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H-CONNECTED INVERSE LIMIT SPACES

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ABSTRACT. The limit space of an inverse sequence of H-connected metric continua in which all bonding maps are open surjective maps is itself H-connected.

1. INTRODUCTION

The question of when a local homeomorphism $f : X \to Y$ of one topological space to another is a global homeomorphism is an interesting problem from the early part of the last century. An early reference appears in 1906 in Hadamard[H]. After generating considerable investigation through the intervening years, see e.g., the bibliographies of Jungck[J] and Heath[He1], the above question prompted the following definition by Jungck in 1983.

Definition 1.1. A connected Hausdorff space Y is H-connected if and only if any proper local homeomorphism $f : X \to Y$ from a connected Hausdorff space X onto Y is a global homeomorphism.

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Observe that any simply connected space is H-connected. In addition, non-simply connected H-connected spaces abound. To see this note that any finite-to-one covering projection of one metric compactum onto another is a proper local homeomorphism. Combine this with the correspondence between covering spaces of a connected finite CW-complex Y and subgroups of its fundamental group $\pi_1(Y)$. Then observe that such a Y is H-connected if and only if $\pi_1(Y)$ has no proper finite index subgroups. Finally, note that the class of finitely presented groups that have no proper finite index subgroups is large: it includes all finitely presented infinite simple groups as well as others, see e.g., Higman[Hi1], [Hi2]. So one may obtain a large class of non-simply connected H-connected n-complexes by starting with a finitely presented group G that has no proper finite index subgroups and constructing your favorite n-complex Y that has $\pi_1(Y) \cong G$. For $n \geq 4$, certain of these constructions, see e.g., Timm[T], yield compact non-simply connected H-connected n-manifolds M with $\pi_1(M)$ isomorphic to G. Slightly more exotic examples of non-simply connected H-connected spaces include the Topologist's sine curve (with its limit bar), the join of the cone on a pair of Hawaiian Earrings, see Spanier[S, 2.5.18], and the pseudo-arc. For a summary of quite general results regarding metric continua and exactly k-to-1 maps between them see Heath[He1].

The main result of this paper provides another method for identifying or constructing H-connected spaces. We show

Theorem 3.1. *If Y_∞ is the inverse limit of an inverse system (Y_n, f_n, N) of the compact H-connected metric spaces Y_n in which the bonding maps $f_n : Y_n \to Y_{n-1}$ are open surjective maps, then Y_∞ is H-connected.*

In addition, the technical lemmas, Lemmas 3.1 and 3.2, developed in the proof of the main result provide what we feel is surprising information on the structure of such inverse sequences and their limits.

Note that there is no hypothesis of contractability or local connectivity on the Y_n. Hence, in this aspect, our result generalizes the main result of Lau[L,Thm.2] where he proves that if $f : X \to Y$ is a local homeomorphism between metric continua and Y is the inverse limit of an inverse sequence of d-compressible metric continua with onto bonding maps, then f is a homeomorphism. A d-compressible space is a special type of a contractible space that possesses a certain amount of local connectivity at at least one point in the space. See Lau[L] for its exact definition. Related results, like those of Tominaga[To] where the domain of the local homeomorphism $f : X \to Y$ is the inverse limit, also require that spaces in the inverse system be locally connected simply connected spaces.

2. Preliminaries and motivation

In what follows all continua are compact connected metric spaces and all maps are assumed to be continuous functions from one topological space to another. In such a space, $N_\epsilon(x)$ denotes the open ϵ-ball about x. The notation N denotes the natural numbers and $N_k = \{1, \ldots, k\}$. A map $f : X \to Y$ is finite-to-one if and only if, the number of points in $f^{-1}(y)$, denoted $|f^{-1}(y)|$, is finite for every $y \in Y$. If there is an $n \in N$ such that $|f^{-1}(y)| = n$ for all $y \in Y$, f is said to be n-to-1. We denote this by $|f| = n$. A map $f : X \to Y$ is proper if and only if $f^{-1}(K)$ is compact for every compact $K \subset Y$ and $f : X \to Y$ is perfect if and only if $f^{-1}(y)$ is compact for every $y \in Y$. The notation (Y_n, f_n, N) is the notation for an *inverse sequence* where for each $n \in N$, Y_n is a metric space and $f_n : Y_n \to Y_{n-1}$ is a map. Note that for an inverse sequence, when $m \leq n$, the bonding map $f_{mn} : Y_n \to Y_m$ is given by $f_{mn} = f_{m+1} \ldots f_{n-1} f_n$. The inverse limit of the sequence will be denoted by either Y_∞ or $\varprojlim(Y_n, f_n, N)$. The projection map, $p_n : Y_\infty \to Y_n$, is the restriction of the product projection map $\pi_n : \prod_m Y_m \to Y_n$ to the subspace Y_∞.

Given two inverse sequences of topological spaces $\mathcal{X} = (X_n, f_n, N)$ and $\mathcal{Y} = (Y_n, g_n, N)$ we define a map of inverse systems $\lambda = (\lambda, \lambda_n)$: $\mathcal{X} \to \mathcal{Y}$ as follows. First, there is an order preserving map $\lambda : N \to N$ such that $\lambda(N)$ is cofinal in N. Furthermore, for each $m \leq n$ in E, there are continuous maps $\lambda_m : \mathrm{X}_{\lambda(m)} \to \mathrm{Y}_m$ and $\lambda_n : \mathrm{X}_{\lambda(n)} \to \mathrm{Y}_n$ such that the following square commutes:

$$\begin{array}{ccc} X_{\lambda(m)} & \overset{\mathrm{f}_{\lambda(m)\lambda(n)}}{\longleftarrow} & X_{\lambda(n)} \\ \lambda_m \downarrow & & \downarrow \lambda_n \\ Y_m & \overset{\mathrm{g}_{mn}}{\longleftarrow} & Y_n \end{array}$$

The $\lambda_m : X_{\lambda(m)} \to Y_m$ are called the component maps of λ. Additional basic results concerning inverse systems and maps between them can be found in Eilenberg and Steenrod [ES], the book by Engelking[E], and Christenson and Voxman[CV]. The book by Engelking contains the most complete development of these properties.

As mentioned in Section 1, the main theorem generalizes the results in Lau[L] and Tominaga[T] in the sense that the spaces Y_n in the inverse system (Y_n, f_n, N) have much weaker connectivity hypotheses imposed on them. This was certainly one of the initial motivations for the work leading up to this paper. A second reason to study inverse systems of H-connected spaces was to add to the calculus of H-connected spaces developed by Jungck[J, Ch. 3]. Jungck's results show how to obtain H-connected spaces by combining H-connected spaces in various ways. In particular, finite products of H-connected compacta are H-connected. So, since an inverse limit generalizes the notion of product, it is reasonable to ask if the inverse limit of an inverse sequence of H-connected compacta is H-connected. Finally, there are examples that show that in certain instances the basic structure of inverse sequences with certain special open bonding maps forces H-connected-like behavior in the limit space. Specifically, there is the following:

Example 2.1. Let $\mathcal{X}$ $=(S^1, f_n, N)$ be an inverse sequence of circles such that for each $n \in N$, $f_n : S^1 \to S^1$ is a k_n-fold covering projection for some prime k_n. Then, the inverse limit $\Sigma = \varprojlim \mathcal{X}$ is a *solenoid* and, according to Fox[F,Ex.2], the solenoids have the property that they possess non-trivial self-covers for every prime that does not appear infinitely often in the sequence $(k_n)_{n \in N}$. Observe that the circle is an *h-connected* space (note the lower case h) which generalizes of the notion of H-connectedness as follows: given any finite sheeted cover $p : X \to M$, if follows that X is homeomorphic to M. By Fox's result, the limit space, the solenoid Σ, has the same self-covering property. Of particular interest in the context of this paper is the proof that for those primes that *do* appear infinitely often in the defining sequence $(k_n)_{n \in N}$, the self-covers of Σ that one obtains in the obvious way, are trivial self-covers. As h-connected spaces are quite abundant (see Robinson and Timm[RT] for a discussion) or even more generally spaces with non-trivial self-covers, it follows that inverse limits of spaces with non-trivial self-covers will frequently exhibit H-connected-like behavior. In particular, the following fact is not hard to prove.

Fact: Assume that $\mathcal{X}$ $=(X, f_n, N)$ is an inverse sequence of metric continua with each $f_n : X_n = X \to X_{n-1} = X$ an m_n-fold covering projection. Assume that λ $=(id, \lambda_n, N)$: $\mathcal{X} \to \mathcal{X}$ is a self-map of the sequence such that there is a finite-to-one covering projection $f : X \to X$ with $\lambda_n = f$ for all $n \in N$. Assume also that there is a subsequence $(n_k)_{k \in N}$ such that $f_{n_k} = \lambda_{n_k} = f$. Then, $\lambda_\infty : X_\infty \to X_\infty$ is a homeomorphism.

We now focus on our objective, namely, to prove that the inverse limit space of a sequence of H-connected metric continua in which the bonding maps are open and onto is an H-connected metric continuum.

The following lemma shows why we demand that the bonding maps be onto and open. (We shall be required to have the projection maps $p_\alpha : Y \to Y_\alpha$ onto and open.) First note that it is well known [E,2.5.B] that for an inverse sequence, the projection maps p_n are onto if the bonding maps f_{mn} are all onto. Recall the following proposition.

Proposition 2.1. ([ES,VIII,3.2]) *Suppose that* (Y_n, f_n, N) *is an inverse system. Then* $p_m = f_{mn} p_n$ *whenever* $n \geq m$.

Reminder. [CV,6.B.8] A basis for the topology of the space Y_∞ is the set β, where $\beta = \{p_n^{-1}(U) : U$ is open in Y_n and $n \in N\}$.

The forward direction in the next lemma is a standard exercise. See, e.g., Engelking[E,2.7.17]. The converse follows in a similar way.

Lemma 2.1. (See also [E,2.7.19]) *Suppose that* $Y = Y_\infty$ *is the inverse limit space of the inverse sequence* (Y_n, f_{mn}, N) *which has all bonding maps* $f_{mn} : Y_n \to Y_m$ *onto. Then the projection map* $p_m : Y \to Y_m$ *is open if and only if the bonding map* f_{mn} *is open for all* $n \geq m$.

Finally, we need the following result. Its proof is also a nice exercise.

Proposition 2.2. *If* $Y = Y_\infty$ *is the inverse limit of a sequence* $\{Y_n\}$ *of metric continua with bonding maps* f_{mn} *onto and* $p_n : Y \to Y_n$ *is the projection map, then for any* $\epsilon > 0$ $\exists m \in N$ *and* $r > 0$ *such that whenever* $W \subset Y_m$ *and* $diam(W) < r$, *then* $diam(p_m^{-1}(W)) < \epsilon$.

3. Main Result

Theorem 3.1. *If* $Y = Y_\infty$ *is the inverse limit of a sequence* $\{Y_n : n \in N\}$ *of nonempty* H*-connected metric continua* Y_n *in which the bonding maps are open, then* Y *is an* H*-connected metric continuum.*

Proof: As is well known, Y is a metric continuum and (as the referee pointed out) since the bonding maps are open and the spaces continua, the bonding maps are also onto. We prove that Y is H-connected. To this end, let $f : X \to Y$ be a proper local homeomorphism (p.l.h.) of a connected T_2 space X onto Y. Since Y is a metric continuum and f is a p.l.h., X is a metric continuum and, by [J, (2.7)], (X, f) is a k-fold covering space for some $k \in N$. We prove the theorem by showing that $k = 1$.

Since X and Y are compact metric spaces, we can and do choose $\delta, \epsilon > 0$ such that whenever V is an open subset of Y and $diam(V) \leq \epsilon$, then

(3.1) $\exists$ disjoint open sets $U_i (i \in N_k)$ with $diam(U_i) < \delta/4$ such that $d(x_i, x_j) > \delta/2$ if $i \neq j$ and $x_i \in U_i, x_j \in U_j$, and $f^{-1}(V) = \cup\{U_i : i \in N_k\}$ where the restriction $f_{|U_i}$ is a homeomorphism onto V for $i \in N_k$.

Moreover, since Y is an inverse limit of metric continua, Proposition 2.2 permits us to conclude that for the ϵ chosen in (3.1) above,

(3.2) $\exists m \in N$ and $r > 0$ such that whenever $W \subset Y_m$ and $diam(W) < r$, then $diam(p_m^{-1}(W)) < \epsilon/2$,

where p_m is the projection map $p_m : Y \to Y_m$ (which is onto since the bonding maps are).

The following lemma yields a relation R on X which, surprisingly, is an equivalence relation. This fact is crucial to our proof.

Lemma3.1. $R = \{(x, x') \in X \times X : p_m f(x) = p_m f(x')$ and $(x, x') < \delta/4\}$ is an equivalence relation.

Proof: Trivially, R is reflexive and symmetric. To see that R is transitive – and hence an equivalence relation – suppose that $(x, x'), (x', x'') \in R$. Then

$p_m f(x) = p_m f(x')$ and $d(x, x') < \delta/4$, and $p_m f(x') = p_m f(x'')$ and $d(x', x'') < \delta/4$.

Hence,

(∗) $p_m f(x) = p_m f(x'')$ and $d(x, x'') < \delta/2$.

Then $diam(\{p_m f(x), p_m f(x'')\}) = 0 < r$, so (3.2) implies $d(f(x), f(x'')) < \epsilon/2$. Hence, (3.1) yields open sets U_i such that

$x, x'' \in f^{-1}(N_{\epsilon/2}(fx))) = \cup\{U_i : i \in N_k\}$, and $diam(U_i) < \delta/4$ for $i \in N_k$.

So $x \in U_i$ and $x'' \in U_j$ for some i, j. If $i \neq j$, then $d(x, x'') > \delta/2$ (see (3.1)) which contradicts (∗). Consequently, $i = j$. Thus $d(x, x'') < \delta/4$; i.e., $(x, x'') \in R$ and R is indeed an equivalence relation. □

Let $\psi : X \to X/R$ be the quotient map; i.e., $\psi(x) = R[x]$, the R-equivalence class of x. Since the bonding maps are onto and open, the projection map $p_m : Y \to Y_m$ is onto and open (by Lemma 2.1). Thus $p_m f : X \to Y_m$ is an open and continuous map of X onto Y_m.

Now define $h : Xx/R \to Y_m$ by $h(\psi(x)) = h(R[x]) = p_m f(x)$. h is clearly a function since $R[x] = R[x']$ implies $p_m f(x) = p_m f(x')$ by definition of the equivalence relation R. So consider the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{p_m f} & Y_m \\ \psi \downarrow & & \uparrow h \\ X/R & = & X/R \end{array}$$

The above diagram commutes since $p_m f = h\psi$. We now prepare to show that h is a proper local homeomorphism of X/R onto Y_m.

For each $y \in Y_m$, let $W_y = N_{r/3}(y) = \{z \in Y : d(z, y) < \epsilon/3\}$; so $diam(W_y) < r$. Therefore, (3.2) implies $diam(p_m^{-1}(W_y)) < \epsilon/2$. Hence (3.1) assures us that

(3.3) for each $y \in Y_m$ $\exists$ a uniquely specified neighborhood W_y of y and disjoint open sets U_i such that $(p_m f)^{-1}(W_y) = f^{-1}(p_m^{-1}(W_y)) = \cup\{U_i : i \in N_k\}$ where $f_{|U_i}$ is a homeomorphism onto $p_m^{-1}(W_y)$ and $diam(U_i) < \delta/4$ for each i. Also $d(x_i, x_j) > \delta/2$ if $x_i \in U_i$, $x_j \in U_j$, and $i \neq j$.

Lemma 3.2. *The sets U_i of (3.3) above satisfy $\psi^{-1}(\psi(U_i)) = U_i$ for $i \in N_k$.*

Proof: Let $y \in Y_m$ and W_y its associated neighborhood. Let $i \in N_k$ and let $x \in \psi^{-1}(\psi(U_i))$. Then $\psi(x) = \psi(x')$) for some $x' \in U_i$, and $R[x] = R[x']$; i.e., $p_m f(x) = p_m f(x')$ and $d(x, x') < \delta/4$. Since $x' \in U_i$, $p_m f(x) = p_m f(x') \in W_y$. So $x \in U_j$ for some j, by (3.3). And since $d(x, x') < \delta/4$, (3.3) implies that $i = j$. Thus, $\psi^{-1}(\psi(U_i)) \subset U_i$. Moreover, $U_i \subset \psi^{-1}(\psi(U_i))$ since $U_i \subset X$, the domain of ψ, and therefore $U_i = \psi^{-1}(\psi(U_i))$. □

Now consider the map $h : X/R \to Y_m$ where $h\psi = p_m f$. Since $p_m f$ is continuous and ψ is a quotient map, h is continuous. Moreover, h is open. To see this, note that since $h\psi = p_m f$ and ψ is onto, $h(A) = (p_m f)(\psi^{-1}(A))$ for $A \subset X/R$. Thus $h(A)$ is open when A is open, since ψ is continuous and $p_m f$ is an open map. Note also that since $p_m f$ is onto, h is onto. Thus $h : X/R \to Y_m$ is a continuous surjective open map.

To see that h is locally one-to-one, let $\psi(x) = R[x] \in X/R$, and let W_y be the specified neighborhood of $y = p_m f(x) = h\psi(x)$ where $(p_m f)^{-1}(W_y) = \cup\{U_i : i \in N_k\}$ as described in (3.3). Then $x \in U_i$ for some i, and $R[x] = \psi(x) \in \psi(U_i)$. But since U_i is open and $U_i = \psi^{-1}(\psi(U_i))$ by Lemma 3.3, $\psi(U_i)$ is open in the quotient space X/R; i.e., $\psi(U_i)$ is a neighborhood of $\psi(x) = R[x]$. We assert that h is one-to-one on $\psi(U_i)$. For if $\psi(x_1) = R[x_1]$ and $\psi(x_2) = R[x_2]$ are elements of $\psi(U_i)$ and $h(R[x_1]) = h(R[x_2])$, then $p_m f(x_1) = p_m f(x_2)$ and $x_1, x_2 \in \psi^{-1}(\psi(U_i)) = U_i$. Since $diam(U_i) < \delta/4$, the definition of R implies $R[x_1] = R[x_2]$, as desired.

We now know that $h : X/R \to Y_m$ is a local homeomorphism of X/R onto the H-connected space Y_m. Moreover, since X is connected and ψ is continuous and surjective, X/R is connected. To appeal to the definition of H-connected we have yet to show that h is proper and that X/R is T_2. We first that show X/R is T_2.

Let $R[x], R[x'] \in X/R$ with $R[x] \neq R[x']$. If $h(R[x]) \neq h(Rx'])$, there are disjoint neighborhoods Vx and Vx' of $h(R[x])$ and $h(R[x'])$ respectively, since Y_m is T_2. Then $h^{-1}(Vx)$ and $h^{-1}(Vx')$ are disjoint neighborhoods of $R[x]$ and $R[x']$ respectively, since h is continuous. On the other hand, if $h(R[x]) = h(R[x'])$ then $p_m f(x) = p_m f(x')$, and we know by (3.3) that there is a specified neighborhood W_y and neighborhoods U_i corresponding to $y = p_m f(x)$ such that
$x, x' \in (p_m f)^{-1}(W_y) = \cup\{U_i : i \in N_k\}$. If $x, x' \in U_i$ for some i, then $d(x, x') < \delta/4$ (by (3.3)) and we have the contradiction $R[x] = R[x']$. Thus $x \in U_i$ and $x' \in U_j$ for some $i, j \in N_k$ with $i \neq j$; therefore,

(3.4) $U_i \cap U_j = \emptyset$.

Since $U_k = \psi^{-1}(\psi(U_k))$ for $k = i, j$ and since ψ is onto, (3.4) implies

(3.5) $\psi(U_i) \cap \psi(U_j) = \emptyset$.

But $\psi(x) = R[x] \in \psi(U_i)$ and $\psi(x') = R[x'] \in \psi(U_j)$. As noted previously, the $\psi(U_i)$ are open in X/R, so (3.5) assures us that we have disjoint neighborhoods of $R[x]$ and $R[x']$ as desired; i.e., X/R is T_2.

To see that $h : X/R \to Y_m$ is proper, first note that since ψ is continuous and X is compact, we know that X/R is compact. But X/R is T_2, so that any closed subset of X/R is compact. Therefore, if M is a compact subset of the compact metric space Y_m, M is closed and thus $h^{-1}(M)$ is closed in X/R since h is continuous. Thus, $h^{-1}(M)$ is compact; i.e., h is a proper map.

We have shown that h is a proper local homeomorphism of the connected T_2 space X/R onto the H-connected space Y_m, and h must therefore be a homeomorphism by the definition of H-connected spaces.

We now have $h\psi = p_m f$, where h is a homeomorphism; specifically, h is one-to-one. Consequently, $h^{-1}(h(A)) = A$ for any subset A of X/R.

Since $Y_m \neq \emptyset$, we can let $y \in Y_m$ and choose $x \in (p_m f)^{-1}(y)$ since f is onto. Let $W = W_y$ be the neighborhood in Y_m specified in (3.3). Now

$(p_m f)^{-1}(W) = f^{-1}(p_m^{-1}(W)) = \cup\{U_i : i \in N_k\}$, where $p_m f(U_i) = W$ for any i. Then

(3.6) $(p_m f)^{-1}(p_m f(U_i)) = (p_m f)^{-1}(W) = \cup\{U_i : i \in N_k\}$, for any i.

Since h is one-to-one, for any $i \in N_k$ we also have:

(3.7) $(h\psi)^{-1}(h\psi)(U_i) = \psi^{-1}(h^{-1}h)(\psi(U_i)) = \psi^{-1}(\psi(U_i)) = U_i$.

But $h\psi = p_m f$, so (3.6) and (3.7) above imply, e.g. that $U_1 = \cup\{U_i : i \in N_k\}$. We assume without loss of generality that the x chosen above is in U_1. Since the U_i are mutually disjoint, we conclude that $U_i = \emptyset$ for $i > 1$. Thus $k = 1$!

We have shown that the local homeomorphism $f : X \to Y$ is one-to-one and is thus a homeomorphism; i.e., Y is H-connected as predicted. □

Corollary 3.3. *Let $\{X_n : n \in N\}$ be a sequence of H-connected metric continua. Then $\prod_{i=1}^{\infty} X_i$ is H-connected.*

Proof: Let $Y_n = \prod_{i=1}^{n} X_i$ and $f_n : Y_{n+1} \to Y_n$ the projection of Y_{n+1} onto the first n factors. Then each f_n is open and so, by Theorem 3.1, $\varprojlim(Y_n, f_n, N)$ is H-connected. That is, the countable infinite product of H-connected metric continua is H-connected.

Example 3.4. Let $G = \langle a_0, a_1, a_2, a_3 : a_{i+1}^{-2} a_i^{-1} a_{i+1} a_i, i = 0, 1, 2, 3\rangle$, where the addition in the subscripts is mod 4. By Higman[Hi1], G has no proper finite index subgroups. Let C_0, C_1, C_2, C_3 denote four pair-wise disjoint copies of the circle

C. Form these four circles into a bouquet B of four circles and denote the join point in the bouquet by P. Complete the construction of a 2-complex $K = K(G)$ such that $\pi_1(K) \cong G$ by attaching four 2-disks to B. Since there is a 1-1 correspondence between the subgroups of $\pi_1(K)$ and covering spaces of K and $\pi_1(K)$ has no proper finite index subgroups, it follows that K is H-connected. We now use our main theorem to show that two different inverse limits involving K are H-connected. Take an infinite sequence $\{K_n : n \in N\}$ such that for each $n \in N$, $K_n = K$. The copy of the join point P in K_n is denoted by P_n. In general, copies of the point $x \in K$ that are in K_n is denoted by x_n. The copy of the circle $C_i \subset K$ that is in K_n is denoted C_{in} and the copy of the generator $a_i \in \pi_1(K)$ that appears in $\pi_1(K_n)$ is denoted a_{in}. Let $X_1 = K_1$. For $n \geq 1$ let $X_{n+1} = X_n \vee K_n$ where the one point join is obtained by identifying P_n in K_n with P_1 in X_n. Let $f_n : X_{n+1} \to X_n$ be the function defined by $f_n(x_j) = x_j$ if $j = 1, \ldots n$, and $f_n(x_{n+1}) = x_n$. Note that f_n is open. By a result in the calculus of H-connectedness in Jungck[J,Ch.3], for each $n \in N$, X_n is H-connected (and so, interestingly, $\pi_1(X_n) \cong G * G * \ldots * G$ has no proper finite index subgroups). Applying the main result it follows that $\varprojlim(X_n, f_n, N)$ is H-connected.

Again begin with the 2-complex $K = K(G)$. Embed K in R^4 in general position. Let $M = M^4(K)$ be the 4-manifold with boundary obtained by taking a regular neighborhood of K. Note that $\pi_1(M) \cong G$ and that the image of the circles $C_i, i = 1, 2, 3, 4$, are geometric representatives of the generators $a_i \in G$, $i = 1, 2, 3, 4$. Now take infinitely many pairwise disjoint copies $\{M_n\}_{n \in N}$ of M. The copy of $C_i \subset M_n$ will be denoted by C_{in}. The copy of the point $x \in M$ that is in M_n will be denoted x_n. Let $X_1 = M_1$. In general, let $X_{n+1} = X_n \underset{\alpha_n}{\cup} M_{n+1}$ where the attaching map $\alpha_n : C_{0(n+1)} \to C_{01}$ glues the two copies of C_0 in X_n and M_{n+1} together via $\alpha_n(x_{n+1}) = x_1$. For each $n \in N$, define $f_n : X_{n+1} \to X_n$ by $f_n(x) = x$, if $x \in X_n$

and $f_n(x_{n+1}) = x_1$. Each f_n is open. So, by the main result, the $\varprojlim(X_n, f_n, N)$ is an H-connected continuum. Note again that there is the interesting group theoretic consequence of the calculus of H-connectedness in Jungck[J,Ch.3] and the correspondence between subgroups of the fundamental group and covering spaces, that the groups $\pi_1(M_n) \cong G \underset{Z}{*} G \underset{Z}{*} \ldots \underset{Z}{*} G$, which are free products of G with amalgamation over certain copies of the integers Z, have no proper finite index subgroups.

4. Retrospect

The following result gives information regarding inverse limits per se and provides another tool for identifying H-connected spaces. And it does suggest that in the process of constructing an inverse limit from a sequence of topological spaces, we "get nothing without paying for it " !

Theorem 4.1. *Suppose that whenever a property P is common to each space Y_n of an inverse system (Y_n, f_n, N) of T_2 spaces Y_n with bonding maps f_n onto, then its inverse limit space Y_∞ has the topological property Q. Then any T_2 topological space Y having property P has property Q.*

Proof: Let Y be a T_2 space having property P, and let Y_∞ be the inverse limit of the inverse system (Y_n, f_n, N) with $Y_n = Y$ and $f_n = id$ (the identity map) for $n \in N$. Then Y_∞ has the topological property Q by hypothesis. Thus, to prove that Y has property Q, it suffices to prove that Y is homeomorphic to Y_∞. But this is well known. □

It is easy to show that in the category of connected T_2 spaces, H-connectedness is a topological property. Note also that any p.l.h. of a connected T_2 space onto a metric continuum is veritably a local homeomorphism between continua. Now consider the following theorem by Lau .

Theorem 2. [L]. *If $f : X \to Y$ is a local homeomorphism between metric continua and Y is an inverse limit (bonding maps onto) of d-compressible continua $\{Y_n : n \in N\}$, then f is a homeomorphism.*

In view of the preceding comments, Theorem 2. says that any proper local homeomorphism of a connected T_2 space onto Y is a homeomorphism; i.e., Y is H-connected. Then Theorem 2 and Theorem 3.1 say that any d-compressible continuum (property P) is H-connected (Property Q).

We conclude by asking

Question. Is Theorem 3.1 true if the bonding maps are onto but not open ?

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