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SOME REMARKS ON g-FUNCTIONS

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ABSTRACT. Considing J.Nagata's question raised in [6], we get an interesting metrization theorem by use of g-function. We shall also consider the decreasing condition of g-function in some metrization theorems.

In the present paper, we shall consider metrization theorems of some generalized metric spaces by means of g-function. All spaces are T_1, and N denotes the set of all natural numbers. X is said to have a *g-function*, if there is a function $g : N \times X \to \tau$, such that $x \in g(n,x)$, where τ denotes the topology of X. We say that g is *decreasing* if $g(n+1,x) \subset g(n,x)$ for each $n \in N$ and each $x \in X$. Let $g^1(n,x) = g(n,x)$ and $g^{i+1}(n,x) = \cup\{g(n,y) : y \in g^i(n,x)\}$ for each $i \in N$.

We consider the following conditions on g-function:

(1) For any $x \in X$ and neighborhood U of x, there is an $n \in N$ such that $x \notin [\cup\{g(n,y) : y \in X - U\}]^-$.
(2) If a sequence $\{x_n : n \in N\}$ converges to $p \in X$, and for all $n \in N$, $x_n \in g(n,y_n)$, then the sequence $\{y_n : n \in N\}$ converges to p.

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(3) If a sequence $\{x_n : n \in N\}$ converges to $p \in X$, and for all $n \in N$, $x_n \in \overline{g(n, y_n)}$, then the sequence $\{y_n : n \in N\}$ converges to p.
(4) If a sequence $\{x_i : i \in N\}$ satisfies that, for some $n \in N$, $x_i \notin g(n, x_j)$ or $x_j \notin g(n, x_i)$, $i < j$, then $\{x_i : i \in N\}$ (as a set) is discrete in X.
(5) If $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$ for each $n \in N$.
(6) For any $Y \subset X$ and each $n \in N$, $\overline{Y} \subset \cup\{\overline{g(n, y)} : y \in Y\}$.
(7) For any $Y \subset X$ and each $n \in N$, $\overline{Y} \subset \cup\{g(n, y) : y \in Y\}$.
(8) For any $Y \subset X$ and each $n \in N$, $\overline{Y} \subset \cup\{g^2(n, y) : y \in Y\}$.
(9) For any $Y \subset X$ and each $n \in N$, $\overline{Y} \subset \cup\{g^k(n, y) : y \in Y\}$, where $k > 2$.
(10) If $x \in g^2(n, x_n)$ for each $n \in N$, then $x_n \to x$.
(11) If $x \in g^{k+1}(n, x_n)$ for each $n \in N$, then $x_n \to x$. Where $k > 0$.

(1) is Heath's characterization of a stratifiable space. (2) is a characterization of k-stratifiable spaces due to Gao [2] and (10) is based on a characterization of σ-spaces due to Heath and Hodel [3]. In [6] J.Nagata proved the following theorem.

Theorem 1. *A regular space X is a Lašnev space iff it is Fréchet and has a g-function satisfying conditions* (2), (4) *and* (5).

Nagata posed the following question: Is it possible to replace (5) in Theorem 1 with any condition of g-function that is to be satisfied by $g(n, x) = \{y \in X : \rho(x, y) < \frac{1}{n}\}$, where (X, ρ) is a metric space ?

Although we could not answer the above question, while working on it, we get the following interesting theorem in comparison with Theorem 1, because it slightly medifies condition (2) essentially.

Theorem 2. *A regular space X is metrizable iff it is Fréchet and has a g-function satisfying conditions* (3) *and* (4).

Before we give the proof of Theorem 2, let us recall some results as follows.

Theorem 3. *A space X is metrizable iff it has a g-function satisfying conditions* (1) *and* (6)([1,6]).

Theorem 4. *A space X is metrizable iff it is Fréchet and has a g-function satisfying conditions* (1) *and* (4) ([1]).

On the other hand, we proved the following theorem which can be compared with Theorem 2, Theorem 3 and Theorem 4.

Theorem 5. *If X is a Lašnev space, then X is Fréchet and X has a g-function satisfying conditions* (2) *and* (6) ([1]).

We shall first prove the following Theorem 6 which may be called a lemma. The theorem is practically known by J.Nagata (A survey of metrization theorem II, Q and A 10(1992) 15-30, Theorem 2, where g(n,x) is not necessarily open). He did not give the proof, so we give our own proof here.

Theorem 6. *A space X is metrizable iff X has a g-function satisfying conditions* (3) *and* (6).

Proof: The necessity of the condition is obvious, we prove only the sufficiency. For each $x \in X$ and $n \in N$, let

$$U_n(x) = X - \{y : x \notin \overline{g(n,y)}\}^-$$

By (6) we know that $U_n(x)$ is a neighborhood of x. We can prove that $\{U_n(x) : n \in N\}$ is a local base of x.

Suppose V is an open set with $x \in V$. If $U_n(x) \subset V$ is not true for every $n \in N$, we can pick $y_n \in U_n(x) - V$. Since $y_n \in U_n(x)$, $x \in \overline{g(n,y_n)}$ for each $n \in N$, and $\{y_n : n \in N\}$ converges to x by (3). This is a contradiction because V is a neighborhood of x.

Furthermore we shall prove that the g-function satisfies the condition (1). Suppose for each $n \in N$,

$$x \in (\cup\{g(n,y) : y \in X - P\})^-$$

where P is a neighborhood of x. Then there is a sequence $\{y_n\} \subset X - P$ with $U_n(x) \cap g(n, y_n) \neq \emptyset$ for each $n \in N$. Pick out $z_n \in U_n(x) \cap g(n, y_n)$. Since $\{U_n(x)\}$ is a local base of x, we have that $\{z_n : n \in N\}$ converges to x. Because $z_n \in g(n, y_n)$, $\{y_n : n \in N\}$ converges to x by the condition (3). This contradicts the fact that P is a neighborhood of x. Now the g-function satisfies (1) and (6),and hence X is metrizable by Theorem 3.

We shall give the proof of Theorem 2.

Proof of Theorem 2: We give the sufficiency by proving that the condition (4) implies (6). Suppose there were a subset $Y \subset X$ and $n \in N$ such that $\overline{Y} \not\subset \cup\{\overline{g(n, y) : y \in Y}\}$, then we can get a point $y^* \in \overline{Y} - Y$, with $y^* \notin \cup\{\overline{g(n, y)} : y \in Y\}$. Since X is Fréchet, there exists a sequence $\{y_m : m \in N\} \subset Y$ with $y_m \to y^*$. Then

$$y^* \notin \cup\{\overline{g(n, y_m)} : m \in N\}.$$

In this case for each $i \in N$, $g(n, y_i) \cap \{y_m : m \in N\}$ is at most a finite set, and thus we can get a subsequence $\{y_{m_i} : i \in N\} \subset \{y_m : m \in N\}$ with $y_{m_j} \notin g(n, y_{m_i})$ for $j > i$. By condition (4) we know that $\{y_{m_i} : i \in N\}$ has no cluster point, but $\{y_{m_i} : i \in N\}$ converges to y^*. This is a contradiction. By Theorem 6, X is metrizable.

Definition 7. *A space X is stratifiable if we can assign a sequence $\{B_n(U) : n \in N\}$ of open sets for each open set U such that*

a). $\cup_{n=1}^{\infty} \overline{B_n(U)} = U$,
b). *If U and V are open sets with $U \subset V$, then $B_n(U) \subset B_n(V)$ for each $n \in N$.*

Lemma 8. *A space X is stratifiable iff X has a decreasing g-function such that, for each compact set C and closed set H with $C \cap H = \emptyset$,*

$$C \cap \overline{\cup\{g(m, y) : y \in H\})} = \emptyset$$

for some $m \in N$.

Lemma 9. *A space X is stratifiable iff X has a decreasing g-function such that, if a sequence $\{x_n : n \in N\}$ converges to p and a closed set H satisfies $\{p, x_1, x_2, \cdots\} \cap H = \emptyset$, then*

$$\{p, x_1, x_2, \cdots\} \cap \overline{\cup\{g(m, y) : y \in H\}} = \emptyset$$

for some $m \in N$.

Theorem 10. *If a space X is stratifiable, then X has a g-function satisfying condition* (3).

Proof: Let g be a g-function satisfying the condition in Lemma 9, that $x_n \to p$ and $x_n \in \overline{g(n, y_n)}$ for each $n \in N$, but $y_n \not\to p$, then there is a subsequence $\{y_{n_k} : k \in N\}$ with $p \notin \overline{\{y_{n_i} : i \in N\}}$, let $H = \overline{\{y_{n_1}, y_{n_2}, \cdots\}}$, then there is an $m \in N$ such that $p \notin \overline{\cup\{g(m, h) : h \in H\}}$. Since $x_n \to p$, we can suppose that

$$\{p, x_1, x_2, \cdots\} \cap \overline{\cup\{g(m, h) : h \in H\}} = \emptyset$$

This contradicts that $x_m \in \overline{g(m, y_m)}$.

Remark. In Theorem 10, we proved actually that if a decreasing g-function satisfies (1), then it satisfies condition (3) (the authors conjecture that condition (3) does not imply the condition (1)). But for a non-decreasing g-function, a similar argument is not true. The following example belongs to [8].

Example. There is a g-function which satisfies (1) but does not satisfy (2), and therefore it does not satisfy (3). Let $X = R^2$, R is the set of real numbers, the topology on X is defined as follows:

(1). If a point is in $X - (R \times \{0\})$, then it is isolated.

(2). For $(r, 0) \in R \times \{0\}$, let

$$B_n(r, 0) = \{(a, b) \in X :\mid a - r \mid < \frac{1}{n}, \mid b \mid < \frac{1}{n}\} - \{(r, b) : 0 < b < \frac{1}{n}\}$$

and $\{B_n(r, 0) : n \in N\}$ be a neighborhood base of $(r, 0)$.

The subspace $X^* = \{(a,b) \in X : b \geq 0\}$ of X is not metrizable. We define a g-function on X^* as follows:

$$g(n,(a,b))=\begin{cases} B_n^*(a,b) & \text{if } b=0 \\ \{(a,b)\} & \text{if } b \geq \frac{1}{n} \\ \{(a,b)\} \cup B_n^*(a-\frac{2}{n},0) \cup B_n^*(a+\frac{2}{n},0) & \text{if } 0 < b < \frac{1}{n} \end{cases}$$

Where $B_n^*(a,b) = X^* \cap B_n(a,b)$.

It is easy to see that g satisfies condition (1), but g does not satisfy (2). In fact, pick $p = (0,0)$, $x_n = (\frac{4}{3n}, \frac{2}{3n})$, $y_n = (0, \frac{2}{3n})$, then $x_n \to p$ and $x_n \in g(n, y_n)$ for each $n \in N$, but $y_n \not\to p$ because that $\{y_1, y_2, \cdots\} \cap U = \emptyset$ for any neighborhood U of p.

In [7], J.Nagata proved that A space X is metrizable iff X has a g-function satisfying conditions (1) and (8).

Y.Ziqiu [8] pointed out that in the above the g-function must be decreasing, and otherwise the above theorem is not true. That is

Theorem 11. *A space X is metrizable iff X has a decreasing g-function satisfying conditions* (1) *and* (8) ([7, 8]).

However we should like to point out that the decreasingness of g-function.

Theorem 12. *A space X is metrizable iff X has a decreasing g-function satisfying conditions* (9) *and* (11) ([8]).

We use Hodel's method in [5] to show that the decreasing hypothesis in Theorem 13 is not required.

Theorem 13. *A space X is metrizable iff X has a g-function satisfying conditions* (9) *and* (11).

Proof: For $p \in X$ and $n \in N$, let

$$U_n(p) = X - \{y : p \notin g^k(n,y)\}^-$$

From (9) we have $p \notin \{y : p \notin g^k(n,y)\}^-$, hence $U_n(p)$ is a neighborhood of p, and we note that if $y \in U_n(p)$, then $p \in g^k(n,y)$. Let $G(n,p) = g(n,p) \cap U_n(p)$, then $G(n,p)$ is a sequence of neighborhoods of p. We shall prove that $G(n,p)$ satisfies (A) and (B) in the following.

(A). If $\{p, x_n\} \subset G(n, y_n)$ for $n \in N$, then $x_n \to p$.
(B). If $G(n,p) \cap G(n,x_n) \neq \emptyset$ for each $n \in N$, then $x_n \to p$.
(A): Let $\{p, x_n\} \subset G(n, y_n)$, then $p \in g(n, y_n)$. Since $x_n \in G(n, y_n)$, $x_n \in U_n(y_n)$. By the above note this implies $y_n \in g^k(n, x_n)$, and hence $p \in g^{k+1}(n, x_n)$ follows from $p \in g(n, y_n)$. Therefore $x_n \to p$.
(B): If $y_n \in G(n,p) \cap G(n,x_n)$, then $y_n \in G(n,p)$, and hence $y_n \in U_n(p)$, which implies $p \in g^k(n, y_n)$ by the above note. Recall that $y_n \in G(n, x_n)$, and hence $y_n \in g(n, x_n)$. Therefore $p \in g^{k+1}(n, x_n)$ and thus $x_n \to p$ by (11). Hence X is metrizable (Theorem 6.1 in [4]).

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References

[1] Gao Zhi min, *Characterizations of metrizable spaces in terms of g-function*, Math.Japonica, **33** (1988), 851-859.
[2] Gao Zhi min, *On g-function separation*, Q. and A. in Gen. Top, **4** (1986), 47-59.
[3] R.W.Heath and R.E.Hodel, *Characterizations of σ-spaces*, Fund. Math, **77** (1973), 271-275.
[4] R.E.Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math.J, **39** (1972), 253-263.
[5] R.E.Hodel, *Metrizability of spaces satisfying Nagata's condition*, Math.Japonica, **47** (1998), 287-293.
[6] J.Nagata, *Characterizations of metrizable and Lašnev spaces in terms of g-function*, Q.and A.in Gen.Top, **4** (1986), 129-141.
[7] J.Nagata, *Metrizability, generalized metric spaces and g-functions*, Comment Math. Univ. Carolinae, **29** (1988), 715-722.
[8] Y.Ziqiu, *On a problem of J.Nagata*, Comment Math. Univ. Carolinae, **30** (1989), 811-815.

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