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A RECURRENT FUNCTION WHICH IS NOT ONE-TO-ONE

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ABSTRACT. A metric continuum is constructed admitting a self-map which is recurrent but not one-to-one.

As is customary, let $\mathbf{Z}$ denote the integers; $\mathbf{N}$, the positive integers, and S^1 the unit circle in the plane $\mathbf{R}^2$. For $A \subseteq, \mathbf{R}$ we shall denote by $B(A, \epsilon)$ the set of points of $\mathbf{R}$ at a distance less than ϵ from set A.

Let X be a topological space. A function $F : X \to X$ is called recurrent provided that for each $x \in X, x$ is in the closure of $\{F^n(x)|n \in \mathbf{N}\}$ (called the forward orbit of x under F.) Jane M. Day has studied these functions in [1] and [2], asking in the latter [2, p 201] whether such a function defined on a compact Hausdorff space must be one-to-one. She specifically asked whether this was true on a totally disconnected compact Hausdorff space.

We do not have a totally disconnected example, but we present here an example of a recurrent function from a metric continuum to itself which is not one-to-one.

To begin the construction, define the real-valued function g by $g(t) = \sin(\cot(\frac{t}{2}))$. This function is periodic with period 2π and is defined and continuous at every real number except for integer multiples of 2π, where it has a $\sin(\frac{1}{x})$ type singularity.

Let $A_o = \{2\pi n|n \in \mathbf{Z}\}$, and for each $k \in \mathbf{N}$, let $A_k = \{x + k|x \in A_o\} = \{2\pi n + k|n \in \mathbf{Z}\}$. Note in particular that $k \in A_k$

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for each k. Define $A = \bigcup_{k=1}^{\infty} A_k$, and $T = \mathbf{R} \setminus A$. Let $f : \mathbf{R} \to \mathbf{R}$ be the translation $f(t) = t - 1$. Clearly, $f(A_k) = A_{k-1}$ for each $k \in \mathbf{N}$, and $f(A) = A \cup A_o = \bigcup_{k=o}^{\infty} A_k$, while $f(T) = T \setminus A_o$. Since 2π is irrational, $A_k \cap A_l = \phi$ for $k \neq l$, and A is dense in $\mathbf{R}$. Now, define $G(t) = \sum_{k=1}^{\infty} \frac{g(t-k)}{2^k}$. Since $t - k = 2n\pi$ if and only if $t = 2n\pi + k, g(t-k)$ has a $\sin\left(\frac{1}{x}\right)$ type of singularity at each $t \in A_k$, and is continuous at each $t \notin A_k$. By uniform convergence, G is continuous at every point of T. Like g, G is periodic with period 2π. Let M_1 denote the graph of G in $\mathbf{R}^2$, and let $M = \overline{M_1}$.

The next two lemmas will give the important facts about the structure of M.

Lemma 1. *Let $s \in T$. Then the only point of M with first coordinate s is the point $(s, (G(s))$.*

Proof: Assume $s \in T$ and $y \neq G(s)$. Let U, V be disjoint open intervals with $G(s) \in U$ and $y \in V$. Since G is continuous at s, there exists an open set $W \subseteq \mathbf{R}$ such that $s \in W$ and $g(W \cap T) \subseteq U$. Consequently, $M_1 \cap (W \times \mathbf{R}) \subseteq W \times U$, and $M \cap (W \times \mathbf{R}) \subseteq \overline{W} \times \overline{U}$. Since $y \in V$ and $V \cap U = \phi, y \notin \overline{U}$. It follows that $(s, y)) \notin M$, and the Lemma is true.

Lemma 2. *Let $a \in A_m$. The set $M \cap (\{a\} \times \mathbf{R})$ is a vertical line segment of length $\frac{2}{2^m}$.*

Proof: Suppose $a \in A_m$. The graph $y = \frac{g(t-m)}{2^m}$ has $\sin\left(\frac{1}{x}\right)$ type behavior at $t = a$, with a limit arc of length $\frac{2}{2^m}$. Since the function defined by deleting the m-th term from the sum defining G is continuous at $t = a$, it is straightforward to check that the graph of G has a limit interval of the same length at $t = a$.

Now, define $f_1 : M_1 \to M_1$ by

$$\begin{aligned} f_1(t, G(t)) &= (t-1, G(t-1)) \\ &= (f(t), G(f(t))). \end{aligned}$$

Lemma 3. *The function $f_1 : M_1 \to M_1$ is uniformly continuous.*

Proof: This proof is provided in detail at the request of two colleagues who did not see how to fill it in. Let d denote the Euclidean metric on $\mathbf{R}^2$, restricted to M_1. Let $\epsilon > 0$ be arbitrary. Choose the required $\delta > 0$ as follows: First choose $n \in \mathbf{N}$ such that $\sum_{k=n+1}^{\infty} \frac{2}{2^k} < \frac{\epsilon}{8}$. Then choose δ_1 sufficiently small that $B(A_1, 2\delta_1) \cap B\left(\bigcup_{k=2}^{n} A_k, 2\delta_1\right) = \phi$. (Each A_k is invariant under translation by 2π, so it suffices to choose δ_1 to have this property in the compact set $[0,4\pi]$.) Let $\delta_2 > 0$ be sufficiently small that whenever x, y are both outside $B(A_1, \delta_1)$ and $|x - y| < \delta_2$, it follows that $|g(x-1) - g(y-1)| < \frac{\epsilon}{4}$. Such a δ_2 exists since $g(t-1)$, by virtue of being periodic, is a uniformly continuous function of t outside $B(A, \delta_1)$. Choose $\delta_3 > 0$ such that if $x, y \notin B\left(\bigcup_{k=2}^{n} A_k, \delta_1\right)$, and $|x - y| < \delta_3$, then $| \sum_{k=2}^{n} \frac{g(x-k)}{2^k} - \sum_{k=2}^{n} \frac{g(y-k)}{2^k} | < \frac{\epsilon}{4}$. Again, this is possible since $\sum_{k=2}^{n} \frac{g(t-k)}{2^k}$ is a uniformly continuous function of t outside of $B\left(\bigcup_{k=2}^{n} A_k, \delta_1\right)$. Let δ be the smallest of $\delta_1, \delta_2, \delta_3$, and $\frac{\epsilon}{4}$.

Then suppose $x, y \in \mathbf{R}$ and $d((x, G(x)), (y, G(y))) < \delta$. Then both
$|x-y| < \delta$ and $|G(x)-G(y)| < \delta$. Observe that is is impossible to have $x \in B(A_1, \delta_1)$ and $y \in B\left(\bigcup_{k=2}^{n} A_k, \delta_1\right)$, (or vice versa) since then $|x - y| \geq \delta_1 \geq \delta$. Hence, there are two cases to consider.

Case I: Assume neither x nor y belongs to $B(A_1, \delta_1)$. Then,

$$
\begin{aligned}
&d(f_1(x, G(x)), f_1(y, G(y))) \\
&\quad = d((x-1, G(x-1)), (y-1, G(y-1))) \\
&\quad \leq |x-1-(y-1)| + |G(x-1) - G(y-1)| \\
&\quad < \frac{\epsilon}{4} + |\sum_{k=1}^{\infty} \frac{g(x-1-k)}{2^k} - \sum_{k=1}^{\infty} \frac{g(y-1-k)}{2^k}| \\
&\quad < \frac{\epsilon}{4} + |\sum_{k=2}^{\infty} \frac{g(x-k)}{2^{k-1}} - \sum_{k=2}^{\infty} \frac{g(y-k)}{2^{k-1}}| \\
&\quad < \frac{\epsilon}{4} + 2|\sum_{k=1}^{\infty} \frac{g(x-k)}{2^k} - \sum_{k=1}^{\infty} \frac{g(y-k)}{2^k} \\
&\qquad - g(x-1) + g(y-1)| \\
&\quad < \frac{\epsilon}{4} + 2|G(x) - G(y)| + |g(x-1) - g(y-1)| \\
&\quad < \frac{\epsilon}{4} + 2\delta + \frac{\epsilon}{4} < \frac{\epsilon}{4} + \frac{2\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}
$$

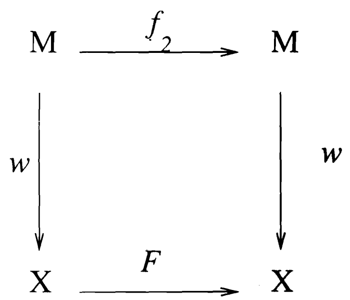
Case II.: Assume neither x nor y belongs to $B\left(\bigcup_{k=2}^{n} A_k, \delta_1\right)$.
Then

$$
\begin{aligned}
&d(f_1(x, G(x)), f_1(y, G(y))) \\
&\quad \leq |x-1-(y-1)| + |G(x-1) + G(y-1)| \\
&\quad < \frac{\epsilon}{4} + 2|\sum_{k=2}^{\infty} \frac{g(x-k)}{2^k} - \sum_{k=2}^{\infty} \frac{g(y-k)}{2^k}| \\
&\quad < \frac{\epsilon}{4} + 2|\sum_{k=2}^{n} \frac{g(x-k)}{2^k} - \sum_{k=2}^{n} \frac{g(y-k)}{2^k}| \\
&\qquad + 2|\sum_{k=n+1}^{\infty} \frac{g(x-k)}{2^k} - \sum_{k=n+1}^{\infty} \frac{g(y-k)}{2^k}| \\
&\quad < \frac{\epsilon}{4} + \frac{2\epsilon}{4} + 2\sum_{k=n+1}^{\infty} \frac{2}{2^k} < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{2\epsilon}{8} = \epsilon.
\end{aligned}
$$

Thus, f_1 is uniformly continuous, as stated.

Corollary 4. *There exists a unique continuous extension of f_1, to all of $M = \overline{M_1}$ (denote it $f_2 : M \to M$.)*

Now, define $w : \mathbf{R} \times \mathbf{R} \to S^1 \times \mathbf{R}$ by the standard wrapping around a cylinder; $w(s,t) = ((\cos s, \sin s), t)$. Let $X = w(M)$, and define $F : X \to X$ by $F(w(s,t)) = w(f_2(s,t))$, for $(s,t) \in M$.



This is well-defined and continuous since M is invariant under translations by 2π units in the first coordinate in $\mathbf{R}^2$. Let $\pi_S : S^1 \times \mathbf{R} \to S^1$ denote the projection and let $\pi_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the projection onto the first factor.

Lemma 5. *A subset $K \subseteq M$ is dense in M if and only if $\pi_1(K)$ is dense in* $\mathbf{R}$.

Proof: If $K \subseteq M$ is dense in M, since $\pi_1(M) = \mathbf{R}$, clearly, $\pi_1(K)$ is dense in $\mathbf{R}$. Suppose conversely that $K \subseteq M$ and $\pi_1(K)$ is dense in $\mathbf{R}$. Let $(s,t) \in M$ and let U_o, V_o be open in $\mathbf{R}$ with $(s,t) \in (U_o \times V_o)$. It suffices to show that $(U_o \times V_o) \cap K \neq \phi$. Let V be open in $\mathbf{R}$ such that $t \in V \subseteq \overline{V} \subseteq V_o$. Since $M = \overline{M_1}$, there exists $(p,q) \in M_1 \cap (U_o \times V)$. Since M_1 is just the graph of $G, q = G(p)$ and $p \in T$. Therefore, G is continuous at p so there exists an open set $U \subseteq \mathbf{R}$ such that $p \in U \subseteq \overline{U} \subseteq U_o$ and such that $G(U \cap T) \subseteq V$. Consequently, $M_1 \cap (U \times \mathbf{R}) \subseteq U \times V$, so that $M \cap (U \times \mathbf{R}) \subseteq \overline{U} \times \overline{V}$, since $M = \overline{M_1}$. Since $\pi_1(K)$ is dense in $\mathbf{R}$, there exists $r \in \pi_1(K) \cap U$. But, $r = \pi_1(r, s)$ for some $(r, s) \in K$, and since $K \subseteq M$, and $r \in \overline{U}, \subseteq U_o$, it follows tht $s \in \overline{V}$. But $\overline{V} \subseteq V_o$, so $(r, s) \in (U_o \times V_o) \cap K$. Thus, $(U_o \times V_o) \cap K \neq \phi$, and it follows that K is dense in M.

Corollary 6. *A set $K \subseteq X$ is dense in $X$is and only if $\pi_S(K)$ is dense in S^1.*

Proposition. For each $x \in X$, the forward orbit $O = \{F^n(x)|n \in \mathbf{N}\}$ is dense in X. Consequently, F is recurrent on X.

Proof: Let $x \in X$. Then $x = w(y)$ for some $y \in M$, and $F^n(x) = F^n(w(y)) = w(f_2^n(y))$. Thus $\pi_S(O) = \{(\cos(\pi_1(y) - n), \sin(\pi_1(y) - n))|n \epsilon \mathbf{N}\}$, which is dense in S^1. The result then follows from the last corollary.

The only task remaining is to prove that F is not one-to-one. Notice that $\{(x, y) \in M|x = 1\}$ is a line segment of length one, by Lemma 2, as is $\{((\cos 1, \sin 1), y)|((\cos 1, \sin 1), y) \in X\}$, while $\{(x, y) \in M|x = 0\}$ is a single point. However, $f_2(\{x, y) \in M|x = 1\}) = \{(x, y) \in M|x = 0\}$, and consequently $F((\cos 1, \sin 1), y) = ((\cos 0, \sin 0), G(0))$ for every y such that $((\cos 1, \sin 1), y) \in X$, so that F carries this closed line segment of length one to a single point, and so is not one-to-one.

References

[1] Jane M. Day, *Semigroups with periodic properties*, Semigroup Forum, **7** (1974), 292-309.

[2] Jane M. Day, *Recurrent functions and Semigroups*, Semigroup Forum **10** (1975), 190-202.

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