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FIBER PROPERTIES OF CLOSED MAPS, AND WEAK TOPOLOGY

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Let X be a space, and let $\mathcal{C}$ be a cover of X. Then, X has the *weak topology* with respect to $\mathcal{C}$, if $A \subset X$ is open in X if and only if $A \cap C$ is open in C for every $C \in \mathcal{C}$. Here, we can replace "open" by "closed". Locally compact spaces, first countable spaces, and many other kinds of topological spaces have the weak topology with respect to covers of compact (or compact metric) subsets.

Let $f : X \to Y$ be a closed map. Then, as is well-known, every $Bf^{-1}(y)$ is countably compact if X is a normal space, and Y is a locally compact space or first countable space, and so on; see [M_1], for example. Here, $Bf^{-1}(y)$ is the boundary of the fiber $f^{-1}(y)$. (Also, many decomposition theorems for f under X being nice properties on X are well-known; see [TY], for example).

In this paper, we shall consider the following question on closed maps in terms of weak topology. Here, if necessary, change "$Bf^{-1}(y)$" by "$f^{-1}(y)$"; or "$y \in Y$" by "$y \in Y - Y_0$, where Y_0 is a ($\sigma-$) discrete closed subset of Y".

Question: Let $f : X \to Y$ be a closed map. Under what conditions on X or Y, does $Bf^{-1}(y)$ have some nice properties for each $y \in Y$?

Let X be a space, and let $\mathcal{C}$ be a cover of X. In this paper, we shall use "*X is determined by $\mathcal{C}$* (or, *$\mathcal{C}$ determines X*)" instead of the usual "X has the weak topology with respect to $\mathcal{C}$", as in [GMT], etc. We recall that X is *dominated by $\mathcal{C}$* (or, *$\mathcal{C}$ dominates X*), if for any subcollection $\mathcal{C}^*$ of $\mathcal{C}$, $\cup\, \mathcal{C}^*$ is a closed subset of X which is determined by $\mathcal{C}^*$.

Let X be a space. Any open cover of X determines X. For a closed cover $\mathcal{C}$ of X, the following hold.

$\mathcal{C}$ dominates $X \Rightarrow \mathcal{C}$ determines X (but, the converse doesn't hold). If $\mathcal{C}$ is increasing and countable, $\mathcal{C}$ determines $X \Leftrightarrow \mathcal{C}$ dominates X.

$\mathcal{C}$ is hereditarily closure preserving (=HCP) $\Rightarrow \mathcal{C}$ dominates $X \Rightarrow \mathcal{C}$ is closure preserving (but, the converse of each implication doesn't hold).

We recall that every CW-complex is dominated by a cover of compact metric spaces, and that every space dominated by paracompact spaces is paracompact.

Let us give other main definitions used in this paper. Let X be a space. Then, X is a *k-space* (resp. *quasi-k-space*; *sequential space*) if X is determined by a cover of compact subsets (resp. countably compact subsets; compact metric subsets). For $x \in X$, let $t(x, X) = \min\{|C| : \text{if } x \in \mathrm{cl}A,\ x \in \mathrm{cl}C \text{ for some } C \subset A\}$, and $t(X) = \sup\{t(x, X) : x \in X\}$. If $t(X) \leq \omega$, X is a *space of countable tightness*, equivalently, X is determined by a cover of countable subsets. A space is an *M-space* if it admits a quasi-perfect map onto a metric space. A map $f : Z \to X$ is *bi-quotient* (resp. *pseudo-open*) if, whenever $\mathcal{V}$ is an open cover of $f^{-1}(x)$ (resp. V is an open set containing $f^{-1}(x)$), then finitely many $f(V)$, with $V \in \mathcal{V}$, cover a nbd (= neighborhood) of x (resp. $f(V)$ is a nbd of x) in X.

For a space X, the following characterizations hold in view of [M_2], etc. For inner definitions of *bi-(quasi-) k-spaces, and singly bi-(quasi-) k-spaces*, see [M_2].

X is a bi-k-space (resp. bi-quasi-k-space) $\Leftrightarrow$ X is a bi-quotient image of a paracompact M-space (resp. M-space).

X is a singly bi-k-space (resp. singly bi-quasi-k-space) $\Leftrightarrow X$ is a pseudo-open image of a paracompact M-space (resp. M-space).

X is a k-space (resp. quasi-k-space; sequential space) $\Leftrightarrow X$ is a quotient image of a paracompact M-space (resp. M-space; metric space).

A space X is an *A-space*, if whenever $c \in \mathrm{cl}(A_n - \{x\})$ with $A_n \supset A_{n+1}$ for $x \in X$ and $n \in N$, there exist $B_n \subset A_n$ such that $\cup\{\mathrm{cl}B_n : n \in N\}$ is not closed in X. If the B_n are closed (resp. singletons) in X, then such a space is an *inner-closed A-space* (*inner-one A-space*). In [MOS], these spaces are defined, and it is shown that these spaces X are precisely spaces with the property that any map onto X belonging to some class C_1 must belong to some other class C_2.

For a space X, we have the following implications, for example.

> X is first countable $\Rightarrow X$ is sequential $\Rightarrow X$ is a k-space with $t(X) \leqq \omega$.
> X is paracompact $M \Rightarrow X$ is bi-$k \Rightarrow X$ is singly bi-$k \Rightarrow X$ is $k \Rightarrow X$ is quasi-k.
> X is bi-quasi-$k \Rightarrow X$ is inner-one $A \Rightarrow X$ is inner-closed $A \Rightarrow X$ is A.

Let α be an infinite cardinal. Then, a space X is *α-compact* if every subset of cardinality α has an accumulation point in X. Every countably compact space is precisely ω-compact, thus, α-compact. Every Lindelöf space is ω_1-compact, and the converse holds among paracompact spaces. A space is *locally α-compact* if each point has a nbd whose closure is α-compact.

Let us say a space X is an *s_α-space*, if for any discrete closed set $D \subset X$ with $|D| \leqq \alpha$, there exists a HCP collection $\mathcal{V}$ of open subsets such that $|\mathcal{V}| = |D|$, and each $V \in \mathcal{V}$ contains a point $p_v \in D$ with the points p_v distinct. s_α-spaces are s_ω-space. Well-separated spaces in the sense of [Mo$_1$] are s_ω-spaces, then, so are normal spaces, countably paracompact spaces, realcompact spaces, or uniformly complete spaces. Collectionwise normal spaces, or M-spaces are s_α-spaces for any α.

Let S_α denote the space obtained from the topological sum of α convergent sequences by identifying all the limit points to a single point. In particular, S_ω is called the *sequential fan.*

Let us use the following letters, $\mathcal{P}$, $\mathcal{P}^*$, and $\mathcal{F}$ in this paper.

$\mathcal{P}$ denotes a point-countable cover *determining* a space;
$\mathcal{P}^*$ denotes a point-countable *closed* cover *determining* a space; and
$\mathcal{F}$ denotes a closed cover *dominating* a space.

We assume that spaces are regular T_1, and maps are continuous and onto.

LEMMAS

In this section, we give some lemmas. First, let us recall that a sequence (A_n) of subsets of a space is a *q-sequence* if $A_n \supset A_{n+1}$, and $K = \cap\{A_n : n \in N\}$ is countably compact such that every nbd of K contains some A_n.

Lemma 1. *Let $f : X \to Y$ be a map such that Y has a cover $\mathcal{F}$ (or $\mathcal{P}$). If (A_n) is a q-sequence in X, then some $f(A_n)$ is contained in a finite union of elements of $\mathcal{F}$ (or $\mathcal{P}$).*

Proof: Since $(f(A_n))$ is a q-sequence in Y, if $y_n \in f(A_n)$, then the sequence $\{y_n : n \in N\}$ has an accumulation point in Y. Thus, the result holds as in the proof of [TZ$_2$; Lemma 2.4] for $\mathcal{F}$, and [T$_5$; Lemma 6] for $\mathcal{P}$.

Lemma 2. (A) *Let X be a singly bi-quasi-k-space. Then,* (a) *and* (b) *below hold.*

(a) *If X has a cover $\mathcal{F} = \{F_\alpha : \alpha \in \lambda\}$, then X has a* HCP *closed cover $\{\mathrm{cl}X_\alpha : \alpha\}$, where $X_\alpha = F_\alpha - \cup\{F_\beta : \beta < \alpha\}$.*
(b) *If X has a cover $\mathcal{P}$, then the set $\{x \in X$: any nbd of x is not contained in a finite union of elements of $\mathcal{P}\}$ is closed discrete in X.*

(B) *Let X be a bi-quasi-k-space, or an inner-closed A-space with $t(X) \leqq \omega$. If X has a cover $\mathcal{F}$ (or $\mathcal{P}$), then, for each $x \in X$, some nbd of x is contained in a finite union of elements of $\mathcal{F}$ (or $\mathcal{P}$).*
(C) *Let X be a quasi-k-space having a* HCP *closed cover $\mathcal{C}$. Then, the set $\{x \in X : \mathcal{C}$ is not locally finite at $x\}$ is closed discrete in X.*

Proof: (A) and (C) are due to [T_8]. As for (B), for X being an inner-closed A-space with $t(X) \leqq \omega$, the result holds by [T_9; Lemma 2.6], for X is an inner-one A-space. For X being a bi-quasi-k-space, suppose the result doesn't hold at $x \in X$. Then, $B = \{X - S : S$ is a finite union of elements of $\mathcal{F}$ (or $\mathcal{P}$) $\}$ is a filter base such that $x \in \mathrm{cl} B$ for any $B \in \mathcal{B}$. Since X is bi-quasi-k, by [M_2; Lemma 3.F.2], there exists a q-sequence (A_n) such that $x \in \mathrm{cl}(B \cap A_n)$ for all $B \in \mathcal{B}$, and all $n \in N$. But, by Lemma 1, some A_n is contained in a finite union S of elements of $\mathcal{F}$ (or $\mathcal{P}$). This is a contradiction. Thus, for X being bi-quasi-k, the result also holds.

Lemma 3. (A) *Let $f : X \to Y$ be a closed map. Let Y be determined by a cover $\{C_\alpha : \alpha\}$. Then, X is determined by a cover $\{f^{-1}(C_\alpha) : \alpha\}$.*

(B) *Let $f : X \to Y$ be a closed map such that X is an s_α-space. Then, every $Bf^{-1}(y)$ is a α-compact if Y is one of the following spaces (where, $\alpha = \omega$ if Y is (i)): (i) inner-closed A-space, (ii) locally α-compact space, (iii) sequential space which contains no closed copy of S_α.*

Proof: (A) is due to [T_1]. For (B), for case (i); (ii); or (iii), see the proof of [M_2; Theorem 9.9]; [T_3; Proposition 1]; [T_4; Lemma 1.5] respectively.

Lemma 4. *For a map $f : X \to Y$, the following (A) $\sim$ (D) hold.*

(A) *Let X be a bi-k-space. Suppose that any closed paracompact M-subspace of Y is locally α-compact. Then, for each*

$x \in X$, $V_x \subset f^{-1}(K)$ for some nbd V_x of x and some α-compact closed subset K of Y. Thus, if $x \in Bf^{-1}(y)$, then $V_x \cap Bf^{-1}(y) \subset Bh^{-1}(y)$, where $h = f|f^{-1}(K)$.

(B) *Let X be a bi-quasi-k-space, or let f be a closed map and X be an inner-closed A-space with $t(X) \leqq \omega$. Suppose that Y has a cover $\mathcal{F}$ (or $\mathcal{P}$). Then, for each $x \in X$, $V_x \subset f^{-1}(S)$ for some nbd V_x of x and some finite union $S = \cup\{C_i : i \leqq n\}$, where $C_i \in \mathcal{F}$ (or $\mathcal{P}$). Thus, if $x \in Bf^{-1}(y)$, the following holds:*

For $\mathcal{P}$, $V_x \cap Bf^{-1}(y) \subset Bh^{-1}(y)$, where $h = f|f^{-1}(S)$.

For $\mathcal{F}$ (resp. $\mathcal{P}^*$), $V_x \cap Bf^{-1}(y) \subset \cup\{Bf_i^{-1}(y) : i \leq n\}$, where $f_i = f|f^{-1}(C_i)$, that is; $Bf^{-1}(y)$ is dominated (resp. determined) by $\{Bf_\alpha^{-1}(y) : y \in C_\alpha \in \mathcal{F}$ (resp. $\mathcal{P}^*$)$\}$, where $f_\alpha = f|f^{-1}(C_\alpha)$

(C) *Let X be a singly bi-quasi-k-space. Let Y have a cover $\mathcal{F}$ (or $\mathcal{P}^*$) $= \{C_\alpha : \alpha\}$. Then, for each $y \in Y$, $Bf^{-1}(y) = \cup\{Bf_\alpha^{-1}(y) : y \in C_\alpha\}$, where $f_\alpha = f|f^{-1}(C_\alpha)$.*

When f is a closed (or pseudo-open) map, the set $\{y \in Y : Bf^{-1}(y) \not\subset \cup\{Bf_\beta^{-1}(y)$ for any finite $\{C_\alpha : \beta\} \subset \mathcal{F}$ (or $\mathcal{P}^*$) with $y \in C_\beta\}$ is closed discrete in Y.

(D) *Let f be a closed map such that X has a cover $\mathcal{F}$ (or $\mathcal{P}$). Suppose that Y is determined by α-compact subsets, or $t(Y) \leqq \alpha$. Then, the set $\{y \in Y : f^{-1}(y) \not\subset \cup\, \mathcal{C}$ for any $\mathcal{C} \subset \mathcal{F}$ (or $\mathcal{P}$) with $|\mathcal{C}| < \alpha\}$ is closed discrete in Y*

Proof: (A) holds in view of the proof of [T_6; Theorem 1.1]. As for (B), for X being bi-quasi-k, suppose the result doesn't hold at $x \in X$. Let $\mathcal{B} = \{X - f^{-1}(S) : S$ is a finite union of elements of $\mathcal{F}$ (or $\mathcal{P}$)$\}$. Then, $\mathcal{B}$ is a filter base such that $x \in \mathrm{cl}B$ for all $B \in \mathcal{B}$. But, by Lemma 1, this is a contradiction as in the proof of Lemma 2(B). Thus, the result holds. For a map f being closed and X inner-closed A with $t(X) \leqq \omega$, the result also holds, because X has a cover $f^{-1}(\mathcal{F})$ (or $f^{-1}(\mathcal{P})$) by Lemma 3(A). For the last paragraph of (B), for $\mathcal{P}^*$, $Bf^{-1}(y)$ is obviously determined by $\{Bf_\alpha^{-1}(y) : y \in C_\alpha \in \mathcal{P}^*\}$. For $\mathcal{F}$, the same remains valid for any $\mathcal{C} \subset \mathcal{F}$ and a closed map $f|f^{-1}(\cup\mathcal{C})$, which shows that $Bf^{-1}(y)$ is dominated by $\{Bf_\alpha^{-1}(y) : y \in C_\alpha \in \mathcal{F}\}$. For (C), let $x \in Bf^{-1}(y)$. Then, $x \in \mathrm{cl}((X - f^{-1}(y)) \cap (\cup\{C_\beta : \beta\}))$ for some finite $\{C_\beta : \beta\} \subset \mathcal{F}$ as in the proof of Lemma 2(B) (by [M_2; Definition 5.F.2] and Lemma 1). Thus, $x \in Bf_\beta^{-1}(y)$ for some $C_\beta \in \mathcal{F}$ with $y \in C_\beta$. Then, $Bf^{-1}(y) = \cup\{Bf_\alpha^{-1}(y) : y \in C_\alpha\}$. For the latter part of (C), since Y is singly bi-quasi-k, by Lemma 2(A) & (C), Y is decomposed into subsets Y_0 and Y_1, where Y_0 is closed discrete in Y, and, for each $p \in Y_1$, some nbd V_p of p in Y is contained in a finite union S_p of elements of $\mathcal{F}$ (or $\mathcal{P}^*$). Let $p \in Y_1$. Then, for any $x \in Bf^{-1}(p), x \in \mathrm{cl}(f^{-1}(V_p) - f^{-1}(p))$, thus, $Bf^{-1}(p) \subset \mathrm{cl}(f^{-1}(S_p) - f^{-1}(y))$. Hence, if $p \in Y_1$, for some finite $\{C_\beta : \beta\} \subset \mathcal{F}$ (or $\mathcal{P}^*$) with $p \in C_\beta$, $Bf^{-1}(p) \subset \cup\{Bf_\beta^{-1}(p) : \beta\}$. Thus, the latter part holds. For (D), review the proof of [TY; Theorem 2.2].

Let (P) be a topological property which is hereditary with respect to closed subsets. Let (P_1) (resp. (P_2)) be the same as (P), but it is also closed under finite (resp. locally finite) unions of closed sets.

Lemma 5. (A) *If Y is singly quasi-k-space having a cover $\mathcal{F}$ of spaces with (P), then Y has a* HCP *closed cover of spaces with (P). Thus, every singly bi-quasi-k-space having a cover $\mathcal{F}$ of spaces with (P_2) is decomposed into a subspace with (P_2) and a discrete closed subspace.*

(B) *Let $f : X \to Y$ be a closed map with X paracompact. Let X be a bi-k-space, or an inner-closed A-space with $t(X) \leqq \omega$. Then* (a), (b), *and* (c) *hold.*

- (a) *If Y has a cover $\mathcal{F}$ (or $\mathcal{P}^*$) of spaces with (P_1), then Y has a* HCP *closed cover consisting of spaces with (P_1).*
- (b) *If Y has a cover $\mathcal{F}$ of spaces with (P_2), and each $Bf^{-1}(y)$ is ω_1-compact, then Y has a countable* HCP *closed cover of spaces with (P_2).*
- (c) *If Y has a cover $\mathcal{P}^* = \{P_\alpha : \alpha\}$ of spaces with (P_2), and each $Bf_\alpha^{-1}(y)$ is ω_1-compact, $f_\alpha = f|f^{-1}(P_\alpha)$, then Y has a countable* HCP *closed cover of spaces with (P_2).*

Proof: (A) holds by Lemma 2(A) & (C). As for (B), for (a), by Lemma 4(B), X has a locally finite open cover $\mathcal{V}$ such that, for each $V \in \mathcal{V}$, $f(V)$ is contained in a finite union of elements of $\mathcal{F}$ (or $\mathcal{P}^*$). Thus, $f(\mathrm{cl}\mathcal{V})(= \{f(\mathrm{cl}V) : V \in \mathcal{V}\})$ is a HCP closed cover consisting of spaces with (P_1). For (b), we can assume that each $f^{-1}(y)$ is Lindelöf. Then, each $f^{-1}(y)$ has a countable subcover $\mathcal{V}_y$ of the open cover $\mathcal{V}$. Since f is closed, there exists a nbd G_y of y with $f^{-1}(G_y) \subset \cup\mathcal{V}_y$. Since Y is paracompact, Y has a locally finite closed refinement $\mathcal{C}$ of $\{G_y : y \in Y\}$. Note that, for each $C \in \mathcal{C}$, C is contained in a countable union of elements of the HCP closed cover $f(\mathrm{cl}\mathcal{V}_y)$, thus, C is determined by a countable HCP closed cover $\mathcal{C}_C = \{C \cap C_i : i \in N\}$, where $\{C_i : i \in N\} \subset f(\mathrm{cl}\mathcal{V}_y)$. But, Y is determined by $\mathcal{C}$. Then, Y is determined by a cover $\cup\{\mathcal{C}_C : C \in \mathcal{C}\}$. For each $n \in N$, let $F_n = \cup\{C \cap C_n : C \in \mathcal{C}\}$. Then, Y is determinded by $\{F_n : n \in N\}$, and F_n are closed subsets of Y having property (P_2). Let $\{Y_n : n \in N\}$ be the collection of all finite unions of the F_n's. Since Y is determined by $\{Y_n : n \in N\}$, by Lemma 4(B), each $x \in X$ has a nbd W_x with $W_x \subset f^{-1}(Y_n)$ for some $n \in N$. For each $n \in N$, let $G_n = \cup\{W_x : W_x \subset f^{-1}(Y_n)\}$. Then, there exists a locally finite open refinement $\{H_n : n \in N\}$ of $\{G_n : n \in N\}$. Hence, $\{f(\mathrm{cl}H_n) : n \in N\}$ is a countable HCP cover of Y such that $f(\mathrm{cl}H_n) \subset Y_n$, thus, $f(\mathrm{cl}H_n)$ has (P_2). Hence, (b) holds. (c) is similarly shown as in (b).

We recall that a cover $\mathcal{C}$ of a space X is a *k-network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \mathcal{C}^* \subset U$ for some finite $\mathcal{C}^* \subset \mathcal{C}$. Spaces with a countable (resp. σ-locally finite) k-network are called $\aleph_0$-*spaces* (resp. $\aleph$-*spaces*).

Lemma 6. (A) *Let X be a k-space with a σ-HCP k-network. If $\{d_\alpha : \alpha \in A\}$ is a closed discrete subset of X, then there exists a discrete closed collection $\{W_\alpha : \alpha \in A\}$ of subsets of X such that each W_α is a weak sequential nbd of d_α; that is any sequence converging to d_α is frequently contained in W_α.*

(B) *let* $f : X \to Y$ *be a closed and open map such that each point of* X *is a* G_δ*-set. Let* $y \in Y$ *be not isolated in* Y*, and* $\{y_n : n \in N\}$ *be a sequence converging to* y*. Then, for each* $x \in f^{-1}(y)$*, there exists a sequence* $\{x_n : n \in N\}$ *converging to* x*, and* $\{f(x_n) : n \in N\}$ *is a subsequence of* $\{y_n : n \in N\}$*.*

Proof: (A) is shown by modifing the proof of [F_0; Theorem 1.3] (indeed, replacing "$\{F_\alpha : \alpha \in A\}$" by " $\{d_\alpha : \alpha \in A\}$" in [$\mathrm{F}_0$], we show that the collection $\{W_\alpha : \alpha \in A\}$ there, is a desired discrete closed collection of weak sequential nbds). For (B), if $x \in f^{-1}(y)$, there exists a sequence $\{G_n : n \in N\}$ of nbds of x with $\mathrm{cl}G_{n+1} \subset G_n$. Since y is not isolated in Y, and $f(G_n)$ are nbds of y, there exists a sequence of $\{x_n : n \in N\}$ in X such that $x_n \in G_n$, and $\{f(x_n) : n \in N\}$ is a subsequence of $\{y_n : n \in N\}$. But, f is closed, then, any subsequence of $\{x_n : n \in N\}$ has an accumulation point. Thus, the sequence $\{x_n : n \in N\}$ converges to x.

Lemma 7. *Let* $X \times Y$ *be a quasi-k-space with* $t(X) \leqq \alpha$*. Then,* X *is an inner-one A-space, otherwise* Y *satisfies the following* (a) *and* (b).

(a) *Any closed paracompact M-subspace is locally* α*-compact.*
(b) *Space containing no closed copy of* S_γ*, where* $\gamma = 2^\alpha$*.*

Proof: Suppose that X is not an inner-one A-space. Then, Y satisfies (a) by [T_2; Proposition 2.4 & Theorem 4.2]. Moreover, to show that Y satisfies (b), let Y contain a closed copy of S_γ. Let F be a closed subset of X such that $F = \mathrm{cl}S$ with $|S| \leqq \alpha$, thus, F has weight $\leqq 2^\alpha$. Suppose that F is not locally countably compact at $p \in F$. Then, there exists a local base $\{U_\beta : \beta < \gamma\}$, $\gamma = 2^\alpha$, at p in F. For each $\beta < \gamma$, there exists a closed discrete subset $D_\beta = \{d_{n\ \beta} : n \in N\}$ of $\mathrm{cl}U_\beta$. Let $T = \{(d_{n\ \beta}, n_\beta) : n \in N, \beta < \gamma\}$, where n_β is the n-th term of the β-th sequence in S_γ. Then, T is not closed in $F \times S_\gamma$. But, $T \cap K$ is finite (hence closed) in K for every countably compact subset K of $F \times S_\gamma$. Then, since $F \times S_\gamma$ is a quasi-k-space, T is closed in $F \times S_\gamma$. This is a contradiction. Thus, every closed subset of X with a dense subset of cardinality $\leqq \alpha$

is locally countably compact. Now, to see X is an inner one A-space, for $x \in X$, let $x \in \mathrm{cl}(A_n - \{x\})$ with $A_n \supset A_{n+1}$. Then, since $t(X) \leqq \alpha$, for each $n \in N$, there exists $C_n \subset A_n - \{x\}$ such that $x \in \mathrm{cl} C_n$ and $|C_n| \leqq \alpha$. Let $C = \cup\{C_n : n \in N\}$. Then, $|C| \leqq \alpha$, so, by the above, $F = \mathrm{cl} C$ is locally countably compact at x in F. Since $x \in \mathrm{cl}_F(C_n - \{x\})$ for each $n \in N$, there exists an infinite set $\{p_n : n \in N\}$ with $p_n \in C_n$, but it is not closed in X. Hence, X is an inner one A-space.

Theorems

Theorem 1. *Let $f : X \to Y$ be a closed map such that X is an s_α-space. Then, the following* (A) *and* (B) *hold.*

(A) *Let X be a bi-k-space. If every closed paracompact M-subspace of Y is locally α-compact, then each $Bf^{-1}(y)$ is locally α-compact.*

(B) *Let Y have a cover $\mathcal{F}$ (or $\mathcal{P}^*$)$= \{Y_\gamma : \gamma\}$, where each Y_γ is one of the following spaces: (i) inner-closed A-space, (ii) locally α-compact space, (iii) sequentail space which contains no closed copy of S_α.*

Then, (a) *and* (b) *below hold.*

(a) *Suppose that X is a bi-quasi-k-space; or an inner-closed A-space with $t(X) \leqq \omega$. Then, each $Bf^{-1}(y)$ is a locally α-compact space dominated by a closed cover $\mathcal{C}$ of α-compact subsets (where, $\alpha = \omega$ if all Y_γ are (i)). For $\mathcal{P}^*$, the closed cover $\mathcal{C}$ can be countable.*

(b) *Suppose that X is a singly bi-quasi-k-space. Then, the set $\{y \in Y : Bf^{-1}(y)$ is not α-compact$\}$ is closed discrete in Y. For $\mathcal{P}^*$, each $Bf^{-1}(y)$ is a countable union of α-compact closed subsets.*

Proof: This holds by means of Lemma 3(B), but also use Lemma 4(A) for (A), and use Lemma 4(B) and Lemma 4(C) for (B).

Note 1: (1) Related to (A), for a closed image Y of a paracompact bi-k-space, any closed paracompact M-subspace of Y

is locally α-compact $\Leftrightarrow$ Y is a closed image of a paracompact bi-k-space which is locally α-compact, where α is regular. In fact, this is shown as in the proof of [T_6; Theorem 1.1], but, for the proof of ($\Rightarrow$), also use a fact that, for any closed map $g : S \rightarrow T$ with S paracompact, each α-compact closed subset of T is a closed image of an α-compact closed subset of S as in the proof of [M_1; Corollary 1.2] by means of Lemma 3(B).

(2) As for (a) in (B), if Y has a cover $\mathcal{P}$ of α-compact spaces, then each $Bf^{-1}(y)$ is locally α-compact in view of Lemma 4(B). (However, we don't know whether it is possible to replace " locally α-compact" by "α-compact").

Related to (b) in (B), we have the following question:

Question: Let $f : X \rightarrow Y$ be a closed map such that Y has a cover $\mathcal{F}$ (or $\mathcal{P}^*$) of metric subsets. If X is paracompact, then, is the set $\{y \in Y : Bf^{-1}(y)$ not compact (or, not locally compact)$\}$ σ-discrete closed in Y?

Theorem 2. *Let $f : X \rightarrow Y$ be a closed map. Then, the following (A) and (B) hold.*

(A) *If X has a cover $\mathcal{F}$ (or $\mathcal{P}^*$) of α-compact spaces, here α is regular, then, the set $\{y \in Y : f^{-1}(y)$ is not α-compact$\}$ is closed discrete in Y. For $\mathcal{F}$ (or $\mathcal{P}^*$ instead of $\mathcal{P}$), we can add a prefix " locally" before "α-compact".*

(B) *If X has a cover $\mathcal{F}$ (or $\mathcal{P}^*$) of metric spaces, then, the set $\{y \in Y : f^{-1}(y)$ is not metric$\}$ is closed discrete in Y.. For $\mathcal{F}$, the set $\{y \in Y : f^{-1}(y)$ is not compact metric$\}$ is σ-discrete closed in Y.*

Proof: (A) holds by Lemma 4(D). For (B), the first half holds by Lemma 4(D), for X is a k-space. For the latter part, since X is dominted by metric subsets, as is well-known, X is a (normal) σ-space. Thus, Y is decompsed into subspaces Y_0 and Y_1, here Y_0 is σ-discrete closed in Y, and for each $y \in Y_1$, $f^{-1}(y)$ is compact by a decomposition theorem on normal σ-spaces; see [0]. Thus, the latter part of (B) holds by the first half.

Note 2. As for the latter part of (B), it is impossible to omit "σ-" even if X is metric. Indeed, in view of [$\mathrm{L_a}$], there exists a closed map $f : X \to Y$ such that X is metric and Y is a countable space, but any $Bf^{-1}(y)$ not locally compact. Thus, for any closed discrete subset D of Y, $Bf^{-1}(y)$ is not locally compact for any $y \in Y - D$. For $\mathcal{P}^*$, we have the following question:

Question: Let $f : X \to Y$ be a closed map with X paracompact. Let X have a cover $\mathcal{P}^*$ of metric spaces. Then, is the set $\{y \in Y : f^{-1}(y)$ is not compact (or, not locally compact) metric$\}$ a σ-discrete closed subset of Y?

Theorem 3. *For a space X, the following* (A) *and* (B) *hold. It is possible to replace "paracompact M-space(s)" by "paracompact bi-k-spaces(s)" (or "metric space(s)", etc) in* (A) *and* (B).

(A) *The following are equivalent.*
   (a) *X is singly bi-quasi-k-space having a cover $\mathcal{F}$ of paracompact M-spaces.*
   (b) *X is a closed image of a paracompact M-space, and X has a cover $\mathcal{F}$ of paracompact M-spaces.*
   (c) *X has a* HCP *closed cover of paracompact M-spaces.*

(B) *The following are equivalent.*
   (a) *X is a closed image of a paracompact M-space, and X has a cover $\mathcal{P}^*$ of M-spaces.*
   (b) *X is closed, Lindelöf (i.e., fibers are Lindelöf) image of a paracompact M-space, and X has a cover $\mathcal{F}$ of M-spaces.*
   (c) *X has a countable* HCP *closed cover of paracompact M-spaces.*

Proof: For (A), (a) $\Rightarrow$ (c) holds by Lemma 2(A). (c) $\Rightarrow$ (b) holds, for X is a closed image of an M-space Z, where Z is the topological sum of the *HCP* closed cover of paracompact M-spaces in (c). (b) $\Rightarrow$(a) is clear. For (B), (a) $\Rightarrow$ (b) holds by means of Lemmas 3(B) and 4(B). (b) $\Rightarrow$ (c) holds by Lemma 5(B), and (c) $\Rightarrow$ (a) is obvious.

Note 3. In [Mo$_2$], CM-spaces are introduced as a modification of Lašnev spaces, and it is shown that, for a space X, X is a quasi-k-and CM-space (resp. paracompact, k-and-CM-space) $\Leftrightarrow$ X is a closed images of M-spaces (resp. paracompact M-spaces).

For a closed map $f : X \to Y$, let us consider the following conditions. (c$_1$) (or (c$_1$)$^+$) and (c$_2$) are independent in view of [L$_1$]. Note that, for any obvious map f from X onto $Y = X/A$ (with A identified), (c$_2$) is satisfied.

(c$_1$) Every $Bf^{-1}(y)$ is locally compact.

(c$_1$)$^+$ Every $Bf^{-1}(y)$ is locally compact, and Lindelöf.

(c$_2$) The set $\{y \in Y : Bf^{-1}(y)$ is not compact$\}$ is closed discrete in Y.

T. Miwa [Mi] asked whether every Lašnev space X/A with X metric has a cover $\mathcal{F}$ of metric spaces. A negative answer is given in [TZ$_2$] by showing that, for a closed map $f : X \to Y$ with X metric, if Y has cover $\mathcal{F}$ of metric spaces, then (c$_1$) is satisfied. Recently, S. Lin [L$_2$] gave the following complete answer to the Miwa's question: For a closed map $f : X \to Y$ with X metric, Y has a cover $\mathcal{F}$ of metric spaces $\Leftrightarrow$(c$_1$) and (c$_2$) are satisfied.

We have the following analogous result to the S. Lin's result.

Theorem 4. *For a closed map $f : X \to Y$ with X a paracompact M-space, the following* (A) *and* (B) *hold.*

(A) *The following are equivalent.*
  (a) *Y has a cover $\mathcal{F}$ of paracompact M-spaces.*
  (b) *Y has a* HCP *closed cover of paracompact M-spaces.*
  (c) (c$_1$) *and* (c$_2$) *are satisfied.*

(B) *The following are equivalent.*
  (a) *Y has a cover $\mathcal{P}^*$ of paracompact M-spaces.*
  (b) *Y has a countable* HCP *closed cover of paracompact M-spaces.*
  (c) (c$_1$)$^+$ *and* (c$_2$) *are satisfied.*

Proof: (A) (a) $\Leftrightarrow$ (b) holds by Theorem 3(A). (b) $\Rightarrow$ (c) holds by Theorem 1(B). Let us prove (c) $\Rightarrow$ (b) holds by refering to the proof of [L$_2$; Theorme 1], where a metric function on X is used. First, X is a paracompact M-space, so there exists a sequence $\{\mathcal{V}_n : n \in N\}$ $(\mathcal{V}_{n+1} < \mathcal{V}_n)$ of locally finite open covers of X such that, for each $x \in X$ and $n \in N$, $\mathrm{cl}(\mathrm{St}(x, \mathcal{V}_{n+1})) \subset \mathrm{St}(x, \mathcal{V}_n)$; and, if $x_n \in \mathrm{St}(x, \mathcal{V}_n)$, the sequence $\{x_n : n \in N\}$ has an accumulation point in X. Thus, for any $A, A_n \subset X$ and any $n \in N$, the following (i) and (ii) hold.

(i) $\mathrm{cl}(\mathrm{St}(A, \mathcal{V}_{n+1})) \subset \mathrm{St}(A, \mathcal{V}_n)$: (ii) if $x \in \mathrm{St}(A_n, \mathcal{V}_n)$, there exist $x_n \in A_n$ such that any subsequence of $\{x_n : n \in N\}$ has an accumulation point in X.

Now, suppose (c) holds. For each $y \in Y$, pick $x_y \in f^{-1}(y)$, and let $X_y = Bf^{-1}(y)$ if $Bf^{-1}(y) \neq \emptyset$, otherwise $X_y \doteq \{x_y\}$. Let $X^* = \cup\{X_y : y \in Y\}$. Then, X^* is closed in X. Thus, $h = f|X^* : X^* \to Y$ is a closed map such that X^* is a paracompact M-space, and $h^{-1}(y) = Bf^{-1}(y)$ or $\{x_y\}$. Thus, we can assume that every $f^{-1}(y)$ is locally compact, and the set $D = \{y \in Y : f^{-1}(y)$ is not compact$\}$ is closed discrete in Y. Let $D = \{y_\alpha : \alpha < \lambda\}$, and let $F_\alpha = f^{-1}(y_\alpha)$. We show the following holds.

Claim: If all F_α are Lindelöf , then Y is determined by the increasing countable closed cover of paracompact M-spaces.

Indeed, any F_α is locally compact and Lindelöf, then F_α is determined by a countable increasing cover $\{K_{\alpha n} : n \in N\}$ of compact subsets with $F_\alpha \neq K_{\alpha n}$. While, $\{F_\alpha : \alpha < \lambda\}$ is a discrete closed collection in X, there exists a discrete open collection $\{G_\alpha : \alpha < \lambda\}$ in X with $F_\alpha \subset G_\alpha$. For each $\alpha < \lambda$, and $n \in N$, let $G_{\alpha n} = G_\alpha \cap \mathrm{St}(F_\alpha - K_{\alpha n}, \mathcal{V}_n)$, $G_n = \cup\{G_{\alpha n} : \alpha < \lambda\}$, $X_n = X - \cup\{G_n : n \in N\}$, $D_n = f(X_n)$, and let $f_n = f|X_n$. Then, each X_n is clearly closed in X with $X_n \subset X_{n+1}$, and also $\{X_n : n \in N\}$ is a cover of X. Indeed, for some $x \in X$, let $x \in X - X_n$ for any $n \in N$. Then, for each $n \in N$, $x \in \mathrm{St}(F_{\alpha(n)} - K_{\alpha(n)n}, \mathcal{V}_n)$ for some $\alpha(n) < \lambda$. Then, by (ii), there exists a sequence $L = \{x_n : n \in N\}$ with $x_n \in F_{\alpha(n)} - K_{\alpha(n)n}$ such that any subsequence of L has an accumulation point in X. Let $A = \{\alpha(n) : n \in N\}$. If A is infinite, $S = L \cap (\cup \{F_{\alpha(n)} : n \in N\}$ contains an infinite discrete closed set, because $\{F_{\alpha(n)} : n \in N\}$ is discrete in X. This is a contradiction. If A is finite, $S = L \cap F_\alpha$ is infinite for some $\alpha < \lambda$, but S is discrete in F_α, because F_α is determined by a countable increasing cover $\{K_{\alpha n} : n \in N\}$ with each $S \cap K_{\alpha n}$ at most finite. This is also a contradiction. Thus, $\{X_n : n \in N\}$ is a cover of X. For each $y \in D_n$, $f_n^{-1}(y) = K_{\alpha n} - G_{\alpha n}$ if $y = y_\alpha$, and $f_n^{-1}(y) = f^{-1}(y) \cap X_n$ if $y \in D_n - D$. Then, f_n is a perfect map. Thus, D_n is a perfect image of a paracompact M-space X_n. Hence, as is well-known, D_n is a paracompact M-space., Let us show that Y is determined by an increasing closed cover $\{D_n : n \in N\}$. Since Y is a k-space. it suffices to show that, for each compact set $C \subset Y$, $C \subset D_n$ for some $n \in N$. So, suppose that, for some compact subset $C \subset Y$, $C \not\subset D_n$ for any $n \in N$. Then, there exists a sequence $\{p_n : n \in N\}$ with $p_n \in C - D_n \subset f(G_n)$. For each $n \in N$, take $q_n \in f^{-1}(p_n) \cap G_n$. Since f is closed and the sequence $\{p_n : n \in N\}$ in C has an accumulation point in Y, the sequence $\{q_n : n \in N\}$ has an accumulation point q in X. Since $G_{n+1} \subset G_n$ for each $n \in N$, $q \in \mathrm{cl} G_n$ for each $n \in N$. But, for each $n \in N$, $\{G_{\alpha n} : \alpha < \lambda\}$ is a discrete collection in X. Then, for each $n \in N (n \geqq 2)$ $q \in \mathrm{cl} G_{\alpha(n)n}$ for some $\alpha(n) < \lambda$, thus, $q \in \mathrm{St}(F_{\alpha(n)} - K_{\alpha(n)n}, \mathcal{V}_{n-1})$ by (i). But, in view of the above, this is a contradiction by (ii). Thus, *Claim* holds.

We complete the proof of (c) $\Rightarrow$ (b). Since any $F_\alpha = f^{-1}(y_\alpha)$ is locally compact and paracompact, F_α has a star-countable cover of compact subsets. Thus, as is well-known, each F_α is the topological sum of $\{L_{\alpha\beta} : \beta < \gamma\}$ of σ-compact (hence, Lindelöf) subsets. Since $\{L_{\alpha\beta} : \alpha < \lambda, \beta < \gamma\}$ is a discrete closed collection in X, there exists a discrete open collection $\{V_{\alpha\beta} : \alpha < \lambda, \beta < \gamma\}$ with $L_{\alpha\beta} \subset V_{\alpha\beta}$. For each $\beta < \gamma$, let $X_\beta = \mathrm{cl}(\cup\{V_{\alpha\beta} : \alpha < \lambda\})$, $Y_\beta = f(X_\beta)$, and let $X_\gamma = Z - \cup\{V_{\alpha\beta} : \alpha < \lambda, \beta < \gamma\}$, and $Y_\gamma = f(X_\gamma)$. For each $\beta \leqq \gamma$ let $f_\beta = f|X_\beta$. For each $\beta \leqq \gamma$, and $y \in Y_\beta$, $f_\beta^{-1}(y) = L_{\alpha\beta}$ if $y = y_\alpha$ and $f_\beta^{-1}(y) = f^{-1}(y) \cap X_\beta$ if $y \in Y_\beta - D$. Thus, $f_\beta^{-1}(y)$ is Lindelöf. Hence, $f_\beta : X_\beta \to Y_\beta$ is a closed map such that X_β is a paracompact M-space, and $f_\beta^{-1}(y)$ is Lindelöf, locally compact, but $\{y \in Y_\beta : f_\beta^{-1}(y)$ is not compact$\}$ is closed discrete in Y_β. Then, by *Claim,* for each $\beta \leqq \gamma$, Y_β can be determined by a countable closed cover of paracompact M-spaces. Thus, by Theorem 3(B) each Y_β has a countable HCP closed cover of paracompact M-spaces. While, since $\{X_\beta : \beta \leqq \gamma\}$ is a HCP closed cover of X, so is a cover $\{Y_\beta : \beta \leqq \gamma\}$ of Y. Hence, it follows that Y as a HCP closed cover of paracompact M-spaces. Thus, (b) holds.

(B) (a) $\Rightarrow$ (b) holds by Theorem 3(B). (b) $\Rightarrow$ (c) holds by Theorem 1(B). For (c) $\Rightarrow$ (a), Y has a cover $\mathcal{F}$ of paracompact M-spaces by (A). While, Y is a closed Lindelöf image of a paracompact M-space, for each $Bf^{-1}(y)$ is Lindelöf. Thus, (c) $\Rightarrow$(a) holds by Theorem 3(B).

Let us recall the following facts. Here, $\chi(y,Y) = \min\{|\mathcal{B}| : \mathcal{B}$ is a local base at y in $Y\}$.

Facts: (A) Let $f : X \to Y$ be a closed (or quotient) map with X metric. For a cover $\mathcal{C}$ of Y, $\mathcal{C}$ determines $Y \Leftrightarrow$ for each sequence L converging to x in X, some $C \in \mathcal{C}$ contains $f(L)$ frequently, and the point $f(x)$.

(B) Let $f : X \to Y$ be a closed map with X metric. Then, (a) If $f^{-1}(y)$ is compact, $\chi(y,Y) \leqq \omega$; (b) If $\chi(y,Y) \leqq \omega$ for all $y \in Y$, Y is metric; and (c) If Y is an M-space, Y is metric.

Proof: For (A), ($\Leftarrow$) is routine, for Y is determined by a cover $\{f(L); L$ is a convergent sequence in X containing the limit point$\}$. For ($\Rightarrow$), note that if the set $f(L)$ doesn't contain the point $f(x)$, $f(L)$ is not closed in Y. For (B), (a) is routine, (b) is well-known. (c) holds by (b), because each point of Y is a G_δ-set, thus, by [M$_2$; Corollary 7.2], $\chi(y,Y) \leqq \omega$ for all $y \in Y$.

Corollary: *Let $f : X \to Y$ be a closed map such that X is metric. Then the following* (A) *and* (B) *hold.*

(A) ([L$_2$]). The following are equivalent.

(a) Y has a cover $\mathcal{F}$ of metric subsets.
(b) Y has a *HCP* closed cover of metric subsets.
(c) (c_1) and (c_2) are satisfied

(B) The following are equivalent.
(a) Y has a countable cover $\mathcal{P}^*$ of metric subsets.
(b) Y has a cover $\mathcal{P}^*$ of metric subsets.
(c) Y has a cover $\mathcal{P}$ of metric subsets, and (c_2) is satisfied.
(d) $(c_1)^+$ and (c_2) are satisfied.

Proof: By means of Theorem 4 and *Fact* (B), it suffices to show that (c) $\Rightarrow$ (b) in (B) holds. So let (c) hold. By (c_2), we can assume that $D = \{y \in Y : f^{-1}(y) \text{ is not compact}\}$ is closed discrete in Y. Let $D = \{y_\alpha : \alpha < \lambda\}$, and let $\{V_\alpha : \alpha < \lambda\}$ be a discrete open collection in Y with $y_\alpha \in V_\alpha$. For the cover $\mathcal{P}$ in (c), let $\mathcal{C} = \{ \text{cl}(P \cap V_\alpha) : \ y_\alpha \in P \in \mathcal{P};\ \alpha < \lambda\} \cup \{f(X - U_{1/n}(f^{-1}(D)) :\ n \in N\}$, here $U_{1/n}(f^{-1}(D)) = \{x \in X : d(x, f^{-1}(D)) < 1/n\}$. d is a metric on X. Then, $\mathcal{C}$ is a point-countable closed cover of Y, and it is routinely shown that $\mathcal{C}$ determines Y by *Fact* (A). We show that each member of $\mathcal{C}$ is metric. For all $y \in Y - D$, $f^{-1}(y)$ are compact. Then, for each $n \in N$, $f(X - U_{1/n}(f^{-1}(D)) \subset Y - D$ is metric by *Fact* (B). For each $P \in \mathcal{P}$ with $y_\alpha \in P \in \mathcal{P}$, $\chi(y, \text{cl}(P \cap V_\alpha)) \leqq \omega$ for all $y \in \text{cl}(P \cap V_\alpha)$, thus, $\text{cl}(P \cap V_\alpha)$ is metric by *Fact* (B). Indeed, let $y \in \text{cl}(P \cap V_\alpha)$. If $y \neq y_\alpha$, then $f^{-1}(y)$ is compact, thus, by *Fact* (B), $\chi(y, \text{cl}(P \cap V_\alpha)) \leqq \omega$. If $y = y_\alpha$, then $\chi(y, P \cap V_\alpha) \leqq \omega$. Thus, since Y is regular, $\chi(y, \text{cl}(P \cap V_\alpha)) \leqq \omega$. Hence, Y has a cover $\mathcal{P}^* (= \mathcal{C})$ of metric subsets, which shows (b) holds.

Note 4. (1) Let us consider the following properties (F) and (P^*) : (F) Space with a cover $\mathcal{F}$ of bi-k-spaces; and (P^*) Space with a cover $\mathcal{P}^*$ of bi-k-spaces.

Let $f : X \to Y$ be a closed map such that X is a paracompact space. If X is locally compact, Y satisfies (F), because Y has a *HCP* closed cover of compact subsets. If X is a bi-k-space, then, by Theorem 1(B), the following implication holds: (*) Y satisfies (F) (resp. $(\mathcal{P}^*)$ $\Rightarrow$ (c_1) (resp. $(c_1)^+$) and (c_2) are satisfied. If X is an M-space, the reverse of (*) holds by

Theorem 4. But, for X being a bi-k-space, we don't know whether the reverse also holds.

(2) For a closed map $f : X \to Y$ such that X is metric, we have the following (a) and (b). Here, (a) shows that (c_2) is essential in (c) of (B) in the previous corollary.

(a) Even if Y has a countable cover $\mathcal{P}$ of metric subsets, Y need not satisfy (F) or $(\mathcal{P}^*)$. Indeed, as in [L$_1$], let $X = \{(x,y) \in R^2 : x \in [0,1],\ 0 \leqq y < x\} \cup \{(0,0)\}$ be a subspace of the Euclidean plane. Let $Y = \{(x,y) \in R^2 : x \in (0,1) - C,\ 0 < y < x\} \cup \{(x,0) : x \in C\}$, $C = \{1/n : n \in N\} \cup \{0\}$, be the quotient space determined by a map $f : X \to Y$ defined by $f(x,y) = (x,0)$ if $x \in C$, otherwise $f(x,y) = (x,y)$. Then, f is a closed map such that X is separable metric, and $(c_1)^+$ is satisfied. But, (c_2) is not satisfied, so Y doesn't satisfy (F) or $(\mathcal{P}^*)$ by the implication (*) in (1). Let $\mathcal{P} = \{Y - \{(1/n,0) : n \in N\}\} \cup \{K_0, K_n : n \in N\}$, where $K_0 = C \times \{0\}$, $K_n = \{f(x,y)) : y \leqq x - 1/n\}$. Then, $\mathcal{P}$ is a countable cover of Y by metric subsets, and $\mathcal{P}$ determines Y by *Fact* (A).

(b) If Y has a cover $\mathcal{P}$ of metric subsets, then every $Bf^{-1}(y)$ is Lindelöf by Lemma 3(B). Indeed, since Y has a cover $\mathcal{P}$ of metric subsets, it is routine that Y contains no closed copy of S_{ω_1} by means of *Fact* (A). But, we don't know the relationship between $(c_1)^+$ and "Y has a cover $\mathcal{P}$ of metric subsets".

Let α be an infinite cardinal. Let us say a space X is a ***c_α-space*** if, for each non-isolated point $p \in X$, any collection $\{U_\beta : \beta < \alpha\}$ of nbds of p is not HCP in X. If $\alpha = \omega$ and the A_β are singletons, such a space is a c-space in the sense of [C]. Quasi-k-spaces, A-spaces, or c-spaces are c_ω-spaces. Locally α-compact spaces are c_α-spaces. If $\alpha < \gamma$, c_α-spaces are c_γ-spaces. Note that spaces determined by c_α-spaces are c_α-spaces. In the following theorm, (A) is a generalization of a result in [C], where X is a uniformly complete space, and Y is a c-space.

Theorem 5. *For a closed and open map $f : X \to Y$, the following* (A) *and* (B) *hold.*

(A) *Let X be an s_α-space. If Y is a c_α-space (or Y is determined by c_α-spaces), then, each $Bf^{-1}(y)$ is a α-compact (indeed, $f^{-1}(y)$ is a α-compact for each non-isolated point $y \in Y$).*

(B) *If X is a k-space with a σ-HCP k-network, then, each $Bf^{-1}(y)$ is compact (indeed, $f^{-1}(y)$ is compact for each non-isolated point $y \in Y$).*

Proof: (A) Let $y \in Y$. If $y \in Y$ is isolated in $Y, Bf^{-1}(y) = \emptyset$ since f is open. Then, we assume that y is not isolated in Y. Supposet that $f^{-1}(y)$ is not α-compact. Then, there exists a discrete closed subset $\{x_\beta : \beta < \alpha\}$ in $f^{-1}(y)$. Since X is an s_α-space, there exists a HCP collection $\{V_\beta : \beta < \alpha\}$ of open subsets with $x_\beta \in V_\beta$. Let $W_\beta = f(V_\beta)$ for each $\beta < \alpha$. Then, since f is closed and open, $\{W_\beta : \beta < \alpha\}$ is a HCP collection of nbds of y in Y. But, the point y is not isolated in Y. Hence, Y is not a c_α-space, a contradiction. Thus, $f^{-1}(y)$ is α-compact.

(B) Let $y \in Y$ be not isolated in Y. Suppose that $f^{-1}(y)$ is not compact. Then, $f^{-1}(y)$ is not countably compact. Then, there exists a discrete closed subset $\{x_n : n \in N\}$ in $f^{-1}(y)$. By Lemma 6(A), there exists a discrete closed collection $\{W_n : n \in N\}$, where each W_n is a weak sequential nbd of x_n in X. Since Y is sequential, there exists a sequence $\{y_n ; n \in N\}$ converging to y with $y_n \neq y$. By Lemma 6(B), there exists a sequence L_1 converging to x_1 in X such that $f(L_1)$ is a subsequence of $\{y_n : n \in N\}$. Since F_1 contains L_1 frequently, we assume that $L_1 \subset F_1$. Similarly, for each $n \in N$, there exists a sequence $L_n \subset W_n$ converging to x_n such that $f(L_n)$ is a subsequence of $f(L_{n-1})$. Then, we can choose points $p_n \in L_n$ such that $\{f(p_n) : n \in N\}$ is a subsequence of $\{y_n : n \in N\}$. Then, $\{p_n : n \in N\}$ is closed discrete in X, thus, so is $\{f(p_n) : n \in N\}$ in Y. But, $\{f(p_n) : n \in N\}$ converges to the point y. This is a contradiction. Thus, $f^{-1}(y)$ is compact.

Note 5. For a closed map $f : X \to Y$, if X is a sequential s_ω-space, and Y contains no closed copy of S_ω, then, each $Bf^{-1}(y)$

is ω-compact by Lemma 3(B). Here, we can replace "sequential s_ω-space" by "k-space with a star-countable k-network", because any k-space with a star-countable k-network is a sequential paracompact space (dominated by k-and-$\aleph_0$-spaces) ([S]). However, related to (B) in Theorem 5, we have the following question:

Question: Let $f : X \to Y$ be a closed map such that X is a k-space with a σ-HCP (or σ-locally finite) k-network (thus, X is sequential). If Y contains no closed copy of S_ω, then, is each $Bf^{-1}(y)$ compact?

Finally, let us consider the products of (quasi-) k-spaces. The following theorem holds by Lemmas 3(B) and 7, and Theorem 1(A).

Theorem 6. *Suppose that $f_i : X_i \to Y_i$ $(i = 1, 2)$ are closed maps such that X_i are bi-k-spaces, X_1 is an s_ω-space and X_2 is an s_α-space, and $t(Y_1) \leqq \alpha$. Let $Y_1 \times Y_2$ be a quasi-k-space. Then, (a) or (b) below holds. When X_2 is an s_γ-space, $\gamma = 2^\alpha$, and Y_2 is sequential, (a) or (b)$^+$ below holds.*

(a) *Every $Bf_1^{-1}(y_1)$ is ω-compact.*

(b) *Every $Bf_2^{-1}(y_2)$ is locally α-compact.*

(b)$^+$ *Every $Bf_2^{-1}(y_2)$ is locally α-compct, and 2^α-compact.*

Note 6. The converse of the above result doesn't hold. Indeed, let $X_1 = X_2 = Y_1$ be the rationals Q, and let $Y_2 = Q/Z$, where Z is the integers. Then, (a), and (b)$^+$ (with $\alpha = \omega$) holds for the obvious closed maps f_i from X_i onto Y_i, but $Y_1 \times Y_2$ is not a quasi-k-space in view of [F]. Related to Theorem 6, when X_i are paracompact bi-k-spaces, and Y_i are sequential spaces, in [T_7], the first author gave some necessary and sufficient conditions on Y_i for $Y_1 \times Y_2$ to be a k-space (equivalently, quasi-k-space) under (CH), etc.

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