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EVERY COXETER GROUP ACTS AMENABLY ON A COMPACT SPACE

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ABSTRACT. Coxeter groups are Higson-Roe amenable, i.e. they admit amenable actions on compact spaces. Moreover, they have finite asymptotic dimension. This answers affirmatively a question from [H-R].

1. HIGSON-ROE AMENABILITY.

An action of a discrete group G on a compact space X is *topologically amenable* [AD-R] if there is a sequence of continuous maps $b^n : X \rightarrow P(G)$ to the space of probability measures on G with the weak*-topology such that for every $g \in G$, $\lim_{n \rightarrow \infty} \sup_{x \in X} \|gb_x^n - b_{gx}^n\|_1 = 0$. Here a measure $b_x^n = b^n(x)$ is considered as a function $b_x^n : G \rightarrow [0, 1]$ and $\|\cdot\|_1$ is the l_1 -norm.

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Definition A discrete countable group G is called Higson-Roe amenable if G admits a topologically amenable action on a compact space or, equivalently, its natural action on the Stone-Ćech compactification βG is topologically amenable.

2. PROPERTY A.

The property A was introduced in [Yu]. For metric spaces of bounded geometry it was reformulated in [H-R] as follows.

Property A A discrete metric space Z has the property A if and only if there is a sequence of maps $a^n : Z \rightarrow P(Z)$ such that

- (1) for every n there is some $R > 0$ with the property that for every $z \in Z$, $\text{supp}(a_z^n) \subset \{z' \in Z \mid d(z, z') < R\}$ and
- (2) for every $K > 0$, $\lim_{n \rightarrow \infty} \sup_{d(z, w) < K} \|a_z^n - a_w^n\|_1 = 0$.

Lemma [H-R] A finitely generated group G is Higson-Roe amenable if and only if the underlying metric space G with a word metric has property A.

A tree T possesses a natural metric where every edge has the length one. We denote by $V(T)$ the set of vertices of T with induced metric. The idea of the proof of the following proposition is taken from [Yu].

Proposition 1. For any tree T the metric space $V(T)$ has property A.

Proof: Let $\gamma_0 : \mathfrak{R} \rightarrow T$ be a geodesic ray in T , i.e. an isometric embedding of the half-line \mathfrak{R} . For every point $z \in V(T)$ there is a unique geodesic ray γ_z issued from z which intersects γ_0 along a geodesic ray. Let $V = V(T) \cap [z, \gamma_z(n)] \subset \text{im}(\gamma_z)$. We define a_z^n as a Dirac measure $\sum_{v \in V} \frac{1}{n+1} \delta_v$ supported uniformly by vertices lying in the geodesic segment $[z, \gamma_z(n)]$. Then condition (1) holds. If $d(z, w) < K$, then the geodesic segments $[z, \gamma_z(n)]$ and $[w, \gamma_w(n)]$ overlap along a geodesic segment of the length $\geq n - 2K$. Then $\sup \|a_z^n - a_w^n\|_1 \leq \frac{2K}{n+1}$ and hence the condition (2) holds. \square

Remark. We do not assume that a tree is locally finite in the above proposition. Thus the geometry is not bounded, and the variant of the property A we use may differ from the genuine property A of [Yu].

The following proposition is obvious.

Proposition 2. *Let $Z = Z_1 \times Z_2$ be the product of discrete metric spaces with the l_1 metric. Assume that Z_1 and Z_2 have property A. Then Z has property A.*

Proposition 3. *Let $W \subset Z$ and let Z have property A. Then W has property A.*

Proof: Let $a^n : Z \rightarrow P(Z)$ be a sequence of maps from the definition of property A for the space Z . We define a sequence $A^n : W \rightarrow P(W)$ by the following rule. Let $r : Z \rightarrow W$ be a retraction which takes every point $z \in Z$ to a closest point $r(z) \in W$. We define $A^n_w = P(r)(a^n_w)$ where $P(r) : P(Z) \rightarrow P(W)$ is the induced map on probability measures. If the support $supp(a^n_w)$ lies in the ball $B_R(w) \subset Z$, then $supp(A^n_w)$ lies in the ball $B_{2R}(w)$. Therefore the condition (1) holds. Let $a^n_z = \sum_{x \in Z} \lambda_x \delta_x$ and $a^n_w = \sum_{x \in Z} \nu_x \delta_x$. Note that $\|A^n_z - A^n_w\|_1 = \|P(r)(a^n_z) - P(r)(a^n_w)\|_1 = \|\sum_{x \in Z} \lambda_x \delta_{r(x)} - \sum_{x \in Z} \nu_x \delta_{r(x)}\|_1 = \sum_{y \in W} |\sum_{z \in r^{-1}(y)} (\lambda_z - \nu_z)| \leq \sum_{x \in Z} |\lambda_x - \nu_x| = \|a^n_z - a^n_w\|_1$. Hence the condition (2) holds. \square

3. COXETER GROUPS

Recall that a Coxeter group (Γ, W) is a group Γ with a distinguished set of generators $w_i \in W$ and relations $w_i^2 = 1 = (w_i w_j)^{m_{ij}}$, where m_{ij} is zero (and then there is no relation between w_i and w_j) or an integer ≥ 2 .

For any Coxeter group (Γ, W) there is a cell complex $C(\Gamma)$ on which Γ acts properly. Its construction, due to M. Davis, is described fully in [D]. It is defined as follows: cells are indexed by right cosets Γ/Γ_S , where Γ_S is a *finite* group generated by a subset S of W . A cell $[\gamma]$ is a face of $[\eta]$ if $[\gamma] \subset [\eta]$ as cosets.

Its vertices correspond to elements of $C(\Gamma)$, edges are indexed by generators etc. Its 1-skeleton is the Cayley graph of (Γ, W) .

The obvious action of Γ on $C(\Gamma)$ coming from the left action of Γ on itself is a reflection group action. Any reflection (i.e. an element conjugated to a generator in W) has its mirror of fixpoints. Any mirror is two sided i.e. its complement has two components. Closures of connected components of the set of all mirrors are called fundamental domains.

Theorem 1. [J] *Every Coxeter group Γ can be isometrically imbedded in a finite product of trees $\prod T_i$ with l_1 metric on it in such a way that the image of Γ under this embedding is contained in the set of vertices of $\prod T_i$.*

Lemma 1. [M] *Let Γ_0 be a normal torsion free subgroup of Γ . Then for each mirror H and every $\gamma_0 \in \Gamma_0$, $H \cap \gamma_0(H)$ is either H or the empty set.*

Observe that such Γ_0 , even with an additional property of being of finite index in Γ , exists by the Selberg Lemma. The proof of the lemma is just two lines, so we repeat it.

Proof: Let h be the reflection in H , and consider the product of reflections $g = h\gamma_0h\gamma_0^{-1}$. If $H \cap \gamma_0(H)$ is nonempty, g is a torsion element. On the other hand since Γ_0 is normal, g is in Γ_0 , hence it is an identity. Thus h commutes with γ_0 , and $H = \gamma_0(H)$. \square

Let \mathcal{H} be the set of orbits for Γ_0 action on the set of all mirrors. Fix an $h \in \mathcal{H}$. Define a graph T_h as follows. Its vertices are connected components of $C(\Gamma) - \bigcup_s \Gamma_0(s)$ where s is any mirror in h . Two vertices are joined by an edge if the components are adjacent in $C(\Gamma)$ i.e. if they intersect after taking closures.

Lemma 2. *T_h is a tree.*

Proof: Any loop λ in T_h lifts to a path in $C(\Gamma)$ which can be closed up to a loop Λ without crossing the h -mirror. Projection

of Λ is again λ . Since $C(\Gamma)$ is contractible (cf. [D]), Λ , hence λ , are homologous to zero thus T_h is a tree. \square

There is an obvious simplicial map from $C(\Gamma)$ to T_h , given on vertices by $g \rightarrow [g]_h$, that is mapping a fundamental domain to the connected component of $C(\Gamma) - \bigcup_s \Gamma_0(s)$ it belongs to. Clearly this map is Γ_0 equivariant. We take the diagonal of the family $\mu : C(\Gamma) \rightarrow \prod_h T_h$ to get a Γ_0 equivariant embedding of the Davis complex into the product of trees.

Lemma 3. *The map μ is a Γ -equivariant embedding*

Proof: First notice that Γ , in fact $\frac{\Gamma}{\Gamma_0}$, acts on \mathcal{H} . An element g maps the tree T_h to $T_{g(h)}$ simplicially. Thus Γ acts on $\prod_h T_h$ by permuting factors of the product. Explicitly $g(x_{h_1}, \dots, x_{h_n}) = (gx_{g^{-1}(h_1)}, \dots, gx_{g^{-1}(h_n)})$ Now equivariance of μ is obvious. To see that it is an embedding, notice that two vertices of $C(\Gamma)$ differ iff they are separated by some mirror say in h ; thus their images in T_h are different. \square

Proof of Theorem 1 In view of Lemmas 1–3 we should only notice that the l_1 -metric on the product of trees restricted to the image of μ agrees with the word metric on Γ . This is clear, as the word metric $d(g, h)$ counts the number of mirrors between chambers gF and hF for any fixed chamber F in the Davis complex.

Theorem A *Every Coxeter group Γ is Higson-Roe amenable.*

Proof: In view of Higson-Roe Lemma it suffices to show that Γ has property A. Propositions 1, 2, 3 and Theorem 1 imply the result. \square

4. ASYMPTOTIC DIMENSION

The following definition is analogous to Ostrand’s characterization of covering dimension.

Definition [Gr] *The asymptotic dimension of a metric space X does not exceed n , as $\dim X \leq n$ if for arbitrary large $d > 0$ there are $n + 1$ uniformly bounded d -disjoint families \mathcal{F}_i of sets in X such that the union $\cup \mathcal{F}_i$ forms a cover of X . A family \mathcal{F} is d -disjoint provided $\min\{\text{dist}(x, y) \mid x \in F_1, y \in F_2, F_1 \neq F_2, F_1, F_2 \in \mathcal{F}\} \geq d$.*

Proposition 4. *For every tree T , as $\dim T \leq 1$.*

Proof: Let $x_0 \in T$ be a fixed point and let $B_{nd}(x_0)$ denote the closed ball of radius nd centered at x_0 . Let $R_{n,i} = B_{nd} \setminus \text{Int}(B_{(n-i)d})$. We define

$\mathcal{F}_1 = \{\text{components of } R_{n,2} \text{ intersected with } R_{n,1} \text{ for } n \text{ odd}\}$
and

$\mathcal{F}_2 = \{\text{components of } R_{n,2} \text{ intersected with } R_{n,1} \text{ for } n \text{ even}\}$.
The diameter of each component of $R_{n,2}$ is less than $4d$ and for every two components in $R_{n,2}$ the distance between their traces on $R_{n,1}$ is greater than $2d$. \square

The following proposition follows from the reformulation of the definition of asymptotic dimension in terms of anti-Čech approximation of metric spaces by polyhedra [Gr].

Proposition 5. *as $\dim(X \times Y) \leq \text{as dim } X + \text{as dim } Y$.*

Theorem B *For every Coxeter group Γ , as $\dim \Gamma < \infty$.*

Proof: Theorem 1 and Propositions 4, 5 imply the proof. \square

Remarks.

1. Theorem B and Lemma 4.3 of [H-R] give an alternative proof of Theorem A.

2. Any of the Theorems A and B implies that Coxeter groups admit a coarsely uniform embedding into Hilbert space. This result is due to [B-J-S].

3. We would like to point out (but not go into details) that the proof we presented for both theorem A and B works in much greater generality: that of "zonotopal complexes of nonpositive curvature" ([D-J-S]), perhaps with an additional

property of "foldability". An interesting subclass here is that of cubical nonpositively curved complexes.

REFERENCES

- [AD-R] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Preprint (1998)
- [B-J-S] M. Bozejko, T. Januszkiewicz and R. J. Spatzier, *Infinite Coxeter groups do not have Kazhdan's property*, J. Operator Theory, **19** (1988), 63-67.
- [D] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, 1983 Ann. of Math. **117** (1983), 293-325.
- [D-J-S] M. Davis, T. Januszkiewicz and R. Scott, *Nonpositive curvature of blowups*, Selecta Mathematica, New Series, **4** (1998), 491-547.
- [Gr] M. Gromov, *Asymptotic invariants of infinite groups*, Cambridge University Press, Geometric Group Theory, vol 2 1993.
- [H-R] N. Higson and J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew Math, **519** (2000), 143-153.
- [J] T. Januszkiewicz, *For Coxeter groups $z^{|g|}$ is a coefficient of a uniformly bounded representation*, Preprint, 1999. <http://www/math.uni.wroc.pl/~tjan>.
- [M] J. J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math. **104** (1976), 235-247.
- [Yu] G. Yu, *The coarse Baum-Connes conjecture for groups which admit a uniform embedding into Hilbert space*, Inventiones Mathematicae, to appear.

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