Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 24, Spring 1999

EXPONENTS AND ALMOST PERIODIC ORBITS

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ABSTRACT. We introduce the group of exponents of a map of the reals into a metric space and give conditions under which this group embeds in the first Čech cohomology group of the closure of the image of the map. We show that this group generalizes the subgroup of the reals generated by the Fourier-Bohr exponents of an almost periodic orbit and that any minimal almost periodic flow in a complete metric space is determined up to (topological) equivalence by this group. We also develop a way of associating groups with any self-homeomorphism of a metric space that generalizes the rotation number of an orientation-preserving homeomorphism of the circle with irrational rotation number.

1. INTRODUCTION

In this paper we shall introduce the group of exponents of a map of $\mathbb{R}$ into a space X. While this group is defined for any such map, it is most natural to consider in the case that X is metric, and we will assume in the following that all topological spaces under consideration are endowed with a metric. We reveal the topological significance of this group by showing

1991 *Mathematics Subject Classification.* Primary 58F25; Secondary 43A60; 22C05.

that under suitable conditions it embeds (sometimes properly) in the first Čech cohomology group of the closure of the image of the map. This group arises naturally from a study of almost periodic orbits, and the results of [LF] together with a thorough examination of the properties of this group allow us to give an intrinsic classification of all the minimal almost periodic flows occurring in complete metric spaces, see Theorem 4. This represents progress in the program of the classification of the limit sets of integral flows initiated by G.D. Birkhoff in 1912, see, e.g., [AM], Chapter 6. More generally, we show that the exponent group for the orbit of a compact minimal set determines the maximal (in a sense made precise) semiconjugate almost periodic flow, and so this group gives an idea of the flow shape of the minimal set.

We also calculate this exponent group for the orbits of the suspensions of orientation–preserving homeomorphisms $S^1 \to S^1$ with irrational rotation number. This demonstrates that if we associate the exponent group of the orbits of the suspension of any homeomorphism $X \to X$ then we obtain groups that in some sense generalize the rotation number.

2. The Exponent Group and its Origin

We begin by introducing the background material on almost periodic functions which naturally leads to the general exponent group. Throughout maps are assumed to be continuous. By a *flow* on X we mean a continuous group action of $(\mathbb{R}, +)$ on X, and $\pi : \mathbb{R} \to (\mathbb{R}/\mathbb{Z}) = S^1$ denotes the standard quotient map. We shall use the following terminology.

Definition 2.1. $S \subset \mathbb{R}$ is *relatively dense* if there is a $\lambda > 0$ such that for any $r \in \mathbb{R}$, $(r, r+\lambda) \cap S \neq \emptyset$.

Definition 2.2. $\tau \in \mathbb{R}$ is an ε – *almost period* of $f : \mathbb{R} \to X$ if

$$\sup_{t \in \mathbb{R}} \{d(f(t), f(t+\tau))\} \leq \varepsilon.$$

There is then the following definition due to H. Bohr.

Definition 2.3. A map $f : \mathbb{R} \to X$ is *almost periodic* if for every $\varepsilon > 0$ the set

$$S_\varepsilon = \{\tau \in \mathbb{R} \mid \tau \text{ is an } \varepsilon - \text{almost period of } f\}$$

is relatively dense.

Similarly, the orbit ϕ_x of the flow $\phi : \mathbb{R} \times X \to X$ is termed *almost periodic* when it is an almost periodic function $\mathbb{R} \to X$.

Definition 2.4. For a map $f : \mathbb{R} \to X$, we define a sequence of reals to be an *f-sequence* if the sequence $\{f(t_i)\}_{i \in \mathrm{N}}$ converges in X and $\{t_i\}$ is defined to be an *$f(0)$-sequence* if $\lim_i \{f(t_i)\} = f(0)$.

We now quote known results translated into our terminology. One has a choice between defining exponents mod2π or mod1, and we always choose the mod1 formulation (i.e., we use the map π as above instead of the map $t \mapsto \exp(it)$).

Theorem 1. *Given an almost periodic function $f : \mathbb{R} \to X$ which is the orbit of a flow in a complete metric space, there exists a countable subgroup $\mathfrak{M}_f$ of $(\mathbb{R}, +)$ such that:*

$$[\{t_i\} \subset \mathbb{R} \textit{ is an } f\textit{-sequence}] \Leftrightarrow [\{\pi(\lambda t_i)\}_i \textit{ converges in } S^1$$

$$\textit{for all } \lambda \in \mathfrak{M}_f],$$

and if X is a Banach space, $\mathfrak{M}_f$ may be taken to be the subgroup of $\mathbb{R}$ generated by the Fourier-Bohr exponents of f, (see, e.g., [LZ] *Chapter 3, Theorems 3-4).*

To apply the quoted theorem we associate points in $\overline{f(\mathbb{R})}$ with their orbits in the flow.

Theorem 2. *For two almost periodic functions $f : \mathbb{R} \to X$ and $g : \mathbb{R} \to Y$, we have: $[\mathfrak{M}_f = \mathfrak{M}_g] \Leftrightarrow [\{f(0) - \text{sequences}\} = \{g(0)\text{-sequences}\}]$; and $\mathfrak{M}_f$ is uniquely determined by f, (see, e.g.,* [LZ], *Chapter 2, Section 2.3).*

Definition 2.5. For an almost periodic function $f : \mathbb{R} \to X$, $\mathfrak{M}_f$ denotes the countable subgroup of $\mathbb{R}$ uniquely determined by f as above.

We now introduce our generalization of this group.

Definition 2.6. For a map $f : \mathbb{R} \to X$, we define the *group of exponents*, denoted $\mathcal{E}_f$, to be

$$\{\alpha \in \mathbb{R} \mid \{\pi(\alpha t_i)\} \text{ converges in } S^1 \text{ for all } f\text{-sequences } \{t_i\}\}.$$

Lemma 2.7. *$\mathcal{E}_f$ is a subgroup of $(\mathbb{R}, +)$.*

Proof: We know $0 \in \mathcal{E}_f \neq \emptyset$. Suppose that $\alpha, \beta \in \mathcal{E}_f$ and that $\{t_i\}$ is an f–sequence. Then $\{\pi((\alpha - \beta) t_i)\} = \{\pi(\alpha t_i) - \pi(\beta t_i)\}$. This sequence converges in S^1 since $-$ is continuous on $S^1 \times S^1$. □

Lemma 2.8. *If f is an almost periodic map into a complete metric space, then $\mathfrak{M}_f = \mathcal{E}_f$.*

Proof: Let f be such a function. Given $\alpha \in \mathfrak{M}_f$, $\{\pi(\alpha t_i)\}$ converges in S^1 for all f–sequences $\{t_i\}$, and so $\mathfrak{M}_f \sqsubset \mathcal{E}_f$. Suppose then that $\alpha \in \mathcal{E}_f - \mathfrak{M}_f$. Let $\mathfrak{M}'_f$ be the subgroup of $(\mathbb{R}, +)$ generated by $\mathfrak{M}_f \cup \{\alpha\}$. Then given any f-sequence $\{t_i\}$, $\{\pi(\lambda t_i)\}$ converges in S^1 for all $\lambda \in \mathfrak{M}'_f$. And if for a given sequence $\{t_i\}$ of real numbers we have that for each $\lambda \in \mathfrak{M}'_f$ the sequence $\{\pi(\lambda t_i)\}$ converges in S^1, then for each $\lambda \in \mathfrak{M}_f$ the sequence $\{\pi(\lambda t_i)\}$ converges in S^1 since $\mathfrak{M}_f \subset \mathfrak{M}'_f$, and so $\{t_i\}$ is an $f-$sequence. Therefore $\mathfrak{M}'_f$ meets the conditions that uniquely determine $\mathfrak{M}_f$, and so $\mathfrak{M}'_f = \mathfrak{M}_f$, contradicting the choice of α. Thus, no such α can exist and $\mathcal{E}_f = \mathfrak{M}_f$. □

Now we shall explore some properties of this more general exponent group. Here $[M; S^1]$ denotes the group of homotopy classes of maps $M \to S^1$, where for convenience we use the group structure induced by point-wise addition of maps.

Definition 2.9. We define a map $f : \mathbb{R} \to X$ to be *stable* if there is an unbounded f-sequence $\{t_i\}$.

Theorem 3. *If $f : \mathbb{R} \to X$ is a stable map and $M = \overline{f(\mathbb{R})}$, the map $\iota : \mathcal{E}_f \to [M; S^1]$ given by*

$\alpha \mapsto [f_\alpha]$, *where* $f_\alpha : M \to S^1$; $f_\alpha(\{\lim f(t_i)\}) = \lim\{\pi(\alpha t_i)\}$

and $[f_\alpha]$ denotes the homotopy class of f_α, is an embedding.

Proof:

1. ι is well-defined:

Let $\alpha \in \mathcal{E}_f$ be given. If $\lim\{f(t_i)\} = \lim\{f(t'_i)\} = x \in M$ are two representations of a point of M, then $\lim\{\pi(\alpha t_i)\} = \xi$ and $\lim\{\pi(\alpha t'_i)\} = \xi'$ both exist by the definition of $\mathcal{E}_f$. Then with $s_{2i} = t_i$ and $s_{2i-1} = t'_i$, $\lim\{f(s_i)\} = x \Rightarrow \lim\{\pi(\alpha s_i)\}$ exists, which is only possible if $\xi = \xi'$. Thus f_α is a well-defined function $M \to S^1$. To see that f_α is continuous, consider a convergent sequence $\{\lim_i\{f(t_i^j)\}\}_j = \{x^j\}_j \to x$. Then for each $j \in \mathbb{N}$ we may choose i_j so that:

$$d\left(f\left(t_{i_j}^j\right), x^j\right) < \frac{1}{j} \text{ and } d_{S^1}\left(f_\alpha\left(x^j\right), \pi\left(\alpha t_{i_j}^j\right)\right) < \frac{1}{j}.$$

Then $\lim_j\{f\left(t_{i_j}^j\right)\}_j = x$ and

$$f_\alpha(x) \stackrel{\text{def}}{=} \lim_j\{\pi\left(\alpha t_{i_j}^j\right)\} = \lim_j\{f_\alpha\left(x^j\right)\},$$

demonstrating that f_α is continuous.

2. ι is a homomorphism:

Let $\alpha, \beta \in \mathcal{E}_f$. Then for $x = \lim\{f(t_i)\} \in M$

$$\begin{aligned} f_{\alpha+\beta}(x) = \lim\{\pi((\alpha+\beta)t_i)\} &= \lim\{\pi(\alpha t_i)\} + \lim\{\pi(\beta t_i)\} \\ &= f_\alpha(x) + f_\beta(x). \end{aligned}$$

3. $\ker \iota = \{0\}$:

Suppose $0 \neq \alpha \in \mathcal{E}_f$ and $[f_\alpha] = [\text{constant map}]$. Since $\pi : \mathbb{R} \to S^1$ is a fibration and since we can lift the constant map, we can lift f_α with a map $g : (M, f(0)) \to (\mathbb{R}, 0)$ making the following diagram commute:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{g}{\nearrow} & \downarrow \pi \\ M & \underset{f_\alpha}{\longrightarrow} & S^1 \end{array}$$

This leads to the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{R} & \overset{\frac{1}{\alpha}}{\to} & \mathbb{R} & \overset{f}{\to} & M & \overset{g}{\to} & \mathbb{R} \\ \downarrow^{\pi} & & & & & \searrow^{f_\alpha} & \downarrow^{\pi} \\ S^1 & & & \overset{id}{\longrightarrow} & & & S^1 \end{array},$$

where $gf\frac{1}{\alpha} : (\mathbb{R},0)\to(\mathbb{R},0); t \mapsto g \circ f\left(\frac{t}{\alpha}\right)$. From this it follows that $gf\frac{1}{\alpha}$ is a lift of id_{S^1}. Such a lift $(\mathbb{R},0)\to(\mathbb{R},0)$ is uniquely determined [S], 2.2 Lemma 4 (specifying that $0 \mapsto 0$ makes the lift unique), and $id_{\mathbb{R}}$ also provides such a lift. Thus $gf\frac{1}{\alpha} = id_{\mathbb{R}}$. Composing both sides of this equality on the right with multiplication by α, we obtain: gf = multiplication by α. Since f is stable, there is an unbounded sequence of numbers $\{t_i\}$ such that $\lim\{f(t_i)\} = x$. And so

$$g(x) = g(\lim\{f(t_i)\}) = \lim\{g \circ f(t_i)\} = \lim\{\alpha t_i\},$$

which is not well-defined since $\{t_i\}$ and hence $\{\alpha t_i\}$ is an unbounded sequence that does not converge. This contradicts the continuity of g. □

Notice that when M is compact f is automatically stable and $\mathcal{E}_f$ is countable since $[M; S^1]$ is then countable. While we shall be principally interested in maps which are orbits of a flow, we give a more general example first.

Example 2.10. For $t \in \mathbb{R}$, $\lfloor t \rfloor \overset{\text{def}}{=}$(the greatest integer less than t) and $\rfloor t \lfloor \overset{\text{def}}{=} t - \lfloor t \rfloor$. Then we define the map $g : \mathbb{R} \to \mathbb{R}^2$;

$$t \mapsto \begin{cases} \left(\rfloor t \lfloor \cdot \frac{1}{2^{\lfloor t \rfloor +1}} + (1 - \rfloor t \lfloor) \cdot \frac{1}{2^{\lfloor t \rfloor}}, \rfloor t \lfloor\right) & \text{if } \lfloor t \rfloor \text{ is odd} \\ \left(\rfloor t \lfloor \cdot \frac{1}{2^{\lfloor t \rfloor +1}} + (1 - \rfloor t \lfloor) \cdot \frac{1}{2^{\lfloor t \rfloor}}, 1 - \rfloor t \lfloor\right) & \text{if } \lfloor t \rfloor \text{ is even} \end{cases}$$

and $X \stackrel{\text{def}}{=} g(\mathbb{R}) \cup \{(0, \frac{1}{2})\}$, which is homeomorphic to

$$\{(x, \sin(1/x)) \mid x \in (0, \infty)\} \cup \{(0,0)\} \subset \mathbb{R}^2.$$

Then $f : \dot{\mathbb{R}} \to X; t \mapsto g(t)$ is stable since $\{f(n + \frac{1}{2})\}_{n\in\mathbb{N}} \to (0, \frac{1}{2})$. And if $\{f(t_i)\} \to x \in X$, we have two cases:

1. $x = f(t)$ for some $t \in \mathbb{R}$.

In this case we have the neighborhood $U \stackrel{\text{def}}{=} \{(w, y) \in X \mid \frac{1}{2^{\lfloor t \rfloor + 1}} \leq w \leq \frac{1}{2^{\lfloor t \rfloor - 1}}\}$ of $f(t)$ and f maps $[\lfloor t \rfloor - 1, \lfloor t \rfloor + 1]$ homeomorphically onto U. Thus, we must have $\{t_i\} \to t$ and so $\{\pi(1 \cdot t_i)\} \to \{\pi(1 \cdot t)\}$ in S^1.

2. $x = (0, \frac{1}{2})$.

In this case, with $\delta_i \stackrel{\text{def}}{=} \rfloor t_i \lfloor$, we have that $\{\delta_i\} \to \frac{1}{2}$, and so $\{\pi(1 \cdot t_i)\} = \{\pi(\delta_i)\} \to \pi(\frac{1}{2})$ in S^1.

Thus, we must have 1 and hence $\mathbb{Z} \subset \mathcal{E}_f$. Let $\alpha = \frac{p}{q} \in \mathbb{Q} - \mathbb{Z}$. We then construct the f-sequence $\{t_i\}$,

$$t_i \stackrel{\text{def}}{=} \begin{cases} qi + \frac{1}{2} & \text{if } i \text{ is odd} \\ qi + \frac{3}{2} & \text{if } i \text{ is even} \end{cases}$$

Then

$$\pi(\alpha \cdot t_i) = \begin{cases} \pi\left(\frac{p}{2q}\right) & \text{if } i \text{ is odd} \\ \pi\left(\frac{p}{2q} + \frac{p}{q}\right) & \text{if } i \text{ is even} \end{cases}$$

and $\pi\left(\frac{p}{2q}\right) \neq \pi\left(\frac{p}{2q} + \frac{p}{q}\right)$ in S^1 since $\frac{p}{q} \in \mathbb{Q} - \mathbb{Z}$, and so $\{\pi(\alpha \cdot t_i)\}$ does not converge in S^1 and $\alpha \notin \mathcal{E}_f$. Suppose then that $\alpha \in \mathbb{R} - \mathbb{Q}$. By Kronecker's Theorem (see, e.g., [LZ], Chapter 3.1) there are then two sequences of integers $\{k_i\}$ and $\{\ell_i\}$ satisfying

1. $\{\pi(\alpha \cdot k_i)\} \to \pi(0)$ in S^1 and $\{\pi(\alpha \cdot \ell_i)\} \to \pi(\frac{1}{3})$ in S^1
2. $k_i, \ell_i > i$ for all $i \in \mathbb{N}$.

Then with

$$t_i \stackrel{\text{def}}{=} \begin{cases} k_i + \frac{1}{2} & \text{if } i \text{ is odd} \\ \ell_i + \frac{1}{2} & \text{if } i \text{ is even} \end{cases},$$

we have that $\{f(t_i)\} \to \left(0, \frac{1}{2}\right)$ while $\{\pi(\alpha \cdot t_i)\}$ does not converge in S^1. Thus, $\mathbb{Z} = \mathcal{E}_f$ and $[X; S^1]$ contains a copy of $\mathbb{Z}$.

It is perhaps of interest to calculate $\check{H}^1(X) \cong [X; S^1]$: we can find a cofinal sequence of nerves consisting of a line attached to the common point of a bouquet of countably infinitely many circles in the weak topology, with each circle collapsing to the line in all refinements after a certain point in the sequence. And so $\check{H}^1(X)$ is the direct limit of a sequence of groups isomorphic with $Hom\left(\oplus_{i=1}^{\infty}\mathbb{Z}, \mathbb{Z}\right) \cong \prod_{i=1}^{\infty} \mathbb{Z}$ with the bonding maps sending a finite number of $\mathbb{Z}$ factors to 0 in the group farther along in the sequence, and so $\check{H}^1(X)$ is isomorphic to the group of sequences of integers with sequences identified which are eventually the same.

2.1 $\mathfrak{M}_f$ Determines the Equivalence Class of ϕ. In the sequel we shall assume all spaces are complete in their endowed metric unless otherwise stated. The following basic result does not seem to be proven anywhere in the literature, but it seems to be implicitly assumed in [LZ]. Cartwright [C] does observe that, "A flow is uniquely determined by its coefficients and its exponents." Notice that the coefficients need not be brought into the picture when determining the (topological) equivalence class of the corresponding almost periodic flow. We shall use the following terminology and notation.

Definition 2.11. The flow ψ on Y is *semiconjugate* to the flow ϕ on X if there exists a surjective map $h : X \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R} \times X & \xrightarrow{\phi} & X \\ {\scriptstyle (\alpha,h)} \downarrow & & \downarrow {\scriptstyle h} \\ \mathbb{R} \times Y & \xrightarrow{\psi} & Y \end{array},$$

where α is multiplication by $a \in \mathbb{R} - \{0\}$, and we write $\alpha \times h : \phi \overset{sc}{\succeq} \psi$. And if h is a homeomorphism and if $a > 0$, we write $\alpha \times h : \phi \overset{equiv}{\approx} \psi$ (or $\phi \overset{equiv}{\approx} \psi$) and we say that ϕ and ψ are *equivalent.*

Theorem 4. *If $f = \phi_x : \mathbb{R} \to X$ is an almost periodic orbit of a flow $\phi : \mathbb{R} \times X \to X$ on (X, d) and $g = \psi_y : \mathbb{R} \to Y$ is an almost periodic orbit of a flow $\psi : \mathbb{R} \times Y \to Y$ on (Y, d') and $\mathfrak{M}_f = \mathfrak{M}_g$, then $id_{\mathbb{R}} \times h : \phi|_{\mathbb{R} \times \overline{f(\mathbb{R})}} \overset{equiv}{\approx} \psi|_{\mathbb{R} \times \overline{g(\mathbb{R})}}$, where h is a topological isomorphism of $\overline{f(\mathbb{R})}$ and $\overline{g(\mathbb{R})}$ with the group structures inherited from the orbits f and g.*

Proof: The group operations on $(\overline{f(\mathbb{R})}, +)$ and $(\overline{g(\mathbb{R})}, +)$ discovered by Nemytski [NS], V, Thm 8.16, are given by $\lim\{f(t_i)\} + \lim\{f(t'_i)\} = \lim\{f(t_i + t'_i)\}$ and $\lim\{g(t_i)\} + \lim\{g(t'_i)\} = \lim\{g(t_i + t'_i)\}$ respectively . The continuity of ϕ then yields that $\phi(t, \lim\{f(t_i)\}) = \phi(\lim\{(t, f(t_i))\}) = \lim\{\phi(t, f(t_i))\} = \lim\{f(t + t_i)\}$. Define $h : \overline{f(\mathbb{R})} \to \overline{g(\mathbb{R})}$ by $\lim\{f(t_i)\} \mapsto \lim\{g(t_i)\}$. Since $\mathfrak{M}_f = \mathfrak{M}_g$, [$\lim\{f(t_i)\}$ exists $\Leftrightarrow$ $\lim\{g(t_i)\}$ exists]. Therefore, h will be well-defined if we can show that when $a = \lim\{f(t_i)\}$ and $a = \lim\{f(t'_i)\}$, then $\lim\{g(t_i)\} = \lim\{g(t'_i)\}$. If $a = \lim\{f(t_i)\}$ and $a = \lim\{f(t'_i)\}$, $f(0) = a - a = \lim\{f(t_i - t'_i)\}$, which implies that $\{t_i - t'_i\}$ is an $f(0)$-sequence and hence a $g(0)$-sequence, see Theorem 2. From this it follows that

$$\lim\{g(t_i)\} - \lim\{g(t'_i)\} = \lim\{g(t_i - t'_i)\} = g(0)$$

and hence that $\lim\{g(t_i)\} = \lim\{g(t'_i)\}$. Thus, h is well-defined and a similar argument shows that $k : \overline{g(\mathbb{R})} \to \overline{f(\mathbb{R})}$; $\lim\{g(t_i)\} \mapsto \lim\{f(t_i)\}$ is well-defined, and $h \circ k = id_{\overline{g(\mathbb{R})}}$ and $k \circ h = id_{\overline{f(\mathbb{R})}}$, demonstrating that h is a bijection.

If

$$a = \lim\{a_j\} = \lim_j \left\{ \lim_i \{f(t_i^j)\} \right\}$$

in $\overline{f(\mathbb{R})}$, then

$$a = \lim_j \left\{ f\left(t^j_{i_j}\right)\right\}$$

for i_j satisfying $d\left(f\left(t^j_n\right), a_j\right) < \frac{1}{j}$ and $d'\left(g\left(t^j_n\right), h\left(a_j\right)\right) < \frac{1}{j}$ for all $n \geq i_j$. Such i_j exist since

$$a_j = \lim_i \left\{ f\left(t^j_i\right)\right\} \text{ and } h\left(a_j\right) = \lim_i \left\{ g\left(t^j_i\right)\right\}.$$

Then

$$h(a) = \lim_j \left\{ g\left(t^j_{i_j}\right)\right\} = \lim_j \left\{ \lim_i \left\{ g\left(t^j_i\right)\right\}\right\} = \lim_j \left\{ h\left(a_j\right)\right\},$$

and so h is continuous and therefore a homeomorphism. And since

$$\begin{aligned} h\left(\lim\{f(t_i)\} + \lim\{f(t'_i)\}\right) &= h\left(\lim\{f(t_i + t'_i)\}\right) = \\ \lim\{g(t_i + t'_i)\} &= \lim\{g(t_i)\} + \lim\{g(t'_i)\} \\ &= h\left(\lim\{f(t_i)\}\right) + h\left(\lim\{f(t'_i)\}\right), \end{aligned}$$

we see that h is a homomorphism and thus a topological isomorphism. Also:

$$\begin{aligned} h\left(\phi\left(t, \lim\{f(t_i)\}\right)\right) &= h\left(\lim\{f\left(t + t_i\right)\}\right) = \lim\{g\left(t + t_i\right)\} = \\ \psi\left(t, \lim\{g(t_i)\}\right) &= \psi\left(t, h\left(\lim\{f(t_i)\}\right)\right), \end{aligned}$$

from which it follows that $id_{\mathbb{R}} \times h$ provides the desired equivalence. □

Notice that the examples X = one trajectory of an irrational flow on the torus and Y = two trajectories of the same irrational flow on the torus show that the theorem does not hold as stated without the completeness requirement.

3. Properties of $\mathcal{E}_f$ in Flows

Theorem 5. *If f and g are the orbits of x and y respectively of the flow ϕ on X and if $y \in \overline{f(\mathbb{R})}$, then $\mathcal{E}_f \subset \mathcal{E}_g$. If both x and y are contained in a minimal set M, then $\mathcal{E}_f = \mathcal{E}_g$.*

Proof: Let $\{f(t_i)\} \to y$ and suppose $\{s_i\}$ is a g-sequence with $\{g(s_i)\} \to \xi$. Then

$$\begin{aligned}\xi &= \lim_i \{\phi(s_i, y)\} = \lim_i \left\{\phi\left(s_i, \lim_j \{\phi(t_j, x)\}\right)\right\} \\ &= \lim_i \{\phi(s_i, \phi(t_{j_i}, x))\} = \lim_i \{\phi(s_i + t_{j_i}, x)\} \\ &= \lim_i \{f(s_i + t_{j_i})\}\end{aligned}$$

for an appropriately chosen subsequence $\{t_{j_i}\}$ of $\{t_j\}$. Then we have that both $\{t_{j_i}\}$ and $\{s_i + t_{j_i}\}$ are f-sequences. Let $\lambda \in \mathcal{E}_f$. Then we have that both $\{\pi(\lambda t_{j_i})\}_i$ and $\{\pi(\lambda(s_i + t_{j_i}))\}_i = \{\pi(\lambda s_i) + \pi(\lambda t_{j_i})\}_i$ converge. But then the sequence $\{\pi(\lambda s_i)\}_i = \{\pi(\lambda s_i) + \pi(\lambda t_{j_i}) - \pi(\lambda t_{j_i})\}_i$ converges since $-$ is continuous on $S^1 \times S^1$. Thus, $\mathcal{E}_f \subset \mathcal{E}_g$. If both x and y are contained in a minimal set M, then $x \in \overline{g(\mathbb{R})} = M$ and $y \in \overline{f(\mathbb{R})} = M$ and so we have both $\mathcal{E}_g \subset \mathcal{E}_f$ and $\mathcal{E}_f \subset \mathcal{E}_g$. □

We now provide an example to show that the containment can be proper.

Example 3.1. Let $X = S^1 \times \mathbb{R}$ be parameterized by (θ, r) and for non–zero real numbers α and β let $\phi(\alpha, \beta)$ be the flow on X generated by the vector field

$$\frac{d\theta}{dt} = \alpha r + \beta(1 - r) \quad \text{and} \quad \frac{dr}{dt} = r(1 - r).$$

Then the orbit g of the point $(\pi(0), 1)$ is given by $t \mapsto (\pi(\alpha t), 1)$ and the orbit g' of the point $(\pi(0), 0)$ is given by $t \mapsto (\pi(\beta t), 0)$. And so $\mathcal{E}_g = \langle \alpha \rangle_{\mathbb{Z}}$ and $\mathcal{E}_{g'} = \langle \beta \rangle_{\mathbb{Z}}$. The orbit f of the point $\left(\pi(0), \frac{1}{2}\right)$ is given by

$$t \mapsto \left(\pi\left(\alpha \ln\left(e^t + 1\right) - \beta \ln\left(e^{-t} + 1\right) + (\beta - \alpha)\ln 2\right), \frac{e^t}{e^t + 1}\right).$$

And so $\{(\pi(0), 1), (\pi(0), 0)\} \subset \overline{f(\mathbb{R})}$. If we choose α and β to be rationally independent then we have by the above $\mathcal{E}_f \subset \mathcal{E}_g \cap \mathcal{E}_{g'} = \{0\}$, and so $\mathcal{E}_f = \{0\}$ and we have proper containment in this case. We mention in passing that we can define exponent groups $\mathcal{E}_f^+$ for semi-orbits $f : [0, \infty) \to X$

in almost exactly the same way as we have defined exponents for orbits, and when we do so, for the semi-orbit of the f in this example we obtain: $\mathcal{E}_f^+ = \langle\alpha\rangle_{\mathbb{Z}}$, the subgroup of $(\mathbb{R},+)$ generated by α .

Lemma 3.2. *If $\alpha \times h : \phi \overset{sc}{\succeq} \psi$, then for any $x \in X$ we have:*

$$\mathcal{E}_{\phi_x} \supset \left\{ a\lambda \mid \lambda \in \mathcal{E}_{\psi_{h(x)}} \right\}.$$

And so $\mathcal{E}_{\phi_x} \supset \mathcal{E}_{\psi_{h(x)}}$ if $a = 1$.

Proof: By hypothesis we have $h \circ \phi_x = \psi_{h(x)} \circ \alpha$. Suppose $\{t_i\}$ is a ϕ_x-sequence. Then $\lim_i \left\{ \psi_{h(x)}(at_i) \right\} = \lim_i \{ h \circ \phi_x(t_i) \}$ and so $\{at_i\}$ is a $\psi_{h(x)}$-sequence. Then for any $\lambda \in \mathcal{E}_{\psi_{h(x)}}$ we have that $\{\pi(a\lambda t_i)\}_i$ converges. □

Corollary 6. *If $(\alpha \times h) : \phi \overset{equiv}{\approx} \psi$, then for any $x \in X$ we have $\mathcal{E}_{\phi_x} = \left\{ a\lambda \mid \lambda \in \mathcal{E}_{\psi_{h(x)}} \right\}$. And so $\mathcal{E}_{\phi_x} = \mathcal{E}_{\psi_{h(x)}}$ if $a = 1$.*

4. Constructing a Flow Dual to a Given Exponent Group

For the sake of uniformity, we shall make use of the terminology and notation found in [F]. Here $\mathbf{T}^\kappa$ denotes the κ–fold product of S^1 with points $\langle x_1, x_2, ...\rangle$, and $\Phi_{\overline{M}}^{\omega}$ is the linear flow on $\sum_{\overline{M}}$

$$(t,x) \mapsto \pi_{\overline{M}}(\omega t) + x,$$

as defined in [LF].

Definition 4.1. Given a countable subgroup of the reals $H = \{h_1 = 0, h_2, ...\}$ with a maximal independent set $B = \{b_i\}_{i=1}^{\kappa}$, we define the *$B$-sequence* of H to be the direct sequence $\{H^i, \beta_i^j\}$ with

$$H^1 = \langle B\rangle_{\mathbb{Z}}, H^2 = \langle B \cup \{h_2\}\rangle_{\mathbb{Z}}, ..., H^n = \langle H^{n-1} \cup \{h_n\}\rangle_{\mathbb{Z}}, ...$$

and with $\beta_i^j : H^i \hookrightarrow H^j$ inclusion (here $\langle S\rangle_{\mathbb{Z}}$ denotes the subgroup of H generated by the set S).

By standard results, H is isomorphic to the direct limit of any B -sequence of H. Also, $\kappa = r_0(H)$ (the torsion-free rank of H), and so the cardinality of B (namely, κ) is uniquely determined by H [F], III, 16.3.

Lemma 4.2. *If $\{H^i, \beta_i^j\}$ is the B-sequence of the countable subgroup of the reals $H = \{h_1 = 0, h_2, ...\}$, and if $B = \{b_i\}_{i=1}^{\kappa}$ has cardinality κ, then for each $i \in \mathbb{N}$ the group H^i is isomorphic to $\bigoplus_{j=1}^{\kappa} \langle b_j^i \rangle_{\mathbb{Z}} \cong \bigoplus_{j=1}^{\kappa} \mathbb{Z}$ for some $b_j^i \in H^i$ and each bonding map β_i^{i+1} can be represented by: (1) an $n \times n$ invertible matrix $(M_i)^T$ with integer entries if $\kappa = n < \infty$ or (2) a map $(M_i)^T \times id$, where M_i is an invertible $n_i \times n_i$ matrix with integer entries if $\kappa = \infty$.*

Proof: We shall only treat the case $\kappa = \infty$, the finite case being handled similarly. Proceeding by induction, as $\{b_j^{i-1}\}_{j=1}^{\infty}$ is a maximal independent set of H, we have that $h_i = r_1 b_1^{i-1} + \cdots + r_k b_k^{i-1}$ for some $k \in \mathbb{N}$ and rationals $r_1, ..., r_k$. With $M' \stackrel{\text{def}}{=} \{b_j^{i-1}\}_{j=1}^{k}$ and $A \stackrel{\text{def}}{=} \langle M' \cup \{h_i\} \rangle_{\mathbb{Z}}$, we have that M' is an independent subset of A and any element of $\langle h_i \rangle_{\mathbb{Z}}$ depends on M'. Thus, M' is a maximal independent subset of A. The finitely generated group A is the direct sum of a finite number (k') of cyclic groups, and the invariance of $r_0(A) = |M'|$ implies that $k' = |M'| = k$: $A \cong \bigoplus_{j=1}^{k} \langle b_j^i \rangle_{\mathbb{Z}} \cong \bigoplus_{j=1}^{k} \mathbb{Z}$, for some $b_1^i, ..., b_k^i$ contained in A. Then for $j > k$, let $b_j^i \stackrel{\text{def}}{=} b_j^{i-1}$. We need to show that $\{b_j^i\}_{j=1}^{\infty}$ is an independent subset of H^i. If we have a relation

$$(*) \;\; n_1 b_1^i + \cdots + n_k b_k^i + \cdots + n_\ell b_\ell^i = 0,$$

where $n_s \in \mathbb{Z}$, then for each $j \leq k$ we have $b_j^i = r_1^j b_1^{i-1} + \cdots + r_k^j b_k^{i-1}$ for rational numbers r_s^j since h_i may be expressed this way and $b_j^i \in \langle M' \cup \{h_i\} \rangle_{\mathbb{Z}}$. Our original relation then leads to a relation of the form $\rho_1 b_1^{i-1} + \cdots + \rho_k b_k^{i-1} + n_{k+1} b_{k+1}^{i-1} + \cdots + n_\ell b_\ell^{i-1} = 0$, where each ρ_s is rational. Multiplying through by

$\lambda \stackrel{\text{def}}{=}$ (lcm of the denominators of the ρ_s), we obtain:

$$\nu_1 b_1^{i-1} + \cdots + \nu_k b_k^{i-1} + \lambda n_{k+1} b_{k+1}^{i-1} + \cdots + \lambda n_\ell b_\ell^{i-1} = 0,$$

where $\nu_s \in \mathbb{Z}$. The independence of $\{b_j^{i-1}\}_{j=1}^{\kappa}$ then yields that

$$\lambda n_{k+1} = \cdots = \lambda n_\ell = 0 \Rightarrow n_{k+1} = \cdots = n_\ell = 0.$$

Our original relation $(*)$ is then of the form $n_1 b_1^i + \cdots + n_k b_k^i = 0$, and since $\{b_j^i\}_{j=1}^k$ is independent we must have $n_1 = \cdots = n_k = 0$. By definition we then have that $\{b_j^i\}_{j=1}^\infty$ is an independent subset of H^i and $H^i \cong \bigoplus_{j=1}^\infty \langle b_j^i \rangle_\mathbb{Z} \cong \bigoplus_{j=1}^\infty \mathbb{Z}$. And if we define the $k \times k$ matrix $M_{i-1} = (a_{rs})$ with integer entries by the condition that for each $r = 1, ..., k$ we have

$$a_{r1} b_1^i + a_{r2} b_2^i + \cdots + a_{rk} b_k^i = b_r^{i-1},$$

then the entries of M_{i-1} are uniquely determined since $\{b_j^i\}_{j=1}^\infty$ is independent. The map β_{i-1}^i may then be said to be represented by $(M_{i-1})^T \times id$. That M_{i-1} is invertible follows from the fact that there are rationals q_{rs} satisfying

$$\begin{pmatrix} q_{11} & \cdots & q_{1k} \\ & \vdots & \\ q_{k1} & \cdots & q_{kk} \end{pmatrix} \begin{pmatrix} b_1^{i-1} \\ \vdots \\ b_k^{i-1} \end{pmatrix} = \begin{pmatrix} b_1^i \\ \vdots \\ b_k^i \end{pmatrix}$$

by the maximality of M' in A. □

Thus, given a B-sequence $\{H^i, \beta_i^j\}$ of a countable subgroup of the reals H, where for each $i \in \mathbb{N}$ $H^i \cong \bigoplus_{j=1}^\kappa \mathbb{Z}$ and the bonding map β_i^{i+1} is represented by $(M_i)^T$ or $(M_i)^T \times id$, there is the Pontryagin dual inverse sequence $\{G_i, \widehat{\beta_i^j}\}$, where for each $i \in \mathbb{N}$, $G_i \cong \mathbf{T}^\kappa$ and $\widehat{\beta_i^{i+1}}$ is represented by M_i or $M_i \times id$, see, e.g., [K]. Each of the bonding maps $\widehat{\beta_i^{i+1}}$ will be epimorphic since the corresponding matrix M_i is invertible. Hence, the following is well-defined.

Definition 4.3. We define the *B-dual* of the B-sequence of the countable subgroup of the reals $H = \{h_1 = 0, h_2, ...\}$ with the bases $\{b_j^i\}$ for the H^i of cardinality κ to be the κ-solenoid

$\sum_{\overline{M}}$ which is the inverse limit of the dual inverse sequence $\{G_i, \alpha_i^j\}$, where $G_i = \mathbf{T}^\kappa$ for all $i \in \mathbb{N}$ and the maps α_i^{i+1} are represented by the transposes of the maps β_i^{i+1} as given with respect to the bases $\{b_j^i\}_{j=1}^\kappa$ and $\{b_j^{i+1}\}_{j=1}^\kappa$.

Theorem 7. *If f is an orbit of an almost periodic flow ϕ on (X,d) with group of exponents $\mathfrak{M}_f = \{h_1 = 0, h_2, ...\}$ and if $B = \{b_1, b_2, ...\}$ is a maximal independent subset of $\mathfrak{M}_f$ and if $\sum_{\overline{M}}$ is the B-dual of the B-sequence of $\mathfrak{M}_f$ with the bases $\{b_j^i\}$ for the H^i as above, then $\phi \overset{equiv}{\approx} \Phi_{\overline{M}}^\beta$, where $\beta = (b_1, b_2, ...)$.*

Proof: Since B is independent, β is irrational and each trajectory of $\Phi_{\overline{M}}^\beta$ is dense in $\sum_{\overline{M}}$. As $\sum_{\overline{M}}$ is compact and distances are preserved by time t maps of $\Phi_{\overline{M}}^\beta$, the orbits of $\Phi_{\overline{M}}^\beta$ are almost periodic, see, e.g., [NS], V, 8.12, and so we need only show that the group of exponents $\mathfrak{M}_g$ for the orbit g of $e_{\overline{M}}$ in the flow $\Phi_{\overline{M}}^\beta$ is the same as $\mathfrak{M}_f$, see Theorem 2. In light of Theorems 2 and 2, we need to then show that

$$[\{\pi_{\overline{M}}(t_i\beta)\} \to e_{\overline{M}}] \Leftrightarrow [\{\pi(\mu t_i)\} \to \pi(0) \text{ in } S^1 \text{ for all } \mu \in \mathfrak{M}_f]$$

Let $\{H^i, \beta_i^j\}$ be the B-sequence for $\mathfrak{M}_f$ and choose bases $\{b_j^i\}_{j=1}^\kappa$ for the groups H^i with $b_j^1 \overset{\text{def}}{=} b_j$ for all j. This at the same time determines the representation of the β_i^{i+1} by matrices $(M_i)^T$. Since the bonding map $\widehat{\beta_i^{i+1}}$ of $\sum_{\overline{M}}$ is represented by M_i (or $M_i \times id$), we have:

$$\pi_{\overline{M}}(t\beta) = (\langle\pi(tb_1^1), \pi(tb_2^1), ...\rangle, \langle\pi(tb_1^2), \pi(tb_2^2), ...\rangle, ...) \in \prod_{i=1}^\infty \mathbf{T}^\kappa$$

Thus, $\cup_i \{b_j^i\}_{j=1}^\kappa$ generates $\mathfrak{M}_g$ as a subgroup of $(\mathbb{R}, +)$. And since $\cup_i \{b_j^i\}_{j=1}^\kappa$ generates $\mathfrak{M}_f$ as a subgroup of $(\mathbb{R}, +)$, we have what we need. □

Notice that this implies that

$$\mathfrak{M}_f = \mathfrak{M}_g \cong \widehat{\sum_{\overline{M}}} \cong \check{H}^1\left(\sum_{\overline{M}}\right) = \check{H}^1(\overline{g(\mathbb{R})}) \cong \check{H}^1(\overline{f(\mathbb{R})}).$$

The following corollary of these results gives a specific form for the inverse limit representation of a compact connected abelian topological group. That some inverse limit representation is possible is guaranteed by a theorem of Pontryagin, [P], Theorem 68.

Corollary 8. *Any metric compact connected abelian topological group is a κ-solenoid for some $\kappa \leq \infty$.*

Proof: This follows from the above and the converse in Nemytskii's Theorem [NS], V, 8.16. □

We are now in a position to give our intrinsic classification of minimal almost periodic flows. The notion of equivalence is generally too strict for classification, and two flows ϕ and ψ on X and Y respectively are said to be *topologically equivalent* (denoted $\phi \overset{top}{\approx} \psi$) when there is a homeomorphism $h : X \to Y$ which maps orbits of ϕ onto orbits of ψ in such a way that the orientation of orbits is preserved. This is the notion of equivalence discussed in [AM], Chapter 6 for the program of Birkhoff. In general, equivalence and topological equivalence are different, but in [LF] we showed that two minimal almost periodic flows are topologically equivalent if and only if they are equivalent, and any two equivalent flows have, up to a non-zero multiplicative factor, the same exponent group, see Corollary 3. And now that we know that any minimal almost periodic flow is equivalent with a linear flow on some κ−solenoid, we are led to the following conclusion.

Theorem 9. *If ϕ and ψ are minimal almost periodic flows with exponent groups $\mathfrak{M}$ and $\mathfrak{N}$ respectively, then*

$$\left[\phi \overset{top}{\approx} \psi\right] \Leftrightarrow \left[\mathfrak{M} = a\mathfrak{N} \text{ for some scalar } a\right].$$

Theorem 10. *If $f = \phi_x$ is the orbit of a flow $\phi : \mathbb{R}\times X \to X$ and if $\overline{f(\mathbb{R})}$ is compact, then there is a map $h_f : \overline{f(\mathbb{R})} \overset{onto}{\to} \sum_{\overline{M}}$ with $id \times h_f : \phi \mid_{\mathbb{R}\times\overline{f(\mathbb{R})}} \overset{sc}{\succeq} \Phi^{\omega}_{\overline{M}}$, where $\sum_{\overline{M}}$ is the B–dual of $\mathcal{E}_f$*

and where $B = \{\omega_i\}$ *is a maximal independent subset of* $\mathcal{E}_f$ *and where* $\omega = (\omega_1, \omega_2, \ldots)$.

Proof: We have the B-sequence of $\mathcal{E}_f$ $\{H^i, \beta_i^j\}$ with the chosen basis for H^i $\{b_j^i\}$, where $b_j^1 = \omega_j$ for all j. We then define $h_f : \overline{f(\mathbb{R})} \to \prod_{\ell=1}^{\infty} \mathbf{T}^\kappa$ as follows, where the maps $f_b : \overline{f(\mathbb{R})} \to S^1$ for $b \in \mathcal{E}_f$ are as in the proof of Theorem 2:

$$x = \lim \{f(t_i)\} \mapsto$$

$$\left(\langle f_{\omega_1}(x), f_{\omega_1}(x), \ldots \rangle, \langle f_{b_1^2}(x), f_{b_1^2}(x), \ldots \rangle, \ldots \right) \in \prod_{\ell=1}^{\infty} \mathbf{T}^\kappa,$$

where κ is the cardinality of B. The limits $\lim_i \left\{\pi\left(b_k^\ell t_i\right)\right\} = f_{b_k^\ell}(x)$ exist for all k, ℓ as shown in Theorem 2. Also, for each $i \in \mathbb{N}$ we have that

$$\left(\langle \pi(\omega_1 t_i), \pi(\omega_2 t_i), \ldots \rangle, \langle \pi\left(b_1^2 t_i\right), \pi\left(b_2^2 t_i\right), \ldots \rangle, \ldots\right) \in \sum\nolimits_{\overline{M}}$$

by the definition of $\sum_{\overline{M}}$. And so $h_f\left(\overline{f(\mathbb{R})}\right) \subset \sum_{\overline{M}}$ since $\sum_{\overline{M}}$ is a closed subset of $\prod_{\ell=1}^{\infty} \mathbf{T}^\kappa$. Thus, our map h_f is well-defined and it is continuous since each $f_{b_k^\ell}$ is continuous. We now show that it is surjective. Since $\Phi_{\overline{M}}^{\omega}$ is an irrational flow, the orbit of $e_{\overline{M}}$ is dense, and so given any point $\sigma \in \sum_{\overline{M}}$ there is a sequence $\{x_i\} \to \sigma$, where

$$\begin{aligned} x_i &= \left(\langle \pi(\omega_1 t_i), \pi(\omega_2 t_i), \ldots \rangle, \langle \pi\left(b_1^2 t_i\right), \pi\left(b_2^2 t_i\right), \ldots \rangle, \ldots\right) \\ &= \Phi_{\overline{M}}^{\omega}(t_i, e_{\overline{M}}) \end{aligned}$$

and $t_i \in \mathbb{R}$. Since $\overline{f(\mathbb{R})}$ is compact, there is a subsequence $\{t_{i_k}\}$ of $\{t_i\}$ which is an f-sequence. Then $h_f(\lim\{f(t_{i_k})\}) = \sigma$ as needed. The commutativity of the diagram as in Definition 2.11 follows from the following: for $y = \lim\{f(t_i)\}$, we have

$$\begin{aligned} h_f \circ \phi(t,y) &= h_f\left(\lim_i \{f(t+t_i)\}\right) \\ &= \left(\langle \lim_i \{\pi(\omega_1 t) + \pi(\omega_2 t_i)\}, \ldots\rangle, \ldots\right) \\ &= (\langle \pi(\omega_1 t), \ldots\rangle, \ldots) + (\langle f_{\omega_1}(y), \ldots\rangle, \ldots) \\ &= \pi_{\overline{M}}(t\omega) + h_f(y) = \Phi^{\omega}_{\overline{M}}(t, h_f(y)). \end{aligned}$$

□

We know that if there is an h with $id_{\mathbb{R}} \times h : \overline{f(\mathbb{R})} \overset{sc}{\succeq} \Phi^{\omega'}_{\overline{N}}$ for some irrational linear flow $\Phi^{\omega'}_{\overline{N}}$ then $\mathcal{E}_f$ contains the exponent group of $\Phi^{\omega'}_{\overline{N}}$, see Lemma 3.2. And so the above provides a semiconjugacy onto a linear flow which is maximal with respect to its exponent group.

5. The Exponent Group Generalizes the Irrational Rotation Number

Definition 5.1. Let $h : X \to X$ be a homeomorphism. Define the equivalence relation $\approx_h$ on $\mathbb{R}\times X$ by

$$[(s,x) \approx_h (t,y)] \Leftrightarrow [\text{there is an } n \in \mathbb{Z}$$

$$\text{with } t = s+n \text{ and } y = h^{-n}(x)]$$

and let $\mathcal{S}_h = (\mathbb{R}\times X)/\approx_h$. The $\approx_h$ class of (s,x) will be denoted $[s,x]_h$.
There is then the flow

$$\sigma_h : \mathbb{R}\times\mathcal{S}_h \to \mathcal{S}_h \ ; \ (t,[s,x]_h) \mapsto [t+s,x]_h \, .$$

We refer to both $\mathcal{S}_h$ and σ_h as the *suspension of h.*

The above formulation of suspension may be found, for example, in [A], but the notion originated with Smale [Sm]. We now proceed to describe the group of exponents of the suspension of an orientation preserving homeomorphism of S^1 with an irrational rotation number, but to do so we will need to relate suspensions with flows on the torus.

Definition 5.2. *For $\theta \in \mathbb{R}$, $R_\theta : S^1 \to S^1$ is the translation given by*

$$x \longmapsto x + \pi(\theta).$$

We can then equate the suspension of R_θ with $\Phi^{(\theta,1)}$ as follows.

Lemma 5.3.

$$\mu_{R_\theta} : \mathcal{S}_{R_\theta} \longrightarrow \mathbf{T}^2$$

$$[s,x]_{R_\theta} \longmapsto \langle x + \pi(s\theta), \pi(s)\rangle$$

is a well-defined homeomorphism and

$$id \times \mu_{R_\theta} : \sigma_{R_\theta} \overset{equiv}{\approx} \Phi^{(\theta,1)}.$$

See, e.g., [KH], Proposition 2.2.2 for a proof.

Given a map $f : S^1 \to S^1$ we may lift the map $f \circ \pi : \mathbb{R} \to S^1$ with a map $F : \mathbb{R} \to \mathbb{R}$ satisfying $f \circ \pi = \pi \circ F$, which is uniquely determined if we require $F(0)$ to be in $[0,1)$. We assume hereafter that such a choice of F is made and we refer to F as the lift of f. And when f is an orientation preserving homeomorphism, $F(x+1) = F(x)+1$, see [KH], Proposition 11.1.1 and so $F(x+k) = F(x)+k$ for $k \in \mathbb{Z}$. And since F^ℓ is a lift of f^ℓ for $\ell \in \mathbb{Z}$ and f^ℓ is an orientation preserving homeomorphism when f is, we have that for $(k,\ell) \in \mathbb{Z}^2$ $F^\ell(x+k) = F^\ell(x)+k$. Also, f is said to be *monotone* (or *strictly monotone*, etc.) when its lift F is monotone (or strictly monotone, etc.) [KH]. Thus, an orientation preserving homeomorphism of S^1 is strictly monotone increasing.

In [KH] 11.1.2 the rotation number is defined as an element of S^1. For our purposes it is more convenient to define it as a real number. Since we have a unique lift associated with any map, this rotation number will be well–defined.

Definition 5.4. Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with the lift F, then the *rotation number* of f is

$$\lim_{|n|\to\infty} \frac{1}{n}\left(F^n(x) - x\right).$$

See [KH] 11.1.1.

Definition 5.5. *A map $g : N \to N$ is a* factor *of $f : M \to M$ if there exists a surjective map $h : M \to N$ such that $h \circ f = g \circ h$. The map h is then called a* semiconjugacy *[KH], 2.3.2. If h is a homeomorphism it is said to provide a* conjugacy *and f and g are said to be* conjugate.

Theorem 11. (Poincaré Classification Theorem) *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number θ.*

1. If f is transitive, then f is conjugate to R_θ.
2. If f is not transitive, then the map R_θ is a factor of f with a non-invertible monotone map $h : S^1 \to S^1$ providing the semiconjugacy [KH], 11.2.7.

So in either case there is a surjective monotone map h with $R_\theta \circ h = h \circ f$.

Lemma 5.6. *If $f : S^1 \to S^1$ is an orientation-preserving homeomorphism with rotation number $\theta \in (\mathbb{R} - \mathbb{Q})$ and if λ is the σ_f-orbit of $x \in \mathcal{S}_f$, then $\mathcal{E}_\lambda \supset \langle \theta, 1 \rangle_{\mathbb{Z}}$.*

Proof: By the above theorem we have a monotone map $h : S^1 \to S^1$ with $R_\theta \circ h = h \circ f$. We then have the map

$$\mathcal{S}(h) : \mathcal{S}_f \to \mathcal{S}_{R_\theta};\ [s,x]_f \mapsto [s, h(x)]_{R_\theta}.$$

This map is well-defined since $[s,x]_f = [s',x']_f$ implies that $s' = s + n$ and $x' = f^{-n}(x)$ for some $n \in \mathbb{Z}$ and:

$$\begin{aligned}\left(\left[s+n, f^{-n}(x)\right]_f\right) &\overset{\mathcal{S}(h)}{\mapsto} \left[s+n, h\left(f^{-n}(x)\right)\right]_{R_\theta} \\ &= \left[s+n, (R_\theta)^{-n}(h(x))\right]_{R_\theta} \\ &= [s, h(x)]_{R_\theta} = \mathcal{S}(h)\left([s,x]_f\right).\end{aligned}$$

We then have the commutative diagram:

$$\begin{array}{ccc} \mathbb{R}\times\mathcal{S}_f & \overset{\sigma_f}{\longrightarrow} & \mathcal{S}_f \\ (id,\mathcal{S}(h)) \downarrow & & \downarrow \mathcal{S}(h) \\ \mathbb{R}\times\mathcal{S}_{R_\theta} & \overset{\sigma_{R_\theta}}{\longrightarrow} & \mathcal{S}_{R_\theta} \end{array}$$

But we also have the map $\mu_{R_\theta} : \mathcal{S}_{R_\theta} \to \mathbf{T}^2$ with $id \times \mu_{R_\theta} : \sigma_{R_\theta} \overset{equiv}{\approx} \Phi^{(\theta,1)}$ (see Lemma 5.3). And so

$$id \times (\mu_{R_\theta} \circ \mathcal{S}(h)) : \sigma_f \overset{sc}{\succeq} \Phi^{(\theta,1)},$$

and by Lemma 3.2 we have $\mathcal{E}_\lambda \supset \langle\theta, 1\rangle_{\mathbb{Z}}$ since this is the group of exponents for each orbit of $\Phi^{(\theta,1)}$. □

To get results on the other inclusion we first introduce some terminology.

Definition 5.7. Given a sequence $\{\xi_i\}$ of points in S^1, we define $\{x_i\} \subset \mathbb{R}$ to be a *lift* of $\{\xi_i\}$ if $\pi(x_i) = \xi_i$ for all $i \in \mathbb{N}$.

Definition 5.8. A sequence $\{\xi_i\}$ of points in S^1 is said to converge to ξ from below (or above) [denoted $\uparrow$ (or $\downarrow$)] if there is a lift of the sequence $\{\xi_i\}$ to a sequence $\{x_i\} \subset \mathbb{R}$ with $\{x_i\} \uparrow$ (or $\downarrow$) x in $\mathbb{R}$, where $\pi(x) = \xi$. We also use $\uparrow$ ($\downarrow$) to indicate that this sort of convergence takes place.

Definition 5.9. We define a stable orbit $f = \phi_x : \mathbb{R} \to X$ of a flow ϕ to be *amphiperiodic* if the following implication holds: for some set of generators $\mathcal{G}$ of $\mathcal{E}_f$

$$[\{\pi(\alpha t_i)\} \uparrow \text{ or } \downarrow \text{ for all } \alpha \in \mathcal{G}] \Rightarrow [\{t_i\} \text{ is an } f\text{–sequence}].$$

Notice that requiring $\{\pi(\alpha t_i)\} \uparrow$ or $\downarrow$ is stricter than requiring that $\{\pi(\alpha t_i)\}$ converges, and so any almost periodic orbit is also amphiperiodic.

Definition 5.10. If f is an amphiperiodic orbit, we define $\theta \in \mathcal{E}_f$ to be *regular* if the following implication holds for some set of generators $\mathcal{G}$ of $\mathcal{E}_f$

$$([\{\pi(\alpha t_i)\} \uparrow \text{or} \downarrow \text{for all } \alpha \in \mathcal{G} - \langle\theta\rangle_{\mathbb{Z}}] \text{ and } [\{\pi(\theta t_i)\}$$

$$\text{converges}]) \Rightarrow (\{t_i\} \text{ is an } f\text{–sequence});$$

otherwise, θ is *singular.*

It then follows that all exponents of an almost periodic orbit are regular.

We now quote a standard result we shall need.

Theorem 12. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with rotation number $\theta \in (\mathbb{R} - \mathbb{Q})$. Then there is a unique minimal set M_f for the dynamical system*

$$\mathbb{Z} \times S^1 \to S^1;\ (n, x) \mapsto f^n(x),$$

which is either S^1 or a perfect and nowhere dense subset (i.e., a Cantor set), see, e.g., [KH], *11.2.5.*

By the Poincaré classification, M_f is S^1 when f is transitive and a Cantor set otherwise. This Cantor set has as its complement a countable union of pairwise disjoint open intervals, each of which gets mapped by h to a single point. The endpoints of these open intervals get mapped to the same point as the open interval by h, while the remainder of M_f is mapped one-to-one by h onto its image, see, e.g., [KH], p. 398. Then the set $\mathcal{M}_f = \left\{\sigma_f\left(t, [0, m]_f\right) \mid t \in \mathbb{R} \text{ and } m \in M_f\right\}$ is a minimal set of the flow σ_f, see, e.g., [Schw]. Any aperiodic C^1 flow on $\mathbf{T}^2$ is topologically equivalent to the suspension of an orientation-preserving circle diffeomorphism with irrational rotation number, see, e.g., [KH] 14.2.3, 0.3 and 11.1.4, and so any minimal set occurring in an aperiodic C^1 flow on $\mathbf{T}^2$ that is a proper subset of $\mathbf{T}^2$ is homeomorphic with some such $\mathcal{M}_f$.

Theorem 13. *If $f : S^1 \to S^1$ is a non-transitive, orientation-preserving homeomorphism with rotation number $\theta \in (\mathbb{R} - \mathbb{Q})$ and if λ is the orbit of a point $x = [s, m]_f \in \mathcal{M}_f$, then $\mathcal{E}_\lambda = \langle \theta, 1 \rangle_{\mathbb{Z}}$ and λ is amphiperiodic and* 1 *is a regular exponent while θ is a singular exponent.*

Proof: Suppose then that for a given sequence $\{t_i\} \subset \mathbb{R}$ we have that $\{\pi(\theta t_i)\} \uparrow$ or $\downarrow$ and that $\{\pi(t_i)\}$ converges. We shall show that $\{t_i\}$ is a λ-sequence. Since we know that $\{\theta, 1\} \subset \mathcal{E}_\lambda$, this will demonstrate that λ is amphiperiodic. As before, we have the monotone map $h : S^1 \to S^1$ with $h \circ f = R_\theta \circ h$ and $id \times \mu_{R_\theta} : \sigma_{R_\theta} \overset{equiv}{\approx} \Phi^{(\theta,1)}$. Then with $\mu_{R_\theta} \circ \mathcal{S}(h)(x) = \langle \xi, \pi(u) \rangle \in \mathbf{T}^2$ and $\mathfrak{t}$ translation by $\langle -\xi, -\pi(u) \rangle$ and g defined to be the map $\mathfrak{t} \circ \mu_{R_\theta} \circ \mathcal{S}(h)$, we have that $id \times g : \sigma_f \overset{sc}{\succeq} \Phi^{(\theta,1)}$ since $\Phi^{(\theta,1)}$ is translation invariant and $g \circ \lambda(t) = \langle \pi(\theta t), \pi(t) \rangle \in \mathbf{T}^2$. Thus, our hypothesis on $\{t_i\}$ guarantees that $\lim_i g \circ \lambda(t_i) = \lim_i \langle \pi(\theta t_i), \pi(t_i) \rangle$ exists and is equal to say $\langle \chi, \pi(\tau) \rangle$. Since h is an onto monotone function, the set $h^{-1}(\xi + \chi)$ is either (**A1**) a point or (**A2**) a closed interval.

(A1): $h^{-1}(\xi + \chi) = \pi(r)$.

In this case we claim that $\{\lambda(t_i)\} \to [\tau + u, \pi(r)]_f$. Let

$$U_\varepsilon \overset{\text{def}}{=} \left\{ [\ell + \tau + u, \pi((r - \varepsilon, r + \varepsilon))]_f \in \mathcal{S}_f \mid \ell \in (-\varepsilon, \varepsilon) \right\}.$$

Then $\{U_\varepsilon \mid \varepsilon > 0\}$ forms a local base at $[\tau + u, \pi(r)]_f$. And for any given $\varepsilon > 0$ the intervals $\pi((r - \varepsilon, r))$ and $\pi((r, r + \varepsilon))$ are not mapped by h to single points, and so their image under the monotone map h contains an open interval. Thus, the image of U_ε under g contains a neighborhood of $g\left([\tau + u, \pi(r)]_f\right) = \langle \chi, \pi(\tau) \rangle$ and so contains the tail of the sequence $\{\langle \pi(\theta t_i), \pi(t_i) \rangle\}$. Thus, the tail of the sequence $\{\lambda(t_i)\}$ is contained in $\widetilde{U_\varepsilon} \overset{\text{def}}{=} (g)^{-1}(g(U_\varepsilon))$. While $\widetilde{U_\varepsilon}$ properly contains U_ε if one of the endpoints $\pi(r - \varepsilon)$ or $\pi(r + \varepsilon)$ is in the closure of a complementary open interval, in this case $\widetilde{U_\varepsilon} - U_\varepsilon$ does not contain points of arbitrarily high index i from the set $\{\lambda(t_i)\}$ since $\widetilde{U_\varepsilon} - U_\varepsilon$ gets mapped to a line segment (or two) in $\mathbf{T}^2$ which is (are) a positive distance from $\langle \chi, \pi(\tau) \rangle$, and so all but finitely many terms of the sequence $\{\lambda(t_i)\}$ are contained in U_ε. Notice that in this case we did not use the full strength of the hypothesis that $\{\pi(\theta t_i)\} \uparrow$ or $\downarrow$, we only needed that $\{\pi(\theta t_i)\}$ converges – a fact we use later.

(A2): $h^{-1}(\xi + \chi) = \pi([\mu, \nu])$.

Suppose without loss of generality that $\{\pi(\theta t_i)\} \uparrow \chi$ and that that h is monotone increasing, then we claim that $\{\lambda(t_i)\} \to [\tau + u, \pi(\mu)]_f$. Since, for $\varepsilon > 0$, h does not collapse the interval $\pi((\mu - \varepsilon, \mu])$ to a point, h maps each such interval onto a set containing an open interval. For each $\varepsilon > 0$ define

$$U_\varepsilon = \left\{ [\ell + \tau + u, \pi((\mu - \varepsilon, \mu])]_f \in \mathcal{S}_f \mid \ell \in (-\varepsilon, +\varepsilon) \right\}.$$

Then $g(U_\varepsilon)$ contains an open subset of the torus and $\langle \chi, \pi(\tau) \rangle$, and, since $\{\pi(\theta t_i)\} \uparrow \chi$, $g(U_\varepsilon)$ contains the tail of the sequence $\{\langle \pi(\theta t_i), \pi(t_i) \rangle\}$. Then U_ε contains a tail of the sequence since $g^{-1}(g(U_\varepsilon)) - U_\varepsilon$ contains only finitely members of the sequence $\{\lambda(t_i)\}$ as above. Notice that if the tail of the sequence $\{\langle \pi(\theta t_i), \pi(t_i) \rangle\}$ also contained points with $\chi + \pi(\theta t_i)$ " $>$ "χ, then the inverse image of such points would not be in U_ε since these inverse images would be "on the other side" of the complementary disk. And since each neighborhood of $[\tau + u, \pi(\mu)]_f$ contains some U_ε, we have that $\{\lambda(t_i)\} \to [\tau + u, \pi(\mu)]_f$. In this case we made full use of our hypotheses.

This shows then that λ is amphiperiodic. It only remains to show $\mathcal{E}_\lambda \subseteq \langle \theta, 1 \rangle_{\mathbb{Z}}$, and the proof of this bears some resemblance to the proof of a related fact for almost periodic orbits (see, e.g., [LZ], Chapter 2.2.3, p. 42-3). Let $\gamma \notin \langle \theta, 1 \rangle_{\mathbb{Z}}$. We have two cases: (**B1**) $\{\gamma, 1, \theta\}$ is rationally independent and (**B2**) $\{\gamma, 1, \theta\}$ is rationally dependent.

(B1): $\{\gamma, 1, \theta\}$ is rationally independent.

Let $\langle \zeta, \pi(0) \rangle \in \mathbf{T}^2$ be a point for which $(g)^{-1}(\langle \zeta, \pi(0) \rangle)$ is a single point, which will be the case whenever ζ is the image under $h(x) - \xi$ of a point x of the Cantor set M_f which in not an endpoint. By Kronecker's Theorem we can construct a sequence $\{t_i\}$ of real numbers with $\{\pi(t_i)\} \to \pi(0)$ and $\{\pi(\theta t_i)\} \to \zeta$ and such that $d_1(\pi(t_i\gamma), \pi(0)) < \frac{1}{i}$ for odd i and $d_1\left(\pi(t_i\gamma), \pi\left(\frac{1}{2}\right)\right) < \frac{1}{i}$ for even i. Then we have that:

$$\{\langle \pi(\theta t_i), \pi(t_i) \rangle\}_i \to \langle \zeta, \pi(0) \rangle.$$

By our choice of $\langle \zeta, \pi(0) \rangle$ and Case (**A1**), we have that $\{t_i\}$ is a λ-sequence (we have that both $\{\pi(\theta t_i)\}$ and $\{\pi(t_i)\}$ converge and this is all we need in this case). Therefore, $\gamma \notin \mathcal{E}_\lambda$ since $\{\pi(t_i\gamma)\}$ does not converge.

(B2): $\{\gamma, 1, \theta\}$ is rationally dependent.

There are thus integers ℓ_1, ℓ_2 and ℓ_3 not all 0 such that $\ell_1 \cdot 1 + \ell_2\theta + \ell_3\gamma = 0$. And since $\{1, \theta\}$ is rationally independent, ℓ_3 cannot be 0. Then we have a representation with the least positive q:

$$\gamma = \frac{p}{q} + \frac{r}{q}\theta, \text{ for integers } p, q \text{ and } r.$$

Let $\alpha_1 \in \pi^{-1}(0)$ and $\alpha_2 \in \pi^{-1}(\zeta)$ and let

$$\alpha_3 = \frac{p}{q}\alpha_1 + \frac{r}{q}\alpha_2,$$

where $\langle \zeta, \pi(0) \rangle$ is as in **(B1)**. Then if for integers ℓ_1, ℓ_2 and ℓ_3 we have that $\ell_1 \cdot 1 + \ell_2\theta + \ell_3\gamma = 0$, we also have

$$\left(\frac{\ell_1}{\ell_3} + \frac{p}{q}\right) + \left(\frac{\ell_2}{\ell_3} + \frac{r}{q}\right)\theta = 0,$$

and so the rational independence of $\{1, \theta\}$ yields that

$$\frac{p}{q} = -\frac{\ell_1}{\ell_3} \text{ and } \frac{r}{q} = -\frac{\ell_2}{\ell_3}.$$

And so

$$\begin{aligned}&\pi(\ell_1\alpha_1 + \ell_2\alpha_2 + \ell_3\alpha_3)\\ &\quad = \pi\left(\ell_1\alpha_1 + \ell_2\alpha_2 + \ell_3\left(-\frac{\ell_1}{\ell_3}\alpha_1 - \frac{\ell_2}{\ell_3}\alpha_2\right)\right) = \pi(0).\end{aligned}$$

Thus, the conditions of the strong version of Kronecker's Theorem (see, e.g., [LZ], Chapter 3.1, p. 37) are met and so we may find for any $\delta > 0$ a real number t satisfying:

$$d_1(\pi(t), \pi(\alpha_1)) < \delta, \ d_1(\pi(\theta t), \pi(\alpha_2)) < \delta$$

$$\text{and } d_1(\pi(\gamma t), \pi(\alpha_3)) < \delta.$$

Now let

$$\alpha_3' = \frac{1}{q} + \frac{p}{q}\alpha_1 + \frac{r}{q}\alpha_2.$$

Then we have that $\pi(\alpha_3') \neq \pi(\alpha_3)$ since q cannot be 1; otherwise, γ would be in $\langle \theta, 1 \rangle_{\mathbb{Z}}$. And if for integers ℓ_1, ℓ_2 and ℓ_3 we have that $\ell_1 \cdot 1 + \ell_2\theta + \ell_3\gamma = 0$, as before we have

$$\frac{p}{q} = -\frac{\ell_1}{\ell_3} \text{ and } \frac{r}{q} = -\frac{\ell_2}{\ell_3}$$

and by our choice of q we must have that q divides ℓ_3. And so:

$$\pi(\ell_1\alpha_1 + \ell_2\alpha_2 + \ell_3\alpha_3') = \pi\left(\ell_1\alpha_1 + \ell_2\alpha_2 + \ell_3\left(\frac{1}{q} - \frac{\ell_1}{\ell_3}\alpha_1 - \frac{\ell_2}{\ell_3}\alpha_2\right)\right)$$
$$= \pi\left(\frac{\ell_3}{q}\right) = \pi(0).$$

Again we have the conditions for the strong version of Kronecker's Theorem and so we may find for any $\delta > 0$ a real number t satisfying:

$$d_1(\pi(t), \pi(\alpha_1)) < \delta, \; d_1(\pi(\theta t), \pi(\alpha_2)) < \delta$$
$$\text{and } d_1(\pi(\gamma t), \pi(\alpha_3')) < \delta.$$

Thus, we may construct a sequence $\{t_i\}$ with $\{\pi(t_i)\} \to \pi(\alpha_1) = \pi(0)$ and $\{\pi(\theta t_i)\} \to \pi(\alpha_2) = \zeta$ and such that

$$d_1(\pi(t_i\gamma), \pi(\alpha_3)) < \frac{1}{i} \text{ for odd } i$$

and

$$d_1(\pi(t_i\gamma), \pi(\alpha_3')) < \frac{1}{i} \text{ for even } i.$$

Then

$$\{\langle \pi(\theta t_i), \pi(t_i) \rangle\}_i \to \langle \zeta, \pi(0) \rangle$$

Then

$$\{\langle \pi(\theta t_i), \pi(t_i)\rangle\}_i \to \langle \zeta, \pi(0)\rangle$$

and by our choice of $\langle \zeta, \pi(0)\rangle$ and Case (**A1**), we have that $\{t_i\}$ is a λ-sequence. Therefore, $\gamma \notin \mathcal{E}_\lambda$ since $\{\pi(t_i\gamma)\}$ does not converge. □

Closer examination of the map g reveals that it is nothing other than h_λ, where h_λ is as in Theorem 4. The complement of $\mathcal{M}_f$ in $\mathcal{S}_f$ consists of "blown up" trajectories of $\Phi^{(1,\theta)}$, see [KH], p. 398, each component of which is homeomorphic to an open disk. To see that the component of $\mathcal{S}_f - \mathcal{M}_f$ containing $\left\{[0,x]_f \in \mathcal{S}_f \mid x \in J\right\}$, where J is one of the countable pairwise disjoint intervals forming a component of $S^1 - M_f$, is homeomorphic with $D = \mathbb{R} \times J$, we construct the one-to-one map $\delta : D \to (\mathcal{S}_f - \mathcal{M}_f)$;

$$(t,x) \mapsto [t,x]_f$$

(that δ is one-to-one follows from the fact that σ_f has no periodic orbits). Brouwer's Theorem on the Invariance of Domain yields that δ is an open map and thus a homeomorphism onto its image, see also [Fok], p. 47. To investigate $\check{H}^1(\mathcal{M}_f)$ we examine a portion of the long exact reduced singular homology sequence for the pair $(\mathcal{S}_f, \mathcal{S}_f - \mathcal{M}_f)$, see, e.g., [Br], p.184-5:

$$\ddot{H}_1(\mathcal{S}_f - \mathcal{M}_f) \to H_1(\mathcal{S}_f) \to H_1(\mathcal{S}_f, \mathcal{S}_f - \mathcal{M}_f) \\ \to \tilde{H}_0(\mathcal{S}_f - \mathcal{M}_f) \to \tilde{H}_0(\mathcal{S}_f).$$

Since each component of $\mathcal{S}_f - \mathcal{M}_f$ is homeomorphic to an open disk, $H_1(\mathcal{S}_f - \mathcal{M}_f) = 0$. And so $\tilde{H}_0(\mathcal{S}_f - \mathcal{M}_f) \cong \oplus_{i=1}^{\kappa-1}\mathbb{Z}$, where as above κ represents the number of orbits blown up to form f, and so there is an exact sequence of groups

$$0 \to \mathbb{Z}^2 \to H_1(\mathcal{S}_f, \mathcal{S}_f - \mathcal{M}_f) \to \oplus_{i=1}^{\kappa-1}\mathbb{Z} \to 0.$$

By Poincaré-Lefschetz Duality (see, e.g., [Br], VI, 8.4), we have an isomorphism $\check{H}^1(\mathcal{M}_f) \cong H_1(\mathcal{S}_f, \mathcal{S}_f - \mathcal{M}_f)$. Thus, for sufficiently large κ, $[\mathcal{M}_f; S^1] \cong \check{H}^1(\mathcal{M}_f)$ has torsion-free rank greater than two and so must properly contain the image of the exponent group $\iota(\mathcal{E}_\lambda)$, which has torsion-free rank two.

Now we proceed to generalize exponent groups to self-homeomorphisms.

Definition 5.11. Let $h : X \to X$ be a homeomorphism. Then for $x \in X$ we define the group $\mathcal{E}_{(x,h)}$ to be the group $\mathcal{E}_f$ for the orbit f of the point $[0,x]_h$ in the suspension σ_h.

Notice that $\mathcal{E}_{(y,h)}$ is the same for all y in the h-orbit of x. And if M is a minimal set of h, all points of M have the same exponent group. By our calculation in Theorem 5, this exponent group generalizes the rotation number in some sense.

Example 5.12. Let $h : U \to \mathbb{R}^2$ be an elementary twist mapping of a domain of $U \subset \mathbb{R}^2$ as described in [AM] 8.3.5.

Such a map occurs naturally as the return map to a cross section of a Hamiltonian flow restricted to a 3–dimensional energy surface. While h itself may not be a self–homeomorphism of U, we may restrict h to the interior V of an invariant circle so that the restricted map h' will be a self–homeomorphism of V, see [AM] 8.3.6. For any invariant circle in V as described in part (i) of Moser's Theorem [AM] 8.3.6, we can calculate the rotation number or the exponent group. For some such h' there will also be minimal Cantor sets on which $\sigma_{h'}$ is equivalent to a linear flow $\Phi^1_{\overline{N}}$ on a 1–sonlenoid $\sum_{\overline{N}}$, see [AM] pp. 583-586. And so if, for example, $\sum_{\overline{N}}$ is the dyadic solenoid, the exponent group for all points in the corresponding Cantor set of V will be the dyadic rationals. Of course, if h' actually is the return map of a flow ϕ, this does not imply that the exponent group of the ϕ–orbits of points in the homeomorphic copy of $\sum_{\overline{N}}$ will be the corresponding subgroup of the rationals: ϕ would be topologically equivalent to $\sigma_{h'}$ but not necessarily equivalent.

We end with a question.

Question: Is there a non-singular compact minimal set M for a flow whose exponent group is $\{0\}$ for all time changes of the flow?

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