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SMOOTHNESS OF HYPERSPACES AND OF CARTESIAN PRODUCTS

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ABSTRACT. We show that for any continua X and Y the smoothness of either the hyperspace $C(X)$ or 2^X or of the Cartesian product $X \times Y$ implies the property of Kelley for X. An example is constructed showing that the converse is not true.

A *continuum* is a compact connected metric space. For a given continuum X with metric d, the symbol 2^X denotes the hyperspace of all nonempty compact subsets of X equipped with the Hausdorff distance H ([16, p. 1] for the definition) and $C(X)$ is the subspace of 2^X composed of all nonempty subcontinua of X. For a given point $p \in X$ the symbol $C(p, X)$ stands for the subspace of $C(X)$ composed of all nonempty subcontinua of X containing the point p. We use the symbol $\mathcal{H}$ for the Hausdorff distance in $C(2^X)$. Given a point $p \in X$ and a positive number r we denote by $B_X(p, r)$ the open ball with center p and radius r and, for $A \subset X$ we define $N_X(A, r) = \bigcup\{B(x, r) : x \in A\}$.

We say that continuum X has the *property of Kelley* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each point $x \in X$, for

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each continuum $K \in C(x, X)$ and for each point $y \in X$ satisfying $d(x, y) < \delta$ there is $L \in C(y, X)$ such that $H(K, L) < \varepsilon$ (see property (3.2) in [11, p. 26]; compare [16, (16.10), p. 538]).

We say that continuum X is *smooth at a point* $p \in X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each $x \in X$, for each continuum $K \in C(p, X)$ such that $x \in K$ and for each $y \in X$ satisfying $d(x, y) < \delta$ there is $L \in C(p, X)$ such that $y \in L$ and $H(K, L) < \varepsilon$. A continuum X is *smooth* if it is smooth at some point.

The concept of smoothness was first defined for fans in [1, p. 7], next extended to dendroids in [3, p. 298]. Many authors have studied smooth dendroids, see e.g. [7], [9], [10], [13], [15], and there were some generalizations of smoothness of dendroids, for example to pointwise smooth dendroids by S. T. Czuba in [5] or to weakly smooth denroids by Lewis Lum in [12]. G. R. Gordh, Jr. has extended the definition of smoothness to continua that are hereditarily unicoherent at some point ([8, p. 52]). The above most general definition (in metric case) is due to T. Maćkowiak [14, p. 81]. S. T. Czuba has shown that for dendroids the property of Kelley implies smoothness [6, Corollary 5, p. 730], and it was shown in [2] that this implication can be extended neither to λ-dendroids (Example 44) nor to arcwise connected continua (Example 45). In this paper we study the implication from smoothness to the property of Kelley for hyperspaces and for Cartesian products.

Theorem 1. *If the hyperspace 2^X or $C(X)$ of a continuum X is smooth, then X has the property of Kelley.*

Proof: We will show the proof in the case of 2^X. The proof for $C(X)$ is similar.

Let a continuum $K \subset X$, a point $p \in K$ and a number $\varepsilon > 0$ be given. Assume that 2^X is smooth at A. If A is not contained in K we additionally assume that $H(P, K) > \varepsilon$ for every $P \in C(A)$. Let $\delta > 0$ be as in the definition of smoothness of 2^X, and let q be a point of X satisfying $d(p, q) <$

δ. It is enough to find a continuum $L \in C(q, X)$ satisfying $H(K, L) \leq \varepsilon$. Consider two cases.

Case 1. A is not contained in K. Let $\mathcal{A}$ be an order arc from $\{p\}$ to K, let $\mathcal{B}$ be an order arc from K to X and let $\mathcal{C}$ be an order arc from A to X. Then, by assumption, $\mathcal{A} \cap \mathcal{C} = \emptyset$. By the smoothness of 2^X at A there is a continuum $\mathcal{M}$ containing $\{q\}$ and A such that $\mathcal{H}(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}, \mathcal{M}) < \varepsilon$. Let $\mathcal{L}$ be the closure of the component of $\mathcal{M} \cap N_{C(X)}(\mathcal{A}, \varepsilon)$ that contains $\{q\}$ and put $L = \bigcup \mathcal{L}$. We will show that $H(K, L) \leq \varepsilon$. Let $B \in \mathcal{L} \cap \mathrm{bd} N_{C(X)}(\mathcal{A}, \varepsilon)$. Since $B \in \mathcal{L}$ there is a continuum $B' \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ with $H(B, B') < \varepsilon$. Because of the choice of ε we have $B' \notin \mathcal{C}$. Since $B \in \mathrm{bd} N_{C(X)}(\mathcal{A}, \varepsilon)$ we have $B' \notin \mathcal{A}$, so $B' \in \mathcal{B}$. Moreower, there is a continuum $B'' \in \mathcal{A}$ with $H(B, B'') = \varepsilon$. Then we have $B'' \subset K \subset B'$ and therefore $H(B, K) \leq \varepsilon$. Observe that $L \subset \mathrm{cl}\, N_X(K, \varepsilon)$ by the definition of $\mathcal{L}$ and $K \subset \mathrm{cl} N_X(B, \varepsilon) \subset \mathrm{cl} N_X(L, \varepsilon)$, so $H(K, L) \leq \varepsilon$. The proof in this case is complete.

Case 2. A is contained in K. Then, by smoothness of 2^X at A there is a continuum $\mathcal{L} \in C(A, 2^X)$ satisfying $\{q\} \in \mathcal{L}$ and $\mathcal{H}(C(K), \mathcal{L}) < \varepsilon$. Then the continuum $L = \bigcup \mathcal{L}$ satisfies all the required conditions, so the proof is complete. □

Theorem 2. *If the Cartesian product $X \times Y$ of nondegenerate continua X and Y is smooth, then each of the continua X and Y has the property of Kelley.*

Proof: Because of the symmetry it is enough to show that X has the property of Kelley. Assume $X \times Y$ is smooth at (p, q) and let $\varepsilon > 0$ be given. Choose $\delta > 0$ as in the definition of smoothness for $X \times Y$.

Let a continuum K, a point $x \in K$ and a point $y \in B_X(x, \delta)$ be given. Let $\{d_1, \ldots, d_n\}$ be an ε-net in K, i.e., for any point $z \in K$ there is an index $i \in \{1, \ldots, n\}$ such that $d(z, d_i) < \varepsilon$. Choose a point $q' \in Y \setminus \{q\}$. For any index $i \in \{1, \ldots n\}$ let $P_i = X \times \{q\} \cup \{d_i\} \times Y \cup K \times \{q'\}$. By smoothness of $X \times Y$ at (p, q) there is a continuum Q_i containing (p, q) and (y, q') such that $H(P_i, Q_i) < \varepsilon$. Denote by L_i the closure of the component of $N_{X\times Y}(K \times \{q'\}, \varepsilon) \cap Q_i$ that contains (y, q'). Then $L_i \subset Q_i \subset N_{X\times Y}(P_i, \varepsilon)$. Let $\pi : X \times Y \to X$ be the projection, and observe that $\pi(L_i)$ contains a point e_i satisfying $d(d_i, e_i) < \varepsilon$. Finally, let $L = \pi(L_1) \cup \cdots \cup \pi(L_n)$ and note that, since $y \in \pi(L_i)$ for every $i \in \{1, \ldots n\}$, the set L is a continuum containing the point y. Moreover, $L \subset N_X(K, \varepsilon)$ and for every point $z \in K$ there is an index $i \in \{1, \ldots, n\}$ such that $d(z, e_i) < 2\varepsilon$, so $H(K, L) < 2\varepsilon$. This finishes the proof of the property of Kelley of X. □

The converses of Theorems 1 and 2 in the case of 2^X are not true as can be seen by the following example.

Example 3. *There is a continuum X with the property of Kelley such that 2^X and $X \times X$ are not smooth.*

Proof: Let S denote the unit circle in the complex plane $\mathbb{C}$. Define functions f and g mapping $\mathbb{H} = [1, \infty)$ into $\mathbb{C}$ by

$$f(t) = (1 + 1/t)\exp(it) \quad \text{and} \quad g(t) = (1 - 1/t)\exp(-it),$$

and let $M = f(\mathbb{H})$ and $L = g(\mathbb{H})$. The space $X = M \cup S \cup L$ is a continuum in $\mathbb{C}$ having the property of Kelley. It is known (see [4, Example, p. 458]) that 2^X does not have the property of Kelley. More precisely, if $\mathcal{F}$ denotes the set of singletons, i.e., $\mathcal{F} = \{\{x\} : x \in X\}$, then there is no continuum $\mathcal{K}$ in 2^X with $\mathcal{H}(\mathcal{K}, \mathcal{F}) < 1/2$ and such that $\mathcal{K}$ contains a two-point set $\{p, q\}$ with $p \in M$ and $q \in L$. We will use this fact to prove that 2^X is not smooth. More generally, we infer that 2^X is not smooth at any point of $\mathcal{F}$. So, assume that 2^X is smooth at a set $A \in 2^X \setminus \mathcal{F}$. Denote $a = f(1)$ and let $\mathcal{A}$ be an order arc from $\{a\}$ to X and $\mathcal{B}$ — an order arc from A to X. Consider a continuum $\mathcal{L} = \mathcal{F} \cup \mathcal{A} \cup \mathcal{B}$. Let $\varepsilon \in (0, 1/6)$ be such that $B_X(a, 3\varepsilon)$ is connected and the Hausdorff distance between A and any singleton is greater than ε. Let δ satisfy the definition of the property of Kelley for this ε. Choose two points $p \in M$ and $q \in L$ such that $d(p, q) < \delta$, i.e., $H(\{p, q\}, p) < \delta$. Then, by smoothness of 2^X at A, there is a continuum $\mathcal{M}$ in 2^X that contains $\{p, q\}$ and A, and satisfies $\mathcal{H}(\mathcal{M}, \mathcal{L}) < \varepsilon$. Note that every component $\mathcal{C}$ of $\mathcal{M} \cap N_{2^X}(\mathcal{F}, \varepsilon)$ contains a point C whose distances to $\mathcal{F}$ and to $\mathcal{A}$ are less than ε. Then there are a point $c \in X$ such that $H(\{c\}, C) < \varepsilon$ and a set $B \in \mathcal{A}$ satisfying $H(B, C) < \varepsilon$. Thus $a \in B \subset N_X(C, \varepsilon) \subset B_X(c, 2\varepsilon)$, so $d(a, c) < 2\varepsilon$, and therefore $C \subset B_X(a, 3\varepsilon)$. Since by the choice of ε the ball $B_{2^X}(\{a\}, 3\varepsilon)$ is connected, the set

$$\mathcal{K} = \mathrm{cl}(\mathcal{M} \cap N_{2^X}(\mathcal{F}, \varepsilon)) \cup \mathrm{cl} B_{2^X}(\{a\}, 3\varepsilon)$$

is a continuum that contains $\{p, q\}$ with $\mathcal{H}(\mathcal{K}, \mathcal{F}) \leq 3\varepsilon < 1/2$. This contradicts the assertion mentioned above.

The proof for $X \times X$ is very similar. It was shown in [17, Example (4.7), p. 297] that $X \times X$ does not have the property of Kelley. More precisely, if $\mathcal{F} = \{(x, x) : x \in X\}$, then there is no continuum $\mathcal{K}$ in $X \times X$ with $\mathcal{H}(\mathcal{K}, \mathcal{F}) < 1/2$ and such that $(p, q) \in \mathcal{K}$ for $p \in M$ and $q \in L$. Assuming that $X \times X$ is smooth at (u, v), put $\mathcal{L} = \mathcal{F} \cup \{(u, x) : x \in X\}$. Choose $\varepsilon > 0$ as before and assume that $\mathcal{M}$ is a subcontinuum of 2^X with $\mathcal{H}(\mathcal{M}, \mathcal{L}) < \varepsilon$ and $(p, q) \in \mathcal{M}$. Then define

$$\mathcal{K} = \mathrm{cl}(\mathcal{M} \cap N_{2^X}(\mathcal{F}, \varepsilon)) \cup \mathrm{cl} B_{X \times X}((a, a), 3\varepsilon).$$

Note that such $\mathcal{K}$ satisfies all of the assumptions mentioned above. This contradicts the specified assertion. The proof is then complete.

Remark 4 The authors do not know if $C(X)$ is smooth when X has the property of Kelley. This is related to the question by S. B. Nadler whether $C(X)$ has the property of Kelley when X has the property of Kelley (see [16, (16.37), p. 558]). A positive answer to the Nadler's question would imply smoothness of $C(X)$ whenever X has the property of Kelley .

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