Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 24, Spring 1999

MUTUAL APOSYNDESIS OF SYMMETRIC PRODUCTS

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ABSTRACT. It is shown that the n^{th} symmetric product of nondegenerate continua is mutually aposyndetic for all $n \geq 3$.

1. INTRODUCTION

Throughout this paper, X denotes a continuum (nondegenerate, compact, connected, metric space), 2^X is the hyperspace of all nonempty closed subsets of X with the Hausdorff metric H_d ([IN], p. 11) and $C(X) = \{K \in 2^X : K \text{ is connected}\}$; $F_n(X) = \{K \in 2^X : K \text{ has at most } n \text{ points}\}$, $n = 1, 2, ...$, $F_n(X)$ is called the n^{th} *symmetric product* of X.

We say that a continuum X is *aposyndetic* provided that for any two different points $x, y \in X$ there is a subcontinuum K of X such that $x \in intK$ and $y \notin K$.

We say that a continuum X is *mutually aposyndetic* provided that for any two different points $x, y \in X$ there exist disjoint subcontinua K and L of X such that $x \in intK$, $y \in intL$.

1991 *Mathematics Subject Classification.* Primary 54B20; Secondary 54F15.

Key words and phrases. Aposyndesis, Mutual aposyndesis, Symmetric products.

Aposyndesis was first studied in connection with hyperspaces by Goodykoontz, who proved that 2^X and $C(X)$ are aposyndetic ([G], Theorem 1). Also, Illanes has some results about aposyndesis in hyperspaces ([I]). Macías has recently proved that if X is a chainable continuum such that its second symmetric product is mutually aposyndetic, then X is homeomorphic to $[0,1]$ ([M], Theorem 15); therefore, the second symmetric product of a continuum is not always mutually aposyndetic.

Illanes asked the author if $F_n(X)$ is mutually aposyndetic when $n \geq 3$. Our purpose here is to answer Illanes' question affirmatively.

We note that our result is the analogue for symmetric products of the following fact about cartesian products: The cartesian product of three nondegenerate continua is mutually aposyndetic ([H], Theorem 2).

Our proof involves a number of technical details. After the proof, we comment about another, natural approach which, unfortunately, does not work.

2. Mutual Aposyndesis in Symmetric products

Let X be a continuum and let $A_1, ..., A_m$ be subsets of X. We let $\langle A_1, ..., A_m\rangle$ denote $\{K \in F_n(X) : K \subset \bigcup_{i=1}^{m} A_i$ and $K \cap A_i \neq \emptyset$ for each $i \in \{1, ..., m\}\}$.

It can be proved easily that if $A_1, ..., A_m$ are closed (open) subsets of X then $\langle A_1, ..., A_m\rangle$ is closed (open) in $F_n(X)$.

Lemma 1. *Let X be a continuum and let $C_1, ..., C_n$ be connected subsets of X. If $m \geq n$, then the set $\langle C_1, ..., C_n\rangle$ is a connected subset of $F_m(X)$.*

Proof: Take two different points $\{x_1, ..., x_r\}, \{y_1, ..., y_s\} \in \langle C_1, ..., C_n\rangle$. For each $i \in \{1, ..., n\}$, there is a point $x_{j_i} \in C_i$ for some $j_i \in \{1, ..., r\}$ and there is a point $y_{k_i} \in C_i$ for some $k_i \in \{1, ..., s\}$. We are going to prove the following fact:

($*$) There is a connected subset of $\langle C_1, ..., C_n\rangle$ which contains both $\{x_1, ..., x_r\}$ and $\{x_{j_1}, ..., x_{j_n}\}$.

To prove $(*)$, assume $\{x_1, ..., x_r\} \neq \{x_{j_1}, ..., x_{j_n}\}$. Then there is a point $x_t \in \{x_1, ..., x_r\} - \{x_{j_1}, ..., x_{j_n}\}$. We know that $x_t \in C_u$ for some $u \in \{1, ..., n\}$. Define $f : \{x_{j_1}\} \times \cdots \times \{x_{j_n}\} \times C_u \longrightarrow \langle C_1, ..., C_n \rangle$ by $f(x_{j_1}, ..., x_{j_n}, c) = \{x_{j_1}, ..., x_{j_n}, c\}$. It easy to see that f is continuous. Consider $\mathcal{A} = f(\{x_{j_1}\} \times \cdots \times \{x_{j_n}\} \times C_u)$. We have that $\mathcal{A}$ is a connected subset of $\langle C_1, ..., C_n \rangle$ and that $\{x_{j_1}, ..., x_{j_n}\}$ and $\{x_{j_1}, ..., x_{j_n}, x_t\}$ are points of $\mathcal{A}$.

If $\{x_1, ..., x_r\} = \{x_{j_1}, ..., x_{j_n}, x_t\}$, then $(*)$ is proved. So, assume that $\{x_1, ..., x_r\} \neq \{x_{j_1}, ..., x_{j_n}, x_t\}$. Then there is a point $x_v \in \{x_1, ..., x_r\} - \{x_{j_1}, ..., x_{j_n}, x_t\}$. Applying the previous argument to x_v, we can find a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{x_{j_1}, ..., x_{j_n}, x_t\}$ and $\{x_{j_1}, ..., x_{j_n}, x_t, x_v\}$.

In this fashion we can construct a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{x_{j_1}, ..., x_{j_n}\}$ and $\{x_1, ..., x_r\}$. Therefore $(*)$ is proved.

Now, define $g_1 : C_1 \times \{x_{j_2}\} \times \cdots \times \{x_{j_n}\} \longrightarrow \langle C_1, ..., C_n \rangle$ by $g_1(c, x_{j_2}, ..., x_{j_n}) = \{c, x_{j_2}, ..., x_{j_n}\}$. We know that g_1 is continuous. Consider $\mathcal{B}_1 = g_1(C_1 \times \{x_{j_2}\} \times \cdots \times \{x_{j_n}\})$. We have that $\mathcal{B}_1$ is a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{x_{j_1}, ..., x_{j_n}\}$ and $\{y_{k_1}, x_{j_2}, ..., x_{j_n}\}$.

Define $g_2 : \{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \cdots \times \{x_{j_n}\} \longrightarrow \langle C_1, ..., C_n \rangle$ by $g_2(y_{k_1}, c, x_{j_3}, ..., x_{j_n}) = \{y_{k_1}, c, x_{j_3}, ..., x_{j_n}\}$. We know that g_2 is continuous. Consider $\mathcal{B}_2 = g_2(\{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \cdots \times \{x_{j_n}\})$. We have that $\mathcal{B}_2$ is a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{y_{k_1}, x_{j_2}, ..., x_{j_n}\}$ and $\{y_{k_1}, y_{k_2}, x_{j_3}, ..., x_{j_n}\}$. Similarly, we define $\mathcal{B}_3, ..., \mathcal{B}_n$.

Using $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$, we can construct a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{x_{j_i}, ..., x_{j_n}\}$ and $\{y_{k_1}, ..., y_{k_n}\}$.

Similar to the proof of $(*)$, we can construct a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{y_{k_1}, ..., y_{k_n}\}$ and $\{y_1, ..., y_s\}$.

Thus, we have constructed a connected subset of $\langle C_1, ..., C_n \rangle$ that contains both $\{x_1, ..., x_r\}$ and $\{y_1, ..., y_s\}$.

Therefore $\langle C_1, ..., C_n \rangle$ is connected. □

Lemma 2. *Let $m \geq 3$. Let X be a continuum, let U, W be nonempty proper open subsets of X and let x, y be two different* *points in X. Then the following set is a connected subset of $F_m(X)$:*

$$\left\langle \overline{U}, \overline{W}, X \right\rangle \cup \langle BdU, BdW, X\rangle \cup \langle BdU, \{x\}, X\rangle \cup \langle \{x\}, \{y\}, X\rangle .$$

Proof: Let $\mathcal{A} = \langle BdU, \{x\}, X\rangle \cup \langle \{x\}, \{y\}, X\rangle$. We will show, first, that $\mathcal{A}$ is connected. Given $q \in BdU$. Then $\{q, x, y\} \in \langle \{q\}, \{x\}, X\rangle \cap \langle \{x\}, \{y\}, X\rangle$ and, hence, $\langle \{q\}, \{x\}, X\rangle \cap \langle \{x\}, \{y\}, X\rangle \neq \emptyset$. Thus, since $\langle \{q\}, \{x\}, X\rangle$ and $\langle \{x\}, \{y\}, X\rangle$ are connected by Lemma 1, $\langle \{q\}, \{x\}, X\rangle \cup \langle \{x\}, \{y\}, X\rangle$ is connected. Therefore, since

$$\mathcal{A} = \left(\bigcup\{\langle \{q\}, \{x\}, X\rangle : q \in BdU\}\right) \cup \langle \{x\}, \{y\}, X\rangle ,$$

we have that $\mathcal{A}$ is connected.

Now, let $\mathcal{B} = \langle BdU, BdW, X\rangle \cup \mathcal{A}$. We will show that $\mathcal{B}$ is connected. Given $q \in BdU$ and $r \in BdW$. Then, noticing that $\{q, r, x\} \in \langle BdU, \{x\}, X\rangle$, we see that $\langle \{q\}, \{r\}, X\rangle \cap \mathcal{A} \neq \emptyset$. Thus, since $\langle \{q\}, \{r\}, X\rangle$ is connected by Lemma 1 and $\mathcal{A}$ is connected (as already proved), $\langle \{q\}, \{r\}, X\rangle \cup \mathcal{A}$ is connected. Therefore, since

$$\mathcal{B} = \left(\bigcup\{\langle \{q\}, \{r\}, X\rangle : q \in BdU, r \in BdW\}\right) \cup \mathcal{A},$$

we have that $\mathcal{B}$ is connected.

Finally, let $\mathcal{C} = \left\langle \overline{U}, \overline{W}, X \right\rangle \cup \mathcal{B}$. We will show that $\mathcal{C}$ is connected. Let C_1 be a component of $\overline{U}$ and let C_2 be a component of $\overline{W}$. Then, by the boundary bumping theorem ([N] , p. 73), we know that there exist $c_1 \in C_1 \cap BdU$ and $c_2 \in C_2 \cap BdW$; hence, $\{c_1, c_2\} \in \langle C_1, C_2, X\rangle \cap \mathcal{B}$. Thus, since $\langle C_1, C_2, X\rangle$ is connected by Lemma 1 and $\mathcal{B}$ is connected (as already proved), $\langle C_1, C_2, X\rangle \cup \mathcal{B}$ is connected. Therefore, since

$$\mathcal{C} = \left(\bigcup \{ \langle C_1, C_2, X \rangle : C_1 \text{ is a component of } \overline{U} \text{ and } C_2 \text{ is a component of } \overline{W} \} \right) \cup \mathcal{B},$$

we have that $\mathcal{C}$ is connected.

Therefore, $\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ is connected. □

Lemma 3. *Let X be a continuum and let $U_1 \subset U_2 \subset \cdots \subset U_n$ be nonempty proper open subsets of X. Then the following set is a connected subset of $F_m(X)$ for each $m \geq n+1$: $\mathcal{D} = \langle \overline{U_1} \rangle \cup \langle BdU_1, \overline{U_2} \rangle \cup \langle BdU_1, BdU_2, \overline{U_3} \rangle \cup \cdots \cup \langle BdU_1, ..., BdU_{n-1}, \overline{U_n} \rangle \cup \langle BdU_1, ..., BdU_n, X \rangle$.*

Proof: Take two elements $A, B \in \mathcal{D}$. We will construct a connected subset of $\mathcal{D}$ which contains A and B. We only consider the case when $A, B \in \langle \overline{U_1} \rangle$; the other cases can be reduced to this case by easy arguments.

Take $A, B \in \langle \overline{U_1} \rangle$, $A = \{a_{01}, ..., a_{0n}\}$, $B = \{b_{01}, ..., b_{0n}\}$; take $C_{11}, ..., C_{1n}$ and $K_{11}, ..., K_{1n}$ to be the components of $\overline{U_1}$ such that $a_{0i} \in C_{1i}$ and $b_{0i} \in K_{1i}$ for each $i \in \{1, ..., n\}$. Consider

$$\langle C_{11}, ..., C_{1n} \rangle \subset \langle \overline{U_1} \rangle \text{ and}$$
$$\langle K_{11}, ..., K_{1n} \rangle \subset \langle \overline{U_1} \rangle .$$

By Lemma 1, $\langle C_{11}, ..., C_{1n} \rangle$ and $\langle K_{11}, ..., K_{1n} \rangle$ are connected. By the boundary bumping theorem ([N], p. 73), we can take $\{a_{11}, ..., a_{1n}\}$ and $\{b_{11}, ..., b_{1n}\}$ such that $a_{1i} \in C_{1i} \cap BdU_1$ and $b_{1i} \in K_{1i} \cap BdU_1$ for each $i \in \{1, ..., n\}$; take $C_{22}, ..., C_{2n}$ and $K_{22}, ..., K_{2n}$ to be the components of $\overline{U_2}$ such that $a_{1i} \in C_{2i}$ and $b_{1i} \in K_{2i}$ for each $i \in \{2, ..., n\}$. Consider

$$\langle \{a_{11}\}, C_{22}, ..., C_{2n} \rangle \subset \langle BdU_1, \overline{U_2} \rangle \text{ and}$$
$$\langle \{b_{11}\}, K_{22}, ..., K_{2n} \rangle \subset \langle BdU_1, \overline{U_2} \rangle .$$

By Lemma 1, $\langle\{a_{11}\}, C_{22}, \ldots, C_{2n}\rangle$ and $\langle\{b_{11}\}, K_{22}, \ldots, K_{2n}\rangle$ are connected; thus, since

$$\{a_{11}, \ldots, a_{1n}\} \in \langle C_{11}, \ldots, C_{1n}\rangle \cap \langle\{a_{11}\}, C_{22}, \ldots, C_{2n}\rangle$$

and

$$\{b_{11}, \ldots, b_{1n}\} \in \langle K_{11}, \ldots, K_{1n}\rangle \cap \langle\{b_{11}\}, K_{22}, \ldots, K_{2n}\rangle\,,$$

we have that the following two sets are connected:

$$\langle C_{11}, \ldots, C_{1n}\rangle \cup \langle\{a_{11}\}, C_{22}, \ldots, C_{2n}\rangle\,,$$
$$\langle K_{11}, \ldots, K_{1n}\rangle \cup \langle\{b_{11}\}, K_{22}, \ldots, K_{2n}\rangle\,.$$

Now, by the boundary bumping theorem ([N], p. 73), we can take $\{a_{22}, \ldots, a_{2n}\}$ and $\{b_{22}, \ldots, b_{2n}\}$ such that $a_{2i} \in C_{2i} \cap BdU_2$ and $b_{2i} \in K_{2i} \cap BdU_2$ for each $i \in \{2, \ldots, n\}$; take $C_{33}, \ldots, C_{3n}$ and $K_{33}, \ldots, K_{3n}$ the components of $\overline{U_3}$ such that $a_{2i} \in C_{3i}$ and $b_{2i} \in K_{3i}$ for each $i \in \{3, \ldots, n\}$. Consider

$$\langle\{a_{11}\}, \{a_{22}\}, C_{33}, \ldots, C_{3n}\rangle \subset \left\langle BdU_1, BdU_2, \overline{U_3}\right\rangle$$

and

$$\langle\{b_{11}\}, \{b_{22}\}, K_{33}, \ldots, K_{3n}\rangle \subset \left\langle BdU_1, BdU_2, \overline{U_3}\right\rangle\,.$$

By lemma 1, $\langle\{a_{11}\}, \{a_{22}\}, C_{33}, \ldots, C_{3n}\rangle$ and $\langle\{b_{11}\}, \{b_{22}\}, K_{33}, \ldots, K_{3n}\rangle$ are connected; thus, since

$$\langle a_{11}, a_{22}, a_{23}, \ldots, a_{2n}\rangle \in \langle\{a_{11}\}, C_{22}, \ldots, C_{2n}\rangle \cap \langle\{a_{11}\}, \{a_{22}\}, C_{33}, \ldots, C_{3n}\rangle$$

and

$$\{b_{11}, b_{22}, b_{23}, \ldots, b_{2n}\} \in \langle\{b_{11}\}, K_{22}, \ldots, K_{2n}\rangle \cap \langle\{b_{11}\}, \{b_{22}\}, K_{33}, \ldots, K_{3n}\rangle$$

we have that the following two sets are connected:

$$\langle C_{11}, \ldots, C_{1n}\rangle \cup \langle\{a_{11}\}, C_{22}, \ldots, C_{2n}\rangle \cup \langle\{a_{11}\}, \{a_{22}\}, C_{33}, \ldots, C_{3n}\rangle\,,$$

$$\langle K_{11}, \ldots, K_{1n}\rangle \cup \langle\{b_{11}\}, K_{22}, \ldots, K_{2n}\rangle \cup \langle\{b_{11}\}, \{b_{22}\}, K_{33}, \ldots, K_{3n}\rangle\,.$$

Repeating the procedure indicated above, we can find connected sets $\mathcal{C}_1$ and $\mathcal{K}_1$ which contain A and B, respectively, of the form

$$\mathcal{C}_1 = \langle C_{11}, ..., C_{1n}\rangle \cup \langle\{a_{11}\}, C_{22}, ..., C_{2n}\rangle \cup \cdots \cup \langle\{a_{11}\}, ..., \{a_{n-1n-1}\}, C_{nn}\rangle$$

and

$$\mathcal{K}_1 = \langle K_{11}, ..., K_{1n}\rangle \cup \langle\{b_{11}\}, K_{22}, ..., K_{2n}\rangle \cup \cdots \cup \langle\{b_{11}\}, ..., \{b_{n-1n-1}\}, K_{nn}\rangle$$

where $a_{ii}, b_{ii} \in BdU_i$ for each $i \in \{1, .., n-1\}$, $C_{11}, ..., C_{1n}$, $K_{11}, ..., K_{1n}$ are components of $\overline{U_1}$, $C_{22}, ..., C_{2n}, K_{22}, ..., K_{2n}$ are components of $\overline{U_2}$, ..., and C_{nn}, K_{nn} are components of $\overline{U_n}$. Now, by the boundary bumping theorem ([N], p. 73), we can take points a_{nn} and b_{nn} such that $a_{nn} \in C_{nn} \cap BdU_n$ and $b_{nn} \in K_{nn} \cap BdU_n$. Consider

$$\mathcal{C}_2 = \langle\{a_{11}\}, ..., \{a_{nn}\}, X\rangle \subset \langle BdU_1, ..., BdU_n, X\rangle$$

and

$$\mathcal{K}_2 = \langle\{b_{11}\}, ..., \{b_{nn}\}, X\rangle \subset \langle BdU_1, ..., BdU_n, X\rangle .$$

By Lemma 1, $\langle\{a_{11}\}, ..., \{a_{nn}\}, X\rangle$ and $\langle\{b_{11}\}, ..., \{b_{nn}\}, X\rangle$ are connected; thus, since

$$\{a_{11}, ..., a_{nn}\} \in \langle\{a_{11}\}, ..., \{a_{nn}\}, X\rangle \cap \langle\{a_{11}\}, ..., \{a_{n-1n-1}\}, C_{nn}\rangle$$

and

$$\{b_{11}, ..., b_{nn}\} \in \langle\{b_{11}\}, ..., \{b_{nn}\}, X\rangle \cap \langle\{b_{11}\}, ..., \{b_{n-1n-1}\}, C_{nn}\rangle$$

we have that $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ are connected. Consider

$$\mathcal{J}_1 = \langle\{b_{11}\}, \{a_{22}\}, ..., \{a_{nn}\}, X\rangle \subset \langle BdU_1, ..., BdU_n, X\rangle ,$$

$$\mathcal{J}_2 = \langle \{b_{11}\}, \{b_{22}\}, \{a_{33}\}, ..., \{a_{nn}\}, X \rangle \subset \langle BdU_1, ..., BdU_n, X \rangle ,$$

$$\vdots$$

$$\mathcal{J}_{n-1} = \langle \{b_{11}\}, ..., \{b_{n-1n-1}\}, \{a_{nn}\}, X \rangle \subset \langle BdU_1, ..., BdU_n, X \rangle ;$$

by Lemma 1, $\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_{n-1}$ are connected; also, since

$$\{a_{11}, ..., a_{nn}, b_{11}\} \in \mathcal{C}_2 \cap \mathcal{J}_1$$

$$\{a_{22}, ..., a_{nn}, b_{11}, b_{22}\} \in \mathcal{J}_1 \cap \mathcal{J}_2$$

$$\vdots$$

$$\{a_{nn}, b_{11}, ..., b_{nn}\} \in \mathcal{J}_{n-1} \cap \mathcal{K}_2$$

we have that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{n-1} \cup \mathcal{K}_2 \cup \mathcal{K}_1$ is a connected set. Hence, we found a connected set that contains both A and B and is contained in $\mathcal{D}$.

Therefore $\mathcal{D}$ is connected. □

Theorem 1. *Let X be a continuum. Then $F_n(X)$ is mutually aposyndetic for each $n \geq 3$.*

Proof: Take two different elements $A, B \in F_n(X)$. We need to construct two disjoint subcontinua $\mathcal{A}$ and $\mathcal{B}$ of $F_n(X)$ such that $A \in int\mathcal{A}$ and $B \in int\mathcal{B}$. We will consider two cases.

Case 1. Suppose that $A \cap B = \emptyset$. Then we can find $U_1 \subset U_2 \subset \cdots \subset U_{n-1}$ and $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ open subsets of X such that $A \subset U_1$, $B \subset V_1$, $\overline{U_{n-1}} \cap \overline{V_{n-1}} = \emptyset$ and for each $i \in \{1, ..., n-2\}$, $\overline{U_i} \subset U_{i+1}$ and $\overline{V_i} \subset V_{i+1}$. Consider $\mathcal{A} = \langle \overline{U_1} \rangle \cup \langle BdU_1, \overline{U_2} \rangle \cup \langle BdU_1, BdU_2, \overline{U_3} \rangle \cup \cdots \cup \langle BdU_1, ..., BdU_{n-2}, \overline{U_{n-1}} \rangle \cup \langle BdU_1, ..., BdU_{n-1}, X \rangle$ and $\mathcal{B} = \langle \overline{V_1} \rangle \cup \langle BdV_1, \overline{V_2} \rangle \cup \langle BdV_1, BdV_2, \overline{V_3} \rangle \cup \cdots \cup \langle BdV_1, ..., BdV_{n-2}, \overline{V_{n-1}} \rangle \cup \langle BdV_1, ..., BdV_{n-1}, X \rangle$, we know that

$\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $F_n(X)$ then by Lemma 3 we have that $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_n(X)$. We will show that $\mathcal{A}$ and $\mathcal{B}$ are disjoint as follows:

First, since $\overline{U_i} \cap \overline{V_j} = \emptyset$ for each $i, j \in \{1, ..., n-1\}$, it follows that for each $k, l \in \{1, ..., n-1\}$

$$\left\langle BdU_1, ..., BdU_k, \overline{U_{k+1}} \right\rangle \cap \left\langle BdV_1, ..., BdV_l, \overline{V_{l+1}} \right\rangle = \emptyset.$$

Second, for each $k \in \{1, ..., n-2\}$, since $\overline{U_{k+1}} \cap \overline{V_j} = \emptyset$ for each $j \in \{1, .., n-1\}$, we have that

$$\left\langle BdU_1, ..., BdU_k, \overline{U_{k+1}} \right\rangle \cap \langle BdV_1, ..., BdV_{n-1}, X \rangle = \emptyset.$$

And third, since $\overline{U_{n-1}} \cap \overline{V_{n-1}} = \emptyset$ and for each for each $i \in \{1, .., n-2\}$, $\overline{U_i} \cap BdU_{i+1} = \emptyset$ and $\overline{V_i} \cap BdV_{i+1} = \emptyset$, it follows that

$$\langle BdU_1, ..., BdU_{n-1}, X \rangle \cap \langle BdV_1, ..., BdV_{n-1}, X \rangle = \emptyset.$$

Therefore $\mathcal{A} \cap \mathcal{B} = \emptyset$. Also, since $A \subset U_1$, $B \subset V_1$ and U_1, V_1 are open subsets of X, we have that $A \in \langle U_1 \rangle \subset \mathcal{A}$ and $B \in \langle V_1 \rangle \subset \mathcal{B}$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are disjoint subcontinua of $F_n(X)$ such that $A \in int\mathcal{A}$ and $B \in int\mathcal{B}$.

Case 2. Suppose that $A \cap B \neq \emptyset$. Since $A \neq B$, we can suppose, without loss of generality, that there exists a point $p \in A - B$. Clearly, there exists a point $q \in A$ different from p. Since X is metric, we can find open subsets U, W and $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ of X and two different points $x, y \in X$ with the following properties: $p \in U$; $q \in W$; $B \subset V_1$; $\overline{U} \cap \overline{V_{n-1}} = \emptyset$; $x, y \notin (\overline{U} \cup \overline{W} \cup \overline{V_{n-1}})$; for each $i \in \{1, ..., n-1\}$, $\overline{W} \cap BdV_i = \emptyset$; and for each $i \in \{1, ..., n-2\}$, $\overline{V_i} \subset V_{i+1}$. Consider $\mathcal{A} = \left\langle \overline{U}, \overline{W}, X \right\rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ and $\mathcal{B} = \left\langle \overline{V_1} \right\rangle \cup \left\langle BdV_1, \overline{V_2} \right\rangle \cup \left\langle BdV_1, BdV_2, \overline{V_3} \right\rangle \cup \cdots \cup \left\langle BdV_1, ..., BdV_{n-2}, \overline{V_{n-1}} \right\rangle \cup \langle BdV_1, ..., BdV_{n-1}, X \rangle$. We know that $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $F_n(X)$; hence, by Lemma 2 and Lemma 3 (correspondingly), we have that $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_n(X)$.We will show that $\mathcal{A}$ and $\mathcal{B}$ are disjoint as follows:

First, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, it follows that for each $k \in \{1,..,n-2\}$

$$(\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle) \\ \cap \langle BdV_1, ..., BdV_k, \overline{V_{k+1}} \rangle = \emptyset.$$

Second, since $x, y \notin \overline{V_{n-1}}$, we have that

$$\langle \{x\}, \{y\}, X \rangle \cap \langle BdV_1, ..., BV_k, \overline{V_{k+1}} \rangle = \emptyset.$$

Third, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, for each $i \in \{1, ..., n-1\}$, $\overline{W} \cap BdVi = \emptyset$ and, for each $i \in \{1, ..., n-2\}$, $\overline{V_i} \cap BdV_{i+1} = \emptyset$, it follows that

$$\langle \overline{U}, \overline{W}, X \rangle \cap \langle BdV_1, ..., BdV_{n-1}, X \rangle = \emptyset$$

and

$$\langle BdU, BdW, X \rangle \cap \langle BdV_1, ..., BdV_{n-1}, X \rangle = \emptyset.$$

And fourth, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$ and $x, y \notin \overline{V_{n-1}}$, we have that

$$(\langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle) \cap \langle BdV_1, ..., BdV_{n-1}, X \rangle = \emptyset.$$

Therefore $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Next, note that $p \in U$, $q \in W$, $B \subset V_1$ and U, W, V_1 are open subsets of X; hence, we have that $A \in \langle U, W, X \rangle \subset \mathcal{A}$ and $B \in \langle V_1 \rangle \subset \mathcal{B}$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are disjoint subcontinua of $F_n(X)$ such that $A \in int\mathcal{A}$ and $B \in int\mathcal{B}$.

Therefore $F_n(X)$ is mutually aposyndetic. □

Symmetric products are related to cartesian products; in particular, letting X^n denote cartesian product, there is a natural map $\pi_n : X^n \to F_n(X)$ given by $\pi_n(x_1, ..., x_n) = \{x_1, ..., x_n\}$. If π_n is open for $n \geq 3$, a simple proof of our theorem might be possible using the Theorem 2 of [H]. It is known that π_2 is open ([M], Lemma 9). However, we now show, π_n never is open for $n \geq 3$:

Proposition 1. *Let X be a continuum. Then π_n is not open for $n \geq 3$.*

Proof: Let p, q be two different points of X, take sequences $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ such that $p_k \to q$, $q_k \to q$ and $p_k \neq q_l$ for every $k, l \in \mathbb{N}$. Let $\varepsilon > 0$ such that $B_\varepsilon(p) \cap B_\varepsilon(q) = \emptyset$. Let $U = B_\varepsilon(p) \times B_\varepsilon(p) \times \cdots \times B_\varepsilon(p) \times B_\varepsilon(q) \subset X^n$. Then U is an open subset of X^n that contains $(p, p, .., p, q)$.

Suppose that π_n is open. Then $\pi_n(U)$ is an open set of $F_n(X)$which contains $\{p, q\}$; since $p_n \to q$ and $q_n \to q$, there exits $N \in \mathbb{N}$ such that $\{p, p_N, q_N\} \in \pi_3(U)$ and $p_N, q_N \in B_\varepsilon(q)$. Note that there are $n!$ points in $\pi_n^{-1}(\{p, p_N, q_N\})$; each of those points has two coordinates in $B_\varepsilon(q)$. Hence, $\{p, p_N, q_N\} \notin \pi_n(U)$ which is a contradiction.

Therefore π_n is not open. □

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