X)$ *is first countable;*

(5) *the set of all non-isolated points of X is compact.*

The situation in the non-abelian case is rather different. We first present the following fact, the proof of which is similar to that of Proposition 4.3.

Proposition 4.6. *Let X be a metrizable space. If $F_4(X)$ is first countable, then X is compact or discrete.*

Proof. Suppose that X is neither compact nor discrete. Then X contains a closed subset $T = S \oplus D$, where S is a non-trivial convergent sequence $\{x_n : n \in \mathbb{N}\}$ with its limit x and $D = \{d_n : n \in \mathbb{N}\}$ is an infinite closed discrete subset of X. Since $F_4(T)$ is a subspace of $F_4(X)$, it suffices to show that $F_4(T)$ is not first countable.

Let $C_k = \{d_k x_n x^{-1} d_k^{-1} : n \in \mathbb{N}\} \cup \{e\}$ for each $k \in \mathbb{N}$. Then each sequence $\{d_k x_n x^{-1} d_k^{-1} : n \in \mathbb{N}\}$ converses to the unit element e and $C_k \cap C_{k'} = \{e\}$ if $k \neq k'$. Let $C = \bigcup_{k=1}^{\infty} C_k$. Since $F_4(T)$ is a k-space by Theorem 4.2, we can prove that C is homeomorphic to the sequential fan S_ω in a fashion similar to that in the proof of Proposition 4.3. Therefore, $F_4(T)$ is not first countable. □

To deduce criteria for metrizability of spaces $F_n(X)$, we consider first the closedness of natural mappings i_n raised by Tkačenko [18]. He proved in [17] that i_n is closed if X^{2n} is normal and countably compact. The following fact gives a condition on a space X when i_n $(n \geq 3)$ are closed.

Lemma 4.7. *Let X be a space. If there are an element $x \in X$, a subset $Y = \{x_\alpha : \alpha \in A\}$ of X such that $x \in \overline{Y} \setminus Y$ and a closed discrete subset $\{d_\alpha : \alpha \in A\}$ in X with $d_\alpha \neq d_{\alpha'}$, if $\alpha \neq \alpha'$, then i_n is not closed for each $n \geq 3$. In particular, if X is metrizable and i_n is closed for some $n \geq 3$, then X is compact or discrete. This is true for both $F(X)$ and $A(X)$.*

Proof. Since the proof for $F(X)$ and for $A(X)$ are similar, we consider only $F(X)$. For each $\alpha \in A$, let $\boldsymbol{x}_\alpha = (x_\alpha, d_\alpha, d_\alpha^{-1})$. Then $E = \{\boldsymbol{x}_\alpha : \alpha \in A\}$ is a closed subset of $\overline{X}^3$. Since $i_3(\boldsymbol{x}_\alpha) = x_\alpha$ for each $\alpha \in A$, $i_3(E)$ is not closed in $F_3(X)$. Hence i_3 is not closed. Since i_3 can be regarded as a restriction of i_n to a closed subset of $\overline{X}^n$ for each $n \geq 3$, we conclude that i_n is not closed for each $n \geq 3$. □

On the other hand, the situation for $n = 2$ is very different. Pestov [14] proved that i_2 is quotient if and only if every neighborhood of the diagonal in X^2 is an element of $\mathcal{U}_X$. Thus, if a space X is paracompact, then i_2 is quotient. We improve this result as follows.

Proposition 4.8. *Let X be a space. The mapping i_2 is closed if and only if every neighborhood of the diagonal in X^2 is an element of $\mathcal{U}_X$. In particular, if X is paracompact, then i_2 is a closed mapping. This holds for both $F_2(X)$ and $A_2(X)$.*

Proof. By Pestov's result cited above it suffices to prove the 'if' part of the proposition. We shall consider the mapping $i_2 : \overline{X}^2 \to F_2(X)$. Clearly, both $F_2(X) \setminus F_1(X)$ and $F_1(X) \setminus \{e\}$ are open in $F_2(X)$. In addition, the restrictions

$$i_2|_{i_2^{-1}(F_2(X)\setminus F_1(X))} : i_2^{-1}(F_2(X) \setminus F_1(X)) \to F_2(X) \setminus F_1(X)$$

and

$$i_2|_{i_2^{-1}(F_1(X)\setminus \{e\})} : i_2^{-1}(F_1(X) \setminus \{e\}) \to F_1(X) \setminus \{e\}$$

are homeomorphisms. (We remark that in the abelian case the restriction

$$i_2|_{i_2^{-1}(A_2(X)\setminus A_1(X))} : i_2^{-1}(A_2(X) \setminus A_1(X)) \to A_2(X) \setminus A_1(X)$$

is a 2 to 1, open and closed mapping.) Let E be a closed set in $\overline{X}^2$. To show that $i_2(E)$ is closed in $F_2(X)$ take $g \in \overline{i_2(E)}$. Assume that $g \in F_2(X) \setminus F_1(X)$. Since $F_2(X) \setminus F_1(X)$ is open in $F_2(X)$ choose an open set V in $F_2(X)$ such that $g \in V \subseteq \overline{V} \subseteq F_2(X) \setminus F_1(X)$. Then $g \in \overline{i_2(E) \cap V} \subseteq F_2(X) \setminus F_1(X)$. Since $i_2(E) \cap V = i_2|_{i_2^{-1}(F_2(X)\setminus F_1(X))}(E \cap {i_2|_{i_2^{-1}(F_2(X)\setminus F_1(X))}}^{-1}(V))$ we can conclude that $g \in i_2(E)$. With the same argument, if $g \in F_1(X) \setminus \{e\}$, then we can show that $g \in i_2(E)$. So, we may assume that $g = e$.

Suppose that $E \cap D = \emptyset$, where $D = \{(x, x^{-1}) : x \in \overline{X}\}$. Since X and X^{-1} are homeomorphic, there is a neighborhood U of the diagonal in $\overline{X}^2$ such that $U \subseteq X^2 \cup (X^{-1})^2 \cup \{(e, e)\}$, $U = U^{-1}$

and $\{(x, y^{-1}) \in \overline{X}^2 : (x, y) \in U\} \cap E = \emptyset$. From our hypothesis it follows that $U \in \mathcal{U}_X$, and hence $W_1(U)$ is a neighborhood of e in $F_2(X)$ by Theorem 2.3. Let $h \in W_1(U)$, then by the definition of $W_1(U)$ either $h = e$ or h has the reduced form $h = xy$ such that $x, y \in \overline{X}$ and $(x, y^{-1}) \in U$. Since $E \cap D = \emptyset$, $e \notin i_2(E)$. On the other hand, the definition of U implies that $(x, y) \notin E$, and hence $h = i_2((x, y)) \notin i_2(E)$ because $h \in F_2(X) \setminus F_1(X)$ and $i_2|_{i_2^{-1}(F_2(X) \setminus F_1(X))}$ is $1-1$. (In the abelian case, since $U = U^{-1}$ it can be also seen that $h \notin i_2(E)$.) It follows that $W_1(X) \cap i_2(E) = \emptyset$, which contradicts our assumption $g = e \in \overline{(i_2(E))}$.

Hence we have $E \cap D \neq \emptyset$. Let $x \in \overline{X}$ be such that $(x, x^{-1}) \in E$. Then $g = e = i_2((x, x^{-1})) \in i_2(E)$. That is, $i_2(E)$ is closed in $F_2(X)$. Consequently i_2 is a closed mapping. □

From the above results we obtain the following characterization of metrizable spaces X such that each $F_n(X)$ is metrizable.

Theorem 4.9. *Let X be a metrizable space. Then the following are equivalent:*

(1) *$F_n(X)$ is metrizable for each $n \in \mathbb{N}$;*

(2) *$F_n(X)$ is first countable for each $n \in \mathbb{N}$;*

(3) *i_n is a closed mapping for each $n \in \mathbb{N}$;*

(4) *$F_4(X)$ is metrizable;*

(5) *$F_4(X)$ is first countable;*

(6) *i_4 is a closed mapping;*

(7) *X is compact or discrete.*

Proof. Proposition 4.6 shows that (5) $\Rightarrow$ (7) and Lemma 4.7 yields (6) $\Rightarrow$ (7). Other implications are clear. □

Finally we discuss the metrizability of $F_3(X)$ and $F_2(X)$. Let X be a metrizable space such that the set of all non-isolated points of X is compact. Since there is a countable base for the universal uniformity $\mathcal{U}_{\overline{X}}$, Remark 3.5 yields that $F_2(X)$ is first countable. Furthermore, since the mapping i_2 is closed by Proposition 4.8, $F_2(X)$ becomes to be metrizable. Note that Proposition 4.3 holds for $F_2(X)$. Then, from these facts, we obtain the following:

For a metrizable space X the following are equivalent:

(1) $F_2(X)$ *is metrizable;*

(2) $F_2(X)$ *is first countable;*

(3) *the set of all non-isolated points of X is compact.*

We shall show that metrizability and first countability of the space $F_3(X)$ are equivalent to the above statements.

Let us recall some properties of the family $\mathcal{W}_n$ in §2. By Theorem 2.3 and Corollary 2.10, we have:

> $\mathcal{W}_2 = \{W_2(U) : U \in \mathcal{U}_{\overline{X}}\}$ is a family of neighborhoods of e in $F_4(X)$, and
>
> $\mathcal{W}_1 = \{W_1(U) : U \in \mathcal{U}_{\overline{X}}\}$ is a neighborhood base at e in $F_2(X)$.

Applying these facts we obtain the following.

Proposition 4.10. *Let X be a metrizable space such that the set C of all non-isolated points of X is compact. Then every point of $X \cup X^{-1}$ has a countable neighborhood base in $F_3(X)$.*

Proof. Let $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be the same sequence in the proof of Theorem 4.4. For every $n \in \mathbb{N}$, let

$$U_n = (\bigcup\{G \times G : G \in \mathcal{G}_n\} \cup \Delta_X) \cup (\bigcup\{G^{-1} \times G^{-1} : G \in \mathcal{G}_n\} \cup \Delta_{X^{-1}}) \cup \{(e, e)\}.$$

Then $\{U_n : n \in \mathbb{N}\}$ is a base for the universal uniformity $\mathcal{U}_{\overline{X}}$ on $\overline{X}$. Fix a point $x \in X \cup X^{-1}$ and put

$$\mathcal{B}_x = \{xW_2(U_n) \cap F_3(X) : n \in \mathbb{N}\}.$$

Then Lemma 3.1 with $n = 1$ and $m = 3$ shows that $\mathcal{B}_x$ is a countable family of neighborhoods of x in $F_3(X)$. To prove this $\mathcal{B}_x$ is as required, let V_1 be an arbitrary neighborhood of e in $F(X)$. Since $F(X)$ is a topological group there is a neighborhood V_2 of e in $F(X)$ such that $(x^{-1}V_2\, x \cdot V_2) \cup (V_2 \cdot V_2) \subseteq V_1$. Since $\mathcal{W}_1$ is a neighborhood base of e in $F_2(X)$, there is $n \in \mathbb{N}$ such that $W_1(U_n) \subseteq V_2$. Then we have

$$(x^{-1}W_1(U_n)\, x \cdot W_1(U_n)) \cup (W_1(U_n) \cdot W_1(U_n)) \subseteq V_1. \qquad (1)$$

Let $g \in xW_2(U_n) \cap F_3(X)$. Then there is $h \in W_2(U_n)$ such that $g = xh$. Of course we may assume that $h \neq e$. By the definition of $W_2(U_n)$, the length of h is 2 or 4. If the length of $h = 2$, then $h \in W_1(U_n)$. Hence $g = xh \in xW_1(U_n) \subseteq xV_1 \cap F_3(X)$. So we assume that the length of h is equal to 4. Since $g = xh \in F_3(X)$, h must have the reduced form $h = x^{-1}x_1x_2x_3$, where $x_1, x_2, x_3 \in \overline{X}$. Since $h \in W_2(U_n)$ there are the following two cases.

Case 1. $(x^{-1}, x_1^{-1}) \in U_n$ and $(x_2, x_3^{-1}) \in U_n$.

In this case, $x^{-1}x_1 \in W_1(U_n)$ and $x_2x_3 \in W_1(U_n)$. Thus, by the property (1), $h = x^{-1}x_1x_2x_3 \in W_1(U_n) \cdot W_1(U_n) \in V_1$. This implies that $g = xh \in xV_1 \cap F_3(X)$.

Case 2. $(x^{-1}, x_3^{-1}) \in U_n$ and $(x_1, x_2^{-1}) \in U_n$.

It follows that $x^{-1}x_3 \in W_1(U_n)$ and $x_1x_2 \in W_1(U_n)$. Since h can be represented as $h = x^{-1}x_1x_2x_3 = x^{-1}x_1x_2xx^{-1}x_3$, by the property (1), $h \in x^{-1}W_1(U_n)x \cdot W_1(U_n) \subseteq V_1$. Hence $g = xh \in xV_1 \cap F_3(X)$.

Thus, $xW_2(U_2) \cap F_3(X) \subseteq xV_1 \cap F_3(X)$, and hence $\mathcal{B}_x$ is a neighborhood base at x in $F_3(X)$. □

The following facts about $F_3(X)$ are well known ([1] and [11]):

Fact. Let X be a space. Then:

1. $F_3(X) = [(F_3(X) \setminus F_2(X)) \cup (F_1(X) \setminus \{e\})] \oplus [(F_2(X) \setminus F_1(X)) \cup \{e\}]$.
2. $F_3(X) \setminus F_2(X)$ is open in $F_3(X)$ and homeomorphic to a subspace of $\overline{X}^3$.
3. $F_2(X) \setminus F_1(X)$ is open in $F_3(X)$ and homeomorphic to a subspace of $\overline{X}^2$.

Again, let X be a metrizable space such that the set of all non-isolated points of X is compact. By Proposition 4.10, each point of $X \cup X^{-1}$ has a countable base in $F_3(X)$. From the above Fact it follows that every point of $(F_3(X) \setminus F_2(X)) \cup (F_2(X) \setminus F_1(X))$ also has a countable neighborhood base in $F_3(X)$. Furthermore, since the universal uniformity $\mathcal{U}_{\overline{X}}$ has a countable base $\mathcal{U}$, Corollary 2.10 implies that e has a countable neighborhood base $\mathcal{B} = \{W_1(U) : U \in \mathcal{U}\}$ in $F_2(X)$. It is clear that every element $W_1(U)$ of $\mathcal{B}$ is contained in the subset $(F_2(X) \setminus F_1(X)) \cup \{e\}$ of $F_2(X)$. Then, by item 1 of the above Fact, $\mathcal{B}$ is a neighborhood base at e in $F_3(X)$. Consequently, $F_3(X)$ is first countable. We will improve this result in Theorem 4.12 by showing that $F_3(X)$ is metrizable.

For a space X and a continuous pseudometric (or metric) d on X, The notation $B_d(x, \varepsilon)$ means the open ball with center at $x \in X$ and radius $\varepsilon > 0$ with respect to d.

Theorem 4.11. *Let X be a metrizable space, and suppose that the set C of all non-isolated points of X is compact. Then $F_3(X)$ is metrizable.*

Proof. Fix a compatible metric d on X and extend d to a continuous invariant metric $\hat{d}$ on $F(X)$ (see Theorem 1 of [7]). The metric $\hat{d}$ is called the Graev's metric on $F(X)$. Then we shall show that $\hat{d}|_{F_3(X)}$ is a metric on $F_3(X)$ that induces the original topology of $F_3(X)$.

It is well known that $F_{n+1}(X) \setminus F_n(X)$ is homeomorphic to a subspace of $\overline{X}^{n+1}$ for each $n \in \mathbb{N}$ (see [1] and [11]). To show the fact, Joiner [11] proved the following:

Let X be a space. Given $n \in \mathbb{N}$, $g = x_1 \cdots x_{n+1} \in F_{n+1}(X) \setminus F_n(X)$ and a neighborhood U_g of g in $F_{n+1}(X) \setminus F_n(X)$, let ρ be a continuous pseudometric on X such that $i_{n+1}(B_\rho(x_1, \delta) \times \cdots \times B_\rho(x_{n+1}, \delta)) \subseteq U_g$ for some $\delta > 0$. Then there exists $\varepsilon > 0$ such that $B_{\hat{\rho}}(g, \varepsilon) \cap (F_{n+1}(X) \setminus F_n(X)) \subseteq i_{n+1}(B_\rho(x_1, \delta) \times \cdots \times B_\rho(x_{n+1}, \delta))$.

Since both $F_3(X) \setminus F_2(X)$ and $F_2(X) \setminus F_1(X)$ are open subsets of $F_3(X)$, the above fact implies that for every $g \in (F_3(X) \setminus F_2(X)) \cup (F_2(X) \setminus F_1(X))$, $\{B_{\hat{d}}(g, 1/n) \cap (F_3(X) \setminus F_2(X)) : n \in \mathbb{N}\}$ and $\{B_{\hat{d}}(g, 1/n) \cap (F_2(X) \setminus F_1(X)) : n \in \mathbb{N}\}$ are neighborhood base at g in $F_3(X)$, respectively. In addition, since $F_2(X)$ is open in $F_3(X)$, Corollary 2.10 implies that $\mathcal{W}_1$ is a neighborhood base at e in $F_3(X)$. If we apply the proof of Theorem 2.3, then it is easy to see that $\{B_{\hat{d}}(e, 1/n) \cap F_3(X) : n \in \mathbb{N}\}$ is a neighborhood base at e in $F_3(X)$.

Thus, we need to show that for every $x \in X \cup X^{-1}$ $\{B_{\hat{d}}(x, 1/n) \cap F_3(X) : n \in \mathbb{N}\}$ is a neighborhood base at x in $F_3(X)$. For each $m \in \mathbb{N}$, let

$$U_m = \bigcup \{B_{\overline{d}}(x, 1/m) \times B_{\overline{d}}(x, 1/m) \; : \; x \in C \cup C^{-1}\} \cup \Delta_X \cup \Delta_{X^{-1}} \cup \{(e, e)\}.$$

Then $\{U_m : m \in \mathbb{N}\}$ is a base for $\mathcal{U}_{\overline{X}}$. The argument of the proof of Theorem 2.3 implies that for every $m, n \in \mathbb{N}$,

$$B_{\hat{d}}(e, 1/m) \cap F_{2n}(X) \subseteq W_n(U_m). \qquad (*)$$

Let $x \in X \cup X^{-1}$ and U_x be a neighborhood of x in $F_3(X)$. Since we have already proved that $\{xW_2(U_m) \cap F_3(X) : m \in \mathbb{N}\}$ is a neighborhood base at x in $F_3(X)$ in the proof of Proposition 4.10, we can choose $m \in \mathbb{N}$ such that $xW_2(U_m) \cap F_3(X) \subseteq U_x$.

Take $g \in F_3(X)$ with $\hat{d}(x,g) < 1/m$. Since $\hat{d}(x,g) = \hat{d}(e, x^{-1}g)$, we have $x^{-1}g \in B_{\hat{d}}(e, 1/m) \cap F_4(X)$. Apply $(*)$ with $n = 2$, to conclude that $x^{-1}g \in W_2(U_m)$. It follows that $g \in xW_2(U_m) \cap F_3(X) \subseteq U_x$. Therefore, we conclude that $\hat{d}|_{F_3(X)}$ induces the original topology of $F_3(X)$, and hence $F_3(X)$ is metrizable. □

Using Theorem 4.12 we will now deduce a criterion for $F_3(X)$ and $F_2(X)$ to be metrizable.

Theorem 4.12. *For a metrizable space X, the following are equivalent:*

(1) *$F_3(X)$ is metrizable;*

(2) *$F_3(X)$ is first countable;*

(3) *$F_2(X)$ is metrizable;*

(4) *$F_2(X)$ is first countable;*

(5) *the set of all non-isolated points of X is compact.*

It is well known that every first countable topological group is metrizable. Though $F_n(X)$ and $A_n(X)$ are not topological groups, we obtain from Theorems 4.5, 4.9 and 4.13 the following result.

Corollary 4.13. *Let X be a metrizable space and $n \geq 2$.*

(1) *If $A_n(X)$ is first countable, then it is metrizable, and*

(2) *if $F_n(X)$ is first countable, then it is metrizable.*

Finally, we answer the question stated in Introduction as follows.

Corollary 4.14. *Let $\mathbb{R}$ be the real line, $\mathbb{Q}$ be the space of rational numbers and $J(\kappa)$ be the hedgehog space of spine κ such that each spine is a sequence which converges to the center point. Then neither $F_n(\mathbb{R})$, $A_n(\mathbb{R})$, $F_n(\mathbb{Q})$, $A_n(\mathbb{Q})$, $F_n(\mathbb{R}\setminus\mathbb{Q})$ nor $A_n(\mathbb{R}\setminus\mathbb{Q})$ are first countable for each $n \geq 2$, and if $n \geq 4$ and $\kappa \geq \omega$, then $F_n(J(\kappa))$ is not first countable. On the other hand, $F_3(J(\kappa))$, $F_2(J(\kappa))$ and $A_n(J(\kappa))$ $(n \in \mathbb{N})$ are metrizable.*

We conclude this paper with a problem.

The family $\mathcal{W}_n = \{W_n(U) : U \in \mathcal{U}_{\overline{X}}\}$ of neighborhood of e in $F_{2n}(X)$ constructed in §2 is not a neighborhood base at e in $F_n(X)$ in general. For example, let X be a non-compact and non-discrete metrizable space such that the set of all non-isolated points is compact. Then $\mathcal{U}_{\overline{X}}$ has a countable base $\mathcal{B}$. Let $\mathcal{W}_{\mathcal{B}} = \{W_2(B) : B \in \mathcal{B}\}$. Then $\mathcal{W}_{\mathcal{B}}$ is a countable subfamily of $\mathcal{W}_2$ such that every element of $\mathcal{W}_2$ includes some element of $\mathcal{W}_{\mathcal{B}}$. On the other hand, from the proof of Proposition 4.6, it follows that e cannot have a countable neighborhood base in $F_4(X)$. Therefore, $\mathcal{W}_2$ is not a neighborhood base at e in $F_4(X)$. It is difficult to know the reduced form of each word in each element of $\mathcal{W}|_{F_{2n}(X)}$, however $\mathcal{W}|_{F_{2n}(X)}$ is a neighborhood base of e in $F_{2n}(X)$. So we raise the following problem. It is definitely a hard problem even for $F_4(X)$. However, some steps towards its solution (for $F_4(X)$ only) were made before (see Theorems 1.4, 1.5 and 2.1 of [19]).

Problem. *Construct a neighborhood base of e in $F_{2n}(X)$ for an arbitrary space X such that the reduced form of each word of each element of the neighborhood base be represented clearly.*

Acknowledgment. The author wishes to thank Professor Gary Gruenhage and Professor Haruto Ohta for carefully listening to and checking our proofs at a seminar. Professor Gruenhage gave me the permission to include an important result (Theorem 4.4) which completes the criterion for each $A_n(X)$ to be metrizable.

Also, the author would like to express his sincere thanks to the referee for the careful reading preliminary version of this paper and many valuable suggestions. In particular, thanks to his help, the author could shorten and clarify the proofs of Theorem 2.3, 2.6 and 4.11.

References

[1] A. V. Agrangel'skiĭ, *Mapping related to topological groups*, Soviet Math. Dokl. **9** (1968) 1011-1015.

[2] A. V. Arhangel'skiĭ, *Classes of topological groups*, Russian Math. Surveys **36** (1981) 151-174.

[3] A. V. Arhangel'skiĭ, *Algebraic objects generated by topological structure*, J. Soviet Math. **45** (1989) 956-978.

[4] A. V. Arhangel'skiĭ, O. G. Okunev and V. G. Pestov, *Free topological groups over metrizable spaces*, Topology Appl. **33** (1989) 63-76.

[5] R. Engelking, *General Topology* (Heldermann, Berlin, 1989).

[6] T. H. Fay, E. T. Ordman and B. V. S. Thomas, The free topological groups over rationals, Gen. Topology Appl. **10** (1979) 33-47.

[7] M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948) 279-324 (in Russian); Amer. Math. Soc. Transl. **8** (1962) 305-364.

[8] A. W. Hager, *Uniformities on a product*, Can. J. Math. **24** (1972) 379-389.

[9] E. Hewitt and K. Ross, *Abstract harmonic analysis I*, Academic Press, New York, (1963).

[10] J. R. Isbell, *Uniform spaces*, Math. Surveys No. 12 (Amer. Math. Soc., Providence, 1964).

[11] C. Joiner, *Free topological groups and dimension*, Trans. Amer. Math. Soc. **220** (1976) 401-418.

[12] A. A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945) 3-64 (in Russian); Amer. Math. Soc. Transl. **8** (1962) 195-272.

[13] C. G. Pestov, *Some properties of free topological groups*, Moscow Univ. Math. Bull. **37** (1982) 46-49.

[14] V. G. Pestov, *Neighborhoods of the identity in free topological groups*, Herald Moscow State Univ. Ser. Math. Mech. **3** (1985) 8-10 (in Russian).

[15] M. G. Tkačenko, *On completeness of free Abelian topological groups*, Soviet Math. Dokl. **27** (1983), 341-345.

[16] M. G. Tkačenko, *On topologies of free groups*, Czech. Math. J. **34** (1984) 541-551.

[17] M. G. Tkačenko, *Strong collectionwise normality and countable compactness in free topological groups*, Sib. Mat. J. **28** (1987) 167-177.

[18] M. G. Tkačenko, *Free topological groups and inductive limits*, Topology Appl. **60** (1994) 1-12.

[19] M. G. Tkačenko, *On group uniformities on the square of a space and extending pseudometrics*, Bull. Austral. Math. Soc. **51** (1995), 309-335.

[20] M. G. Tkačenko, *On group uniformities on the square of a space and extending pseudometrics II*, Bull. Austral. Math. Soc. **52** (1995), 41-61.

[21] V. V. Uspenskiĭ, *Free topological groups of metrizable spaces*, Math. USSR Izvestiya **37** (1991) 657-680.

[22] K. Yamada, *Characterizations of a metrizable space X such that every $A_n(X)$ is a k-space*, Topology Appl. **49** (1994) 75-94.

[23] K. Yamada, *Tightness of free abelian topological groups and of finite products of sequential fans*, To appear in Topology Proceedings.

Department of Mathematics, Faculty of Education, Shizuoka University, Shizuoka, 422 Japan

E-mail address: eckyama@ipc.shizuoka.ac.jp

Current address: Department of Mathematics, 218 Parker Hall, Auburn University, AL 36849-5310 U.S.A.

E-mail address: yamadko@mail.auburn.edu