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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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ON THE PLANE FIXED POINT PROBLEM

VLADIMIR N. AKIS

ABSTRACT. We prove every smooth map of a planar domain with non-negative Jacobian and isolated singularities must have a fixed point in every invariant continuum that does not separate the plane. The language and techniques used are from basic complex analysis and differential topology in dimension two.

1. INTRODUCTION.

L. E. J. Brouwer established several theorems that are cornerstones in fixed point theory. The best known is the following [Br1].

Theorem 1. [Brouwer Fixed Point Theorem]. *Every continuous map of the n-dimensional cube into itself has a fixed point.*

A topological space S has the *fixed point property* if every continuous map $f : S \to S$ has a fixed point. Since this property is invariant under homeomorphisms, every topological cube has the fixed point property. It was suspected for some time that the intersection $\cap K_j$ of a nested sequence ($K_{j+1} \subset K_j$) of topological cubes must also have the property. However, this was proved to be false for $n > 2$. K. Borsuk first provided in [Bo1] a fixed point free self homeomorphism of the intersection of a nested sequence of topological cubes in $\mathbb{R}^3$. Other examples followed for $n = 3$. Our favorite is R. J. Knill's fixed point free self map of the cone over the disk with a spiral [K], [A1]. The case for $n = 2$ remains a classical unsolved problem called the *Plane Fixed Point Problem.* To our knowledge the first printed reference to the problem appeared in a 1930 paper of W. L. Ayres [Ay]. It is believed however that the problem was known even earlier as folklore. More on its history can be found in [Brow], [H2], [KW], and [St].

A compact connected metric space is called a *continuum*. We note that a continuum $M \subset \mathbb{R}^2$ is homeomorphic to the intersection of a nested sequence of topological disks if and only if M is *non-separating* ($\mathbb{R}^2 - M$ is connected). We should also note that such spaces may be very intricate. The Mandelbrot set and the components of filled Julia sets of polynomial maps are examples of non-separating plane continua (see for example [CG] or [M]). The intricate nature of these sets has been revealed recently by a plethora of computer generated fractal images. A measure of the "intricacy" is the extent to which neighborhoods in M are disconnected, and thus they fail to resemble the Euclidean space. In particular, a *locally connected* space (one with arbitrarily small connected neighborhoods at every point), is considered to be "nice". An important unsolved problem in complex dynamics is to determine whether the Mandelbrot set is locally connected [M]. K. Borsuk proved that every locally connected non-separating plane continuum is a retract of the plane [Bo2]. This implies that such continua must have the fixed point property. C. L. Hagopian improved Borsuk's result by showing that every arcwise connected non-separating plane continuum has the fixed point property [H1]. Earlier, H. Bell [B2], S. Iliadis [I], and K. Sieklucki [S] had proved independently, if there were a non-separating continuum $M \subset \mathbb{R}^2$ admitting a fixed point free map, then its boundary ∂M would contain an *indecomposable* subcontinuum. Hagopian showed this cannot happen if M is arcwise connected.

An *indecomposable continuum* is one that cannot be expressed as the union of two proper subcontinua. This definition may seem innocent enough to an inexperienced reader until she/he tries to construct an example. As it turns out, an indecomposable continuum is locally disconnected everywhere (it is very intricate). The simplest planar example is the so called Knaster U-continuum described in [HY] and [Ku]. The most intricate cases are given by indecomposable continua in which all subcontinua are indecomposable. In particular, the *pseudoarc* is an indecomposable continuum that is homeomorphic to each of its subcontinua. However, the pseudoarc has a nice property that implies the fixed point property; it is a *chainable continuum* . By that we mean a continuum homeomorphic to the inverse limit space $\varprojlim\{f_j, I\}$ where each f_j is a self map of the unit interval $I = \{t : 0 \leq t \leq 1\}$. Every chainable continuum is homeomorphic to a non-separating plane continuum. O. H. Hamilton [Ha] proved that all chainable continua have the fixed point property. More recent work of P. Minc [Mi1] generalized both Hamilton's and Hagopian's results mentioned above.

In view of Hamilton's result, it was natural to ask whether all *tree-like* continua must also have the fixed point property. Such continua are homeomorphic to an inverse limit space of trees T_j (acyclic graphs):

$$\varprojlim\{f_j, T_j\} = \{(x_1, x_2, \cdots) : x_j = f_j(x_{j+1})\} \subset \prod T_j.$$

Every tree-like continuum is homeomorphic to the intersection of a nested sequence of topological cubes in $\mathbb{R}^3$. David Bellamy [Be] was first to construct a tree-like continuum with a fixed point free self map. More recent work has produced several variations of Bellamy's construction (see for example [Mi2], [OR]).

Another branch in the history of the planar problem involves non-separating plane continua invariant by homeomorphisms of the plane. The Cartwright-Littlewood theorem states that an orientation preserving homeomorphism of the plane that maps a non-separating continuum onto itself has a fixed point in the continuum [CL]. A one page proof of this fact was provided later by M. Brown in [Bro], using the following theorem also attributed to Brouwer:

Theorem 2. [Brouwer [Br2]]. *If h is an orientation preserving homeomorphism of the plane that keeps invariant a (non-empty) bounded set, then h must have a fixed point (possibly not in the set).*

H. Bell showed in [B3] that the Cartwright-Littlewood Theorem is also true for orientation reversing homeomorphisms of the plane. The question about homeomorphisms of M that do not extend to the plane has not been resolved. Recently we [A2] proved an analytic map of a planar region that keeps invariant a non-separating plane continuum has a fixed point in the continuum. This is related to the Cartwright-Littlewood theorem in the sense that an analytic map is locally an orientation preserving homeomorphism at all regular points. In this paper we generalize the result in [A2] by solving the problem for maps $f : M \to M$ that can be extended to a planar domain U whose restriction to $U - M$ is smooth with non-negative Jacobian and isolated critical points.

2. THE COVARIANT INDEX.

The index of a vector field on a simple closed curve is a useful tool for proving fixed point theorems. Let $\mathbb{D}$ denote the open unit disk consisting of all complex numbers with modulus less than one. Its boundary is the unit circle S^1. Recall that the degree of a continuous map $f : S^1 \to S^1$, denoted by *deg* f, is the order of the homotopy class of f in $\pi_1(S^1)$. The integer *deg* f represents the number of times the image of f is wrapped around S^1. Given a simple closed curve C in the complex plane $\mathbb{C}$, parametrized by a continuous map $\sigma : S^1 \to C$, and given a continuous vector field X that does not vanish on C, the *index of* X *on* C is defined by

$$i(X, C) = deg\ \frac{X(\sigma)}{|X(\sigma)|}.$$

It represents the number of revolutions made by $X(z)$ as z traverses C in the counter-clockwise direction. Suppose that C is continuously deformed into another simple closed curve C' by a homotopy σ_t and $X(z) \neq 0$ for every z in the image of σ_t. Then $deg\ \frac{X(\sigma_t)}{|X(\sigma_t)|}$ is a continuous function of an interval into the integers. Thus it must be constant. Therefore, $i(X,C) = i(X,C')$. Hence, if C can be continuously deformed into another simple closed curve C', and if $i(X,C) \neq i(X,C')$, then X vanishes somewhere in the image of the homotopy from C to C'.

Proof of Theorem 1 for n = 2. Let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be continuous. If f has a fixed point on S^1, then there is nothing to prove. Otherwise, the vector field $X_f = f - id$ does not vanish on S^1 and $i(X_f, S^1) = 1$. The maps $\sigma_t\left(e^{i\theta}\right) = te^{i\theta}$ deform S^1 to the constant curve 0. But $i(X_f, 0) = \deg \frac{X(0)}{|X(0)|} = 0$. Therefore, $X_f(z) = 0$ for some $z \in \mathbb{D}$.□

Let $\sigma : I \to \mathbb{C}$ denote a parametrization of a C^1 curve $C = \sigma(I)$ such that σ' does not vanish. The unit tangent vector $T(z) = \frac{\sigma'(t)}{|\sigma'(t)|}$, together with the unit normal vector $N(z) = -iT(z)$, form an orthonormal basis $\{N(z), T(z)\}$ for the tangent space $T_z\mathbb{C}$ at each point $z = \sigma(t)$. The pair of vector fields $\{N, T\}$ is called the *Frenet frame field*, in contrast to the *Euclidean frame field*, defined by $E_1(z) = (1,0)_z, E_2 = (0,1)_z$.

Definition. Let X be a C^1 non-zero vector field on a C^1 closed curve C. The *covariant index of* X on C, also called the *variation* of X on C, is defined to be

$$v(X,C) = i(X',C),$$

the index of the vector field $X' = x_1 + ix_2$, where x_1, x_2 are functions on C satisfying $X = x_1N + x_2T$.

The covariant index represents the number of revolutions made by $X(z)$ as z traverses C from the point of view of an observer riding on the Frenet frame field.

The numerical relationship between the covariant index and the usual index is easily obtained by considering the functions α, β, ϕ of C into $[0, 2\pi)$, where α is the angle from E_1 to X, β is the angle from N to X, and ϕ is the angle from E_1 to N (See Figure 2.1).

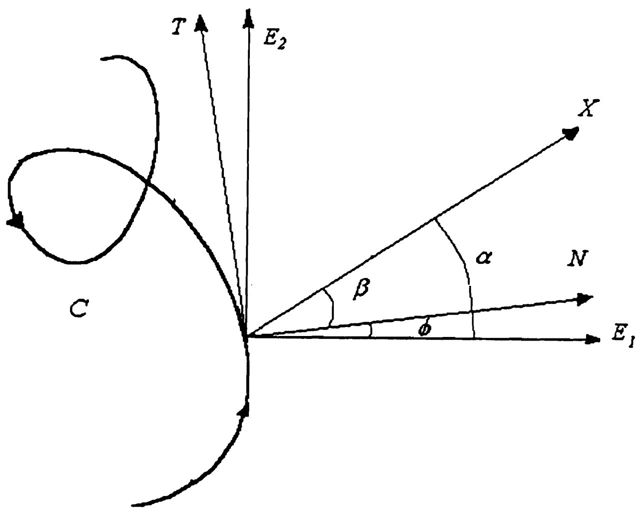

Figure 2.1

We have

$$i(X,C) = \frac{1}{2\pi}\int_C d\alpha, \;\; v(X,C) = \frac{1}{2\pi}\int_C d\beta, \;\; \text{and}$$

$$deg\, G = i\,(N,C) = \frac{1}{2\pi}\int_C d\phi,$$

where G denotes the Gauss map of C. And $\alpha = \beta + \phi$ implies

$$\int_C d\alpha = \int_C d\beta + \int_C d\phi.$$

Therefore,

$$i(X,C) = v(X,C) + deg\, G.$$

If C is a simple closed curve, then $\int_C d\phi = 2\pi$, and we have that

$$i(X, C) = v(X, C) + 1. \tag{2.1}$$

We have seen that the index of a vector field is invariant among simple closed curves homotopic on a region where the vector field has no zeros. It also follows directly from the definitions that homotopic non-zero vector fields on a simple closed curve have the same index. Therefore, we extend the definition of the covariant index to a continuous non-zero vector field X on a simple closed curve C, by defining it to be $v(Y, C')$, the covariant index of a C^1 vector field Y homotopic to X on a C^1 simple closed curve C' homotopic to C. Thus, the above equations hold for every continuous vector field X and every simple closed curve C.

Corollary 1. *Let D denote the bounded complementary domain of positively oriented simple closed curve C. If a continuous map $f : \overline{D} \to \mathbb{C}$ is fixed point free, then $v(X_f, C) = -1$, where $X_f = f - id$.*

3. Local variation and orientation.

Under certain constraints imposed on a function $f : \overline{D} \to \mathbb{C}$, the covariant index of $X_f = f - id$ on C can be decomposed into a sum of local variations $Var(f, A)$, where A denotes an arc in C. We proceed to define $Var(f, A)$.

We adopt the following terminology: A *closed arc* $A = ab \subset \mathbb{C}$ with initial point a and final point b is the homeomorphic image of the closed interval from 0 to 1, where a, b denote the images of $0, 1$ respectively. An *open arc* is a closed arc minus its initial and final points, or a simple closed curve minus one of its points. When it is clear from the context, we will often refer to an open or closed arc, or to a point $a = b$, as the *arc* $A = ab$. A ray $R \subset \mathbb{C}$ is the homeomorphic image of the half open interval $[0, 1)$.

Definition. Let an open arc $ab = A \subset \mathbb{C}$, and a continuous function $f : \overline{A} \to \mathbb{C}$ be given such that $f(\overline{A}) \cap \overline{A} = \emptyset$. A ray $R \subset \mathbb{C}$ is *admissible for* (f, A) if the initial point of R is the only point in $A \cap R$, its final point is at infinity, and neither $f(a)$ nor $f(b)$ are in R. The union of $f(A)$ with a curve from $f(b)$ to $f(a)$ that is contained in $\mathbb{C} - \overline{A} \cup R$ defines a closed curve whose winding number around A does not depend on the particular choice of the arc from $f(b)$ to $f(a)$. We denote this number by $V_R(f, A)$ and we call it the *variation of* (f, A) *with respect to* R.

Suppose the interior of R is contained in the unbounded complementary domain of a simple closed curve C containing A. We define an extension $\overline{f} : C \to \mathbb{C} - \overline{A}$ of f by mapping $C - A$ injectively onto a closed arc from $f(b)$ to $f(a)$ that is contained in $\mathbb{C} - \overline{A} \cup R$. Then

$$V_R(f, A) = deg\ \overline{f} = \ \overline{f}_*(1)\,,$$

where $\overline{f}_*$ denotes the induced homomorphism on the group of integers.

If R_1, R_2 are disjoint admissible rays for (f, A), and $f(a)$, $f(b)$ are in the same complementary connected domain of $R_1 \cup A \cup R_2$, then

$$V_{R_1}(f, A) = V_{R_2}(f, A) = V_R(f, A),$$

for every admissible ray R whose interior is a subset of the complementary connected domain of $R_1 \cup A \cup R_2$ not containing $f(a)$ and $f(b)$.

Suppose that C is a positively oriented continuous simple closed curve in $\mathbb{C}$, and D is the bounded complementary domain of C. Let f be a map of $\overline{D} = C \cup D$ into $\mathbb{C}$, where $f|_C$ is fixed point free. Let h be a homeomorphism of the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ into the extended complex plane $\widehat{\mathbb{C}}$ satisfying $h(0) = \infty$ and $h(S^1) = C$. The Riemann Mapping Theorem implies that h can be chosen to be conformal in $\mathbb{D}$. For each $s = h(e^{i\theta}) \in C$, the *external* *ray* R_s is defined by

$$R_s = \{h(te^{i\theta}) : 0 < t \leq 1\}.$$

Definition. Let $\mathcal{L}$ denote the collection of all open arcs $A = ab \subset C$ such that, $f(a), f(b) \in \overline{D}$ and $\overline{A} \cap f(\overline{A}) = \emptyset$. We observe that if $A \in \mathcal{L}$, then R_s is admissible for $(f|_{\overline{A}}, A)$ for each $s \in A$. Hence $V_{R_s}(f|_{\overline{A}}, A)$ is constant for all $s \in A$. We denote this number by $Var(f, A)$ and call it the *variation of* f *at* A.

Lemma 1. [Bell's Equation [B1]]. *Suppose* $f|_C$ *is fixed point free and that* C *can be partitioned into finitely many subarcs* $L_i \in \mathcal{L}$. *Then*

$$i(X_f, C) = \sum Var(f, L_i) + 1.$$

Proof: If $L_i \subset \overline{D}$ then $Var(f, L_i) = 0$. Suppose $Var(f, L_i) \neq 0$. Let a' be the last point in L_i and let b' be the first point in L_i, with the property that $aa' \cup bb' \subset L_i$ and $f(aa') \cup f(bb') \subset \overline{D}$. We define a fixed point free continuous function g on C. For every L_i not contained in $\overline{D}$, g maps $a'b'$ homeomorphically onto the unique subarc of C that is disjoint from L_i, with initial point $f(a') = g(a')$ and final point $f(b') = g(b')$. Elsewhere g agrees with f. Hence, $g(C) \subset \overline{D}$. Therefore, $i(g - id, C) = 1$. On the other hand, $i(X_f, C) = i(g - id, C) + \sum_i Var(f, L_i)$. □

Note that if f satisfies the hypotheses of Lemma 1, then (2.1) implies that the covariant index of X_f on C can be expressed as the sum

$$v(X_f, C) = \sum_i Var(f, L_i). \tag{3.1}$$

Also note that the integer $\sum_i Var(f, L_i)$ does not depend on the particular parametrization h. Furthermore, the condition $L_i \in \mathcal{L}$ holds if the diameter of L_i is strictly less than

$inf\{|f(z) - z| : z \in C\}$ and if the images of the endpoints of L_i under f are in $\overline{D}$.

Suppose f is a complex valued function defined on a domain U (open, connected) of the complex plane that has no fixed points in the invariant non-separating continuum $M \subset U$. The following lemma will be used to construct simple closed curves C with bounded complementary domain D satisfying the hypotheses of Lemma 1. Its proof can be found in [Bo].

Lemma 2. [Borsuk Lemma]. *Let M be a non-separating plane continuum. For every $\epsilon > 0$ there exists a piecewise linear simple closed curve C with bounded complementary domain D satisfying the following three conditions:*

(i) $M \subset \overline{D}$.
(ii) *If $x \in \overline{D}$ then $dist(x, M) \leq \epsilon$.*
(iii) *C is the union of a finite number of arcs $L_1, L_2, \dots, L_n$ with ends belonging to M, with interiors lying in $\mathbb{C} - M$, and with diameters less than ϵ.*

Lemma 3. *Suppose $f : U \to \mathbb{C}$ has no fixed points in an invariant continuum $M \subset U$. There exists $\epsilon_0 > 0$ such that if $\epsilon_0 > \epsilon > 0$ and C, D are as in Lemma 2, then $\{L_1, L_2, \dots, L_n\} \subset \mathcal{L}$ and $v(X_f, C) = \sum_i Var(f, L_i) = -1$.*

Proof: By choosing a smaller domain U, we may assume $\overline{U}$ is a topological disk and f has no fixed points in an open set containing $\overline{U}$. Therefore, $inf\{|f(z) - z| : z \in \overline{U}\} > 0$. Note that $f|_{\overline{U}}$ is uniformly continuous. Therefore for all $\eta > 0$ there exists $\delta(\eta)$ such that $|f(z_1) - f(z_2)| < \eta$ whenever $|z_1 - z_2| < \delta$ and $z_1, z_2 \in \overline{U}$. Let C, D be as in Lemma 2 with $\epsilon < \epsilon_0 = min\left(\delta(\eta), inf\{|f(z) - z| : z \in \overline{U}\}\right)$ where $\eta = \frac{1}{2} diam\, M$. Suppose there exists $z \in L_i$ such that $f(z) \in L_i \cap f(L_i)$. Then we obtain

$$|f(z) - z| < diam L_i < \frac{1}{2} inf\{|f(z) - z| : z \in \overline{U}\},$$

a contradiction. Furthermore, since f keeps M invariant it follows that the endpoints of L_i are mapped by f into

$M \subset \overline{D}$. Therefore, $L_i \in \mathcal{L}$. The rest follows from Corollary 1 and (3.1). □

So far in this section we have confined our discussion to continuous maps. Next we will introduce smoothness to the objects in the definition of variation to obtain a method for computing $V_R(f, A)$. This method involves the sense of the angle in which $f(A)$ crosses R at finitely many points. To be more precise, we must first define the concept of *transversality* as it applies to smooth planar curves.

Definition. Suppose $\gamma : U \to \mathbb{C}$ and $\sigma : V \to \mathbb{C}$ are smooth parametrizations of planar curves, where U, V are open subsets in $\mathbb{R}$. We say γ and σ are *transverse*, if for every $s \in U$ and $t \in V$ such that $\gamma(s) = \sigma(t) = z$, the vectors $\gamma'(s)$, $\sigma'(t)$ span $T_z\mathbb{C}$, the tangent space at z. Two piecewise smooth curves are transverse if they intersect only in the interiors of smooth transverse subarcs.

For the general definition of transversality we refer the reader to [Hi].

Definition. Given two linearly independent vectors v, w$\in T_z\mathbb{C}$ the *orientation* of (v, w) is 1 or -1 depending on the sign of the determinant of the matrix whose columns are the vectors v, w.

Lemma 4. *Let $f : \overline{A} \to \mathbb{C}$ be a piecewise smooth function defined on the closure of a smooth open arc A. Suppose a smooth ray R is admissible for (f, A) and R is transverse to $f(A)$. Then $R \cap f(A)$ is finite, and*

$$V_R(f, A) = \sum_{z \in f^{-1}(R)} sign\ df_z,$$

where $sign\ df_z$ denotes the orientation of $(\zeta'(t), (f \circ \gamma)'(s))$, where ζ, γ denote smooth parametrizations of R and A, respectively, and $z = \gamma(s)$, $f(z) = \zeta(t)$.

Proof: That $R \cap f(A)$ is finite follows from transversality. An appropriate diffeomorphism h of the plane maps R onto the non-negative imaginary axis, and A onto an interval of reals containing zero so that $h(a) < h(b)$. Thus, the restrictions of h to R and A are maps of oriented segments into the reals that preserve orientation. Yet h itself may preserve or reverse the orientation of the plane. Let $f^* = h \circ f \circ h^{-1}$, then $h(R)$ is admissible for $(f^*, h(A))$. Let π_1 denote the retraction of the plane onto the lower half plane that collapses $\{(x,y) : 0 \le y\}$ to $(x,0)$. Let $g : h(A) \to \mathbb{R}$ denote $\pi_1 \circ f^*$. Then

$$V_R(f,A) = V_{h(R)}(f^*, h(A)) = \sum_{x \in g^{-1}(0)} \text{sign } g'(x) = \sum_{f^*(x) \in h(R)} \text{sign } df^*_x.$$

Now let γ be the restriction of h^{-1} to the real axis, and let σ be the restriction of h^{-1} to the imaginary axis. Therefore, $x = h(z)$ and $y = h \circ f(z)$. Now, $f^*(x) \in h(R)$ if and only if $f(z) = f \circ h^{-1}(x) \in R$. Finally, note that the linear transformation $Dh(f(z))$ maps $(f \circ h^{-1})'(x)$ to $Df^*(x)$, and $(h^{-1})'(y)$ to y. See also 1.8 in [BG]. □

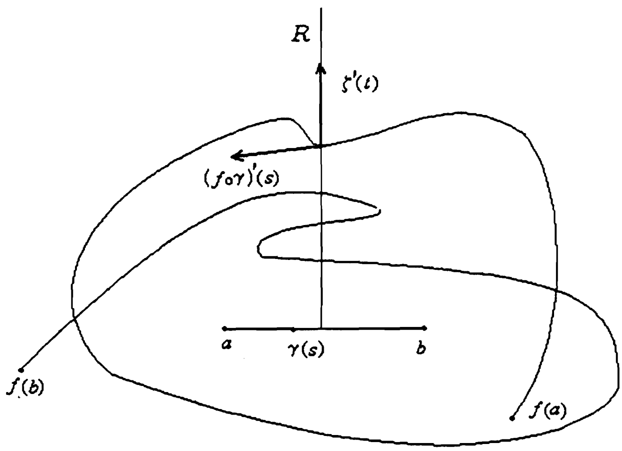

Figure 3.1

In Figure 3.1 we represent objects found in Lemma 3.4. The horizontal interval between a and b represents the arc A. For $z = \gamma(s)$ in the figure, sign $df_z = 1$, and $V_R(f, A) = 2$.

4. Proof of the main theorem.

Theorem. *Suppose f is a continuous function defined on a domain U in the plane and f keeps invariant a non-separating continuum $M \subset U$. If the restriction of f to $U - M$ is smooth with non-negative Jacobian and isolated critical points, then M must contain a fixed point of f.*

Proof: Suppose M contains no fixed points of f. By replacing U with a smaller domain if necessary, we may assume that U is bounded and simply connected. Let V equal to $U - M$ minus the all critical points of f. Let ϵ, C and D be as in Lemma 3 with ϵ so small that $f(\overline{D}) \subset U$, and $C = \bigcup_{i=1}^{n} L_i$ such that each L_i is smooth with interior lying in V. Hence, $\{L_1, L_2, \ldots, L_n\} \subset \mathcal{L}$ and

$$\sum_{i=1}^{n} Var(f, L_i) = -1.$$

We will get a contradiction by showing that

$$Var(f, L_i) \geq 0, \text{ for all } i = 1, 2, \cdots, n.$$

Let R denote a smooth ray containing no critical values of f, with initial point in $L_i - M$, final point at infinity, with interior R° contained in $\mathbb{C} - \overline{D}$, and such that $R \cap U$ is an arc. By applying the Transversality Theorem, we choose R transverse to $f(L_i)$.

In Figure 4.1 the shaded area represents an open set of M. The horizontal line segment L_i is part of the bold polygonal curve representing C. The bold vertical line is the visible part of the ray R. And the thin curve labeled $f(L_i)$ represents the image of L_i.

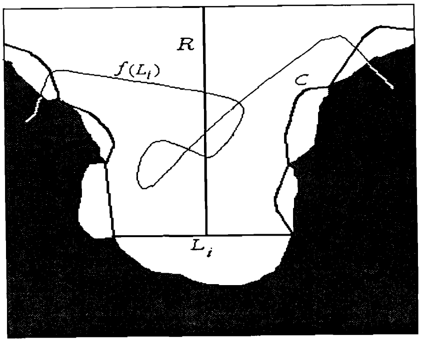

Figure 4.1

Note that

$$f|_V : V \to \mathbb{C}$$

is an immersion. Therefore, $f^{-1}(R^\circ)$ is a one dimensional submanifold of V. As the Jacobian $J(f) > 0$ on V, it follows that $f^{-1}(R^\circ)$ is the union of finitely many arcs R_κ in $V \subset U - M$. The initial point of each R_κ is mapped by f to the initial point of R, and the final point of R_κ is mapped to $\partial U \cap R$. Note that the final point of R_κ must be a point of ∂U, for it cannot be in $\overline{D}$ as $f(\overline{D}) \subset U$, and it cannot be a critical point of f since R contains no critical values of f. By defining $P_\kappa = L_i \cap R_\kappa$, we create a partition $\{P_\kappa\}$ of $f|_{L_i}^{-1}(R)$. By Lemma 3.4,

$$Var(f, L_i) = V_R(f, L_i) = \sum_\kappa \sum_{z \in P_\kappa} \text{sign } df_z.$$

In the above df_z means $d(f|_{L_i})_z$.

Now note that f preserves the sense of the angle between tangent vectors to L_i and R_κ at each point $z \in P_\kappa$, since the Jacobian $J(f) > 0$ on V and $P_\kappa = L_i \cap R_\kappa \subset V$. Therefore, for smooth parametrizations γ, σ of L_i and R_κ

respectively,

$$\sum_{z\in P_\kappa} \text{sign } df_z = \sum_{z\in P_\kappa} \text{sign } dz,$$

where sign dz denotes the orientation of $(\sigma'(t), \gamma'(s))$, and sign df_z denotes the orientation of $((f\circ\sigma)'(t), (f\circ\gamma)'(s))$. Next we observe that $\sum_{z\in P_\kappa} \text{sign } dz \geq 0$. In fact, note that when R_κ crosses L_i at $z \in P_\kappa$, either R_κ enters D at z if sign $dz = -1$, or R_κ exits D at z if sign $dz = 1$. With its final point in ∂U, R_κ must eventually stay outside $\overline{D}$. It follows that there are at least as many elements $z \in P_\kappa$ with sign $dz = 1$ as there are with sign $dz = -1$. See Figure 4.2.

Therefore,

$$Var(f, L_i) = \sum_{\kappa\in\mathcal{K}} \sum_{z\in P_\kappa} \text{sign } df_z \geq 0.\square$$

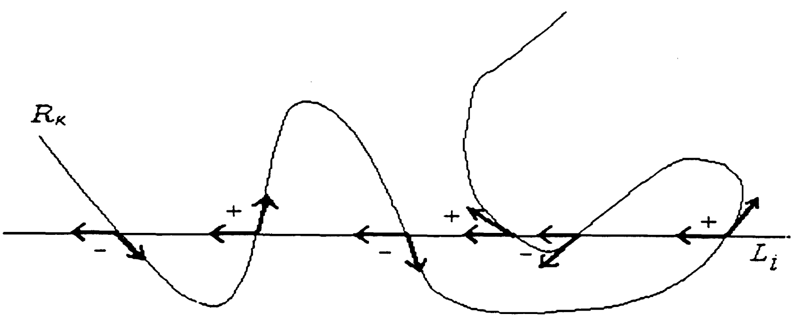

Figure 4.2

Corollary. *Every smooth map of a planar domain with non-negative Jacobian and isolated singularities must have a fixed point in every invariant continuum that does not separate the plane.*

Corollary. *Every holomorphic map of a planar domain with must have a fixed point in every invariant continuum that does not separate the plane.*

In conclusion we ask the following:

Problem 1. Let f denote a smooth complex valued function defined on a plane domain. Must f have a fixed point in every invariant non-separating continuum?

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CALIFORNIA STATE UNIVERSITY, LOS ANGELES,CA 90032
E-mail address: `vakis@calstatela.edu`