Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 24, Summer 1999, 1–34

MEASURES ON CANTOR SPACE

Ethan Akin

Abstract

We construct an uncountable family of topologically distinct measures on the Cantor set. These measures are all rigid, i.e. the identity is the only measure preserving automorphism. On the way to the construction we classify the order isomorphism classes of full, monotonic measures on an ordered Cantor space.

0. Introductory Confusions

Perhaps I should not publicly admit how recently I learned about the classic work of Oxtoby and Ulam or how stunned I was by their results. In the fall of 1998, Joe Auslander pointed out to me the paper [9] in which they proved that any measure μ on a Euclidean ball B which is *full* (i.e. $\mu(U) > 0$ for every nonempty open subset U of B), *nonatomic* (i.e. $\mu(\{p\}) = 0$ for every point p of B) and trivial on the boundary sphere (i.e. $\mu(\partial B) = 0$) is homeomorphic to the Lebesgue measure λ on B with $\mu(B) = \lambda(B)$ via a homeomorphism $h : B \to B$ which restricts to the identity on the boundary. They used this *Homeomorphic Measures Theorem* together with the rich supply of measure preserving automorphisms of B to prove the Oxtoby-Ulam Theorem. This says that the ergodic homeomorphisms form a residual subset of the space of measure preserving automorphisms of B. This work still stimulates current research. For example, at the summer, 1999 Topology Conference,

Mathematics Subject Classification: 28A33, 28C15, 37A05, 54F05
Key words: Cantor Space, Measures on Cantor Space, Oxtoby-Ulam Theorem, Measures on Ordered Spaces

H. Murat Tuncali showed me a paper extending the Oxtoby-Ulam Theorem to Menger manifolds [5].

Let me try to explain my surprise. Consider the following measure on the unit disk B in the plane:

$$\mu = .1\lambda_2 + .9\lambda_1 \tag{0.1}$$

where λ_2 is ordinary two dimensional Lebesgue measure on the disk and λ_1 is linear Lebesgue measure on some concentric circle C of radius less than 1. Each measure is normalized so that $\lambda_1(B) = \lambda_2(B) = 1$. By the Homeomorphic Measures Theorem there exists an automorphism h of B such that $h_*\mu = \lambda_2$. Then the image $h(C)$ is a Jordan curve in the disk whose two dimensional Lebesgue measure is .9. Amazing.

I have been told that I should not be amazed. After all, I am familiar with Cantor subsets of the unit interval whose one dimensional Lebesgue measure is ϵ close to 1 (e.g. remove "middle $\frac{\epsilon}{3}$" intervals instead of "middle thirds"). True enough, but about the Cantor set I will believe anything. Consider the ho-hum character of the analogue of the Homeomorphic Measures Theorem for the Cantor set: If μ_1 and μ_2 are full, nonatomic measures on the Cantor set X with $\mu_1(X) = \mu_2(X)$ then there exists a homeomorphism $h : X \to X$ such that $h_*\mu_1 = \mu_2$.

My next surprise, as I wandered into this area, was the discovery that the proposition stated just above is false.

My stumbles provided some amusement for Raj Prasad when I recounted them to him. Prasad was one of Oxtoby's last students and colleagues, and he is, with Steve Alpern, the author of a beautiful forthcoming book which provides an exposition and extension of the Oxtoby-Ulam Theorem [2]. He politely restrained his laughter and gently informed me that among Oxtoby's associates the failure of the Homeomorphic Measures Theorem for Cantor Sets was well-known. In fact, they published papers proving homeomorphic equivalence in certain special cases, e.g. [8].

In retrospect, one can see a simple reason why so many measures on the Cantor set X are topologically distinct. For a measure μ on X we define $S(\mu)$ to be the set of values of μ at the clopen subsets of X. Because there are only countably many clopen subsets of X the set $S(\mu)$ is countable. If μ is a nonatomic probability measure (i.e. $\mu(X) = 1$) then $S(\mu)$ is a countable, dense subset of the unit interval, I, containing $0 = \mu(\emptyset)$ and $1 = \mu(X)$. Clearly, the set $S(\mu)$ is a topological invariant of μ. That is, if $h_*\mu_1 = \mu_2$ for a homeomorphism h then $S(\mu_1) = S(\mu_2)$. Using an uncountable collection of distinct countable, dense subsets of I we will construct uncountably many measures on X which are homeomorphically distinct in a very strong sense.

Theorem 0.1. *On a Cantor space X there exists a set $\mathcal{M}^*$ of full, nonatomic probability measures on X such that for $\mu_1, \mu_2 \in \mathcal{M}^*$ and U_1, U_2 proper clopen subsets of X*

$$\mu_1(U_1) = \mu_2(U_2) \Rightarrow \mu_1 = \mu_2 \text{ and } U_1 = U_2 \tag{0.2}$$

$\mathcal{M}^*$ can be constructed with cardinality that of the continuum.

It follows not only that the measures of $\mathcal{M}^*$ are homeomorphically distinct but that each one is rigid, i.e. the only automorphism preserving a measure in $\mathcal{M}^*$ is the identity, 1_X:

Corollary 0.2. *Let $h : X \to X$ be a continuous map. For measures μ_1, μ_2 in $\mathcal{M}^*$*

$$h_*\mu_1 = \mu_2 \Rightarrow \mu_1 = \mu_2 \text{ and } h = 1_X. \tag{0.3}$$

Proof. If U_2 is any proper clopen subset of X then with $U_1 = h^{-1}(U_2)$, $h_*\mu_1 = \mu_2$ implies

$$\mu_1(U_1) = \mu_2(U_2) < 1 \tag{0.4}$$

and so U_1 is a proper clopen subset of X. Thus, (0.2) implies $\mu_1 = \mu_2$ and $h^{-1}(U_2) = U_2$ for every proper clopen subset U_2 of X. Hence, $h = 1_X$ □

1. Measures on Cantor Spaces

The spaces we will consider are all assumed to be nonempty, compact and metrizable, and so with a countable base. Such a space is called a *Cantor space* when it is perfect and zero-dimensional. So a Cantor space is a nonempty, compact, second countable Hausdorff space with no isolated points and whose clopen subsets form a base for the topology.

Lemma 1.1. *Assume that $\mathcal{B}$ is a base for the topology of a space X. Any clopen subset is a finite union of elements of $\mathcal{B}$. In particular, the collection of clopen subsets of X is countable.*

Proof. The elements of $\mathcal{B}$ contained in any open subset U of X form an open cover of U. If U is also closed and hence compact, we can extract a finite subcover whose union is U. If we begin with a countable base and close under finite unions we obtain a countable family of open sets which includes all the clopens. □

If $h : X_1 \to X_2$ is a continuous map then clearly

$$U_2 \text{ clopen in } X_2 \Rightarrow h^{-1}(U_2) \text{ clopen in } X_1. \tag{1.1}$$

It follows that the collection of clopen subsets is a topologically distinguished countable base for a Cantor space. A homeomorphism between Cantor spaces associates not only the topologies but the bases of the clopens as well.

Of special importance for a Cantor space is the rich supply of partitions. A *partition* $\mathcal{P}$ of a space X is a finite, pairwise disjoint cover of X by clopen sets. Another partition $\mathcal{P}_1$ *refines* $\mathcal{P}$ if each member of $\mathcal{P}_1$ is contained in some member of $\mathcal{P}$. Thus, $\mathcal{P}_1$ induces a partition of each element of P of $\mathcal{P}$, exactly those elements of $\mathcal{P}_1$ contained in P.

We will call a nonempty, closed, perfect, nowhere dense subset of the unit interval $I = [0, 1]$ a *Cantor set*, e.g. the original "middle thirds" Cantor set. Of course, with the topology induced from $\mathbf{R}$ a Cantor set is a Cantor space.

The other important example of a Cantor space is the *shift space* on an alphabet. An *alphabet* A is a finite set with at least two elements. An n-tuple $w \in A^n$ is then called a *word* of length n. $A^{\mathbf{N}}$ is the space of infinite sequences in A enumerated by $\mathbf{N}$, the set of positive integers. From the inclusion of $\{1, \dots, n\}$ into $\mathbf{N}$ we obtain the projection map:

$$\pi^n : A^{\mathbf{N}} \longrightarrow A^n$$

$$\pi^n(x)_i = x_i \qquad i = 1, \dots, n. \tag{1.2}$$

Equipped with the product topology $A^{\mathbf{N}}$ is a Cantor space. For each word w in A^n the *cylinder set* $(\pi^n)^{-1}(w)$ is clopen and $\mathcal{P}^n = \{(\pi^n)^{-1}(w) : w \in A^n\}$ is a partition of $A^{\mathbf{N}}$. $\cup_{n=1}^{\infty} \mathcal{P}^n$ is a base for the topology.

Especially important is the shift map:

$$s : A^{\mathbf{N}} \longrightarrow A^{\mathbf{N}}$$

$$s(x)_i = x_{i+1} \qquad i \in N. \tag{1.3}$$

The shift map is a continuous surjection.

All of these examples are really just different versions of the "same" space. This is the important principle which we will call: *The Uniqueness of Cantor*.

Proposition 1.2. *Any two Cantor spaces are homeomorphic. In particular, any two nonempty, clopen subsets of a Cantor space are homeomorphic.*

Proof. As this proof is well-known, we will just recall it with a sketch.

It suffices to show that any Cantor space X equipped with a metric is homeomorphic to one particular version. We use $\{0, 1\}^{\mathbf{N}}$.

Inductively construct a sequence of partitions $\mathcal{P}_n$ on X. Begin by choosing for some positive integer k_1, a partition $\mathcal{P}_1$ of X into

2^{k_1} distinct sets of diameter at most 1. Index the elements of $\mathcal{P}_1$ by the words in $\{0,1\}^{s_1}$ with $s_1 = k_1$.

At the n^{th} stage we have a partition $\mathcal{P}_n$ indexed by the words in $\{0,1\}^{s_n}$ for some positive integer s_n. We can choose a positive integer k_{n+1} so that *each* $P \in \mathcal{P}_n$ can be partitioned into $2^{k_{n+1}}$ distinct sets of diameter at most $\frac{1}{(n+1)}$. These are then labelled by the $\{0,1\}^{s_n}$ index of P followed by a word in $\{0,1\}^{k_{n+1}}$. Varying P in $\mathcal{P}_n$ we obtain the partition $\mathcal{P}_{n+1}$ indexed by $\{0,1\}^{s_{n+1}}$ with $s_{n+1} = s_n + k_{n+1}$.

Because the diameters in $\mathcal{P}_n$ tend to zero as n tends to infinity, there is a bijection between the points of X and the infinite words in $\{0,1\}^{\mathbf{N}}$. This is the required homeomorphism.

Finally, any closed, perfect subset of a Cantor space, and so any clopen subset, is itself a Cantor space. $\square$

By a *measure* μ on a space X we will mean a Borel probability measure so that $\mu(X) = 1$. Such a measure is called *full* if every nonempty open set has positive measure and *non-atomic* if singletons, and hence countable subsets have measure zero.

Lemma 1.3. *Assume that X is a space with metric d and μ is a nonatomic measure on X. For every $\epsilon > 0$ there exists $\delta > 0$ so that for every Borel subset B of X*

$$diam(B) \leq \delta \Rightarrow \mu(B) < \epsilon, \tag{1.4}$$

where the diameter, $diam(B)$ is $sup\{d(x_1,x_2) : x_1, x_2 \in B\}$.

Proof. If $\{B_n\}$ is a sequence of Borel sets with $\text{diam}(B_n)$ tending to zero, then there is a limit point $p \in X$ whose every neighborhood contains some B_n. If $\mu(B_n) \geq \epsilon$ for all n then $\mu(\{p\}) \geq \epsilon$ and so $\{p\}$ is an atom for μ. $\square$

On a finite set A a measure is determined by its value on the singletons and so by a *distribution function*

$$p : A \longrightarrow [0,1], \text{ such that}$$

$$\sum_{a \in A} p(a) = 1. \tag{1.5}$$

A *positive distribution* satisfies, in addition,

$$p(a) > 0 \quad \text{for all } a \in A. \tag{1.6}$$

Equivalently, the associated measure on A is full.

If $\mathcal{P}$ is a partition of X then a measure μ on X induces a distribution on $\mathcal{P}$ by $P \mapsto \mu(P)$ for $P \in \mathcal{P}$. If μ is full then this is a positive distribution.

Associated with a positive distribution p on an alphabet A there is a full, nonatomic measure $\beta(p)$ on the shift space A^N called the *Bernoulli measure* for p. It is determined by its value on the cylinder sets. For a word $w \in A^n$:

$$\beta(p)((\pi^n)^{-1}(w)) = \Pi_{i=1}^n p(w_i). \tag{1.7}$$

Instead of the function p we will often just list the values. For example, $\beta(\frac{1}{2}, \frac{1}{2})$ is the measure for the space of outcomes of an infinite sequence of independent flips of a fair coin.

If $h : X_1 \to X_2$ is continuous and μ_1 is a measure on X_1 then *image measure* $h_* \mu_1$ on X_2 is defined by

$$h_* \mu_1(B) = \mu_1(h^{-1}(B)) \tag{1.8}$$

for all Borel subsets B of X_2. The measure $h_* \mu_1$ is full iff μ_1 is full and h is surjective. $h_* \mu_1$ is nonatomic if the preimage of singletons in X_2 have measure zero in X_1, which requires that μ_1 be nonatomic. Clearly, if h is injective and μ_1 is nonatomic then $h_* \mu_1$ is nonatomic.

The Bernoulli measures $\beta(p)$ on A^N are invariant with respect to the shift map s of (1.3). That is:

$$s_* \beta(p) = \beta(p), \tag{1.9}$$

for every positive distribution p. These measures are all ergodic with respect to s and so they are all mutually singular.

For a measure μ on a space X define the *clopen values set*:

$$S(\mu) = \{\mu(U) : U \text{ clopen in } X\}. \tag{1.10}$$

Proposition 1.4. *If $h : X_1 \to X_2$ is a continuous map and μ_1 is a measure on X_1, then*

$$S(h_*\mu_1) \subset S(\mu_1) \tag{1.11}$$

with equality if h is a homeomorphism.

Proof. If U_2 is clopen in X_2 then $h^{-1}(U_2)$ is clopen in X_1 so the inclusion is obvious from (1.8). If h is a homeomorphism we obtain all clopens in X_1 this way and equality follows. □

For any measure μ on any space X the set $S(\mu)$ is contained in the unit interval, I, and it includes $0 = \mu(\emptyset)$ and $1 = \mu(X)$. By Lemma 1.1, $S(\mu)$ is countable. Of course, if X is connected the only clopens are $\emptyset$ and X and so then $S(\mu) = \{0, 1\}$. For Cantor spaces the set becomes more interesting.

Proposition 1.5. *If X is a Cantor space and μ is a nonatomic measure on X then $S(\mu)$ is a countable, dense subset of I containing* 0 *and* 1.

Proof. To prove density we choose a metric d for X. Given $\epsilon > 0$ we apply Lemma 1.3 to choose $\delta > 0$. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a partition of X by sets of diameter less than δ. The lemma implies that $\mu(P_i) < \epsilon$ for $i = 1, \dots, m$. Hence, some point of the finite subset of $S(\mu)$

$$\{\mu(\cup_{i=1}^k P_i) : k = 1, \dots m\} \tag{1.12}$$

is ϵ close to any arbitrarily chosen point of I. □

We will call measures μ_1 on X_1 and μ_2 on X_2 *homeomorphic* when there exists a homeomorphism $h : X_1 \to X_2$ such that $h_*\mu_1 = \mu_2$. Proposition 1.4 says that for homeomorphic measures $S(\mu_1) = S(\mu_2)$, i.e. this countable set is a topological invariant of the measure.

We apply this to Bernoulli measures. For any integer $m \geq 2$ we will write $\beta(\frac{1}{m})$ for the Bernoulli measure on the shift space $A^{\mathbf{N}}$ where A is an alphabet with cardinality m and the distribution function p on A is given by $p(a) = \frac{1}{m}$ for all $a \in A$. Thus, we use $\beta(\frac{1}{2})$ for what we earlier denoted by $\beta(\frac{1}{2}, \frac{1}{2})$. We speak of *the* Bernoulli measure, being sloppy about just which set is used as the alphabet. Similarly, interpreted properly we have:

$$\beta(\tfrac{1}{m_1 m_2}) = \beta(\tfrac{1}{m_1}) \times \beta(\tfrac{1}{m_2}), \tag{1.13}$$

and

$$\beta(\tfrac{1}{m^k}) = \beta(\tfrac{1}{m}) \tag{1.14}$$

for all integers with $m_1, m_2, m \geq 2$ and $k \geq 1$.

Equation (1.13) says that if A_α is an alphabet of cardinality m_α $(\alpha = 1, 2)$ then $A = A_1 \times A_2$ is an alphabet of cardinality $m_1 m_2$. The resulting bijection between $A^{\mathbf{N}}$ and the product $A_1^{\mathbf{N}} \times A_2^{\mathbf{N}}$ is a homeomorphism mapping $\beta(\frac{1}{m_1 m_2})$ onto the product measure.

Similarly, if A has cardinality m then A^k has cardinality m^k and we can identify $A^{\mathbf{N}}$ with $(A^k)^{\mathbf{N}}$ by regarding the sequences of letters in the former set as sequences of words of length k. This homeomorphisms maps $\beta(\frac{1}{m})$ to $\beta(\frac{1}{m^k})$.

Proposition 1.6. *Let m, m_1, m_2 be integers larger than one.*

(a) The set $S(\beta(\frac{1}{m}))$ consists of those rational numbers in I which can be expressed as a fraction with denominator dividing a power of m. That is,

$$S(\beta(\tfrac{1}{m})) = \{\tfrac{a}{m^n} : n \in \mathbf{N},\ a = 0, \dots, m^n\}. \tag{1.15}$$

(b) If m_1 and m_2 have the same prime divisors, i.e. $p|m_1$ iff $p|m_2$ for all primes p, then $\beta(\frac{1}{m_1})$ and $\beta(\frac{1}{m_2})$ are homeomorphic and so $S(\beta(\frac{1}{m_1})) = S(\beta(\frac{1}{m_2}))$.

(c) If m_1 and m_2 do not have the same prime divisors then $S(\beta(\frac{1}{m_1})) \neq S(\beta(\frac{1}{m_2}))$ and so $\beta(\frac{1}{m_1})$ and $\beta(\frac{1}{m_2})$ are not homeomorphic measures.

Proof. (a) For any $n \in \mathbf{N}$, $\{\frac{a}{m^n} : a = 0, 1, \dots, m^n\}$ are exactly the values of the measure $\beta(\frac{1}{m})$ on finite unions of the partition $\mathcal{P}^n$ by words of length n. By Lemma 1.1 every clopen set is a union of elements of $\mathcal{P}^n$ for some n. So (1.15) follows.

(b) This is an easy exercise using (1.13) and (1.14).

(c) For any prime p, (1.15) implies

$$\frac{1}{p} \in S(\beta(\frac{1}{m})) \Leftrightarrow p|m. \tag{1.16}$$

So if $p|m_1$ by $p \not| m_2$ then $\frac{1}{p} \in S(\beta(\frac{1}{m_1}))$ but not $S(\beta(\frac{1}{m_2}))$. □

Proposition 1.7. *Let $\beta(\frac{1}{3}, \frac{2}{3})$ be the Bernoulli measure on the shift space with a two element alphabet and distribution function with values $\frac{1}{3}$ and $\frac{2}{3}$. The measure $\beta(\frac{1}{3}, \frac{2}{3})$ is not homeomorphic to $\beta(\frac{1}{3})$ but*

$$S(\beta(\tfrac{1}{3}, \tfrac{2}{3})) = S(\beta(\tfrac{1}{3})). \tag{1.17}$$

Proof. Use $\{0, 1\}$ as the two element alphabet, with $p(0) = \frac{1}{3}$ and $p(1) = \frac{2}{3}$. Let $X = \{0, 1\}^{\mathbf{N}}$ and $\mu = \beta(\frac{1}{3}, \frac{2}{3})$.

To show μ is not homeomorphic to $\beta(\frac{1}{3})$ it suffices to show that if $\mathcal{P} = \{P_1, P_2, P_3\}$ is a three element partition of X then it cannot happen that $\mu(P_\alpha) = \frac{1}{3}$ for $\alpha = 1, 2, 3$.

By Lemma 1.1, $\mathcal{P}^n$ refines $\mathcal{P}$ for sufficiently large n. Let $\overline{0}$ be the word of length n with every letter 0. From (1.7) we have:

$$\mu((\pi^n)^{-1}(\overline{0})) = \frac{1}{3^n}, \tag{1.18}$$

while if $w \neq \overline{0}$ then

$$\mu((\pi^n)^{-1}(w)) = \frac{a}{3^n} \text{ with } 2|a. \tag{1.19}$$

If $(\pi^n)^{-1}(\overline{0}) \subset P_1$ then $\mu(P_2)$ and $\mu(P_3)$ have a 2 in their numerator when reduced to lowest terms.

To prove (1.17) we show that for every $n \in N$ and $a = 0, \dots, 3^n$ there exists a clopen set U in X such that $\mu(U) = \frac{a}{3^n}$. The equation then follows from (1.15). This is clear for $n = 1$. Proceed by induction on n.

First we write

$$\frac{a}{3^n} = \frac{1}{3} \cdot \frac{a_0}{3^{n-1}} + \frac{2}{3} \cdot \frac{a_1}{3^{n-1}} \tag{1.20}$$

with $a_0, a_1 \leq 3^{n-1}$. If $a \leq 2 \cdot 3^{n-1}$ then we obtain (1.20) with $a_1 = [\frac{a}{2}]$ and $a_0 = 0$ or 1. If $a \geq 2 \cdot 3^{n-1}$ then we obtain (1.20) with $a_1 = 3^{n-1}$ and $a_0 = a - 2 \cdot 3^{n-1} \leq (3-2) \cdot 3^{n-1}$.

By induction hypothesis there exist clopen subsets U_α of X such that $\mu(U_\alpha) = a_\alpha$ $(\alpha = 0, 1)$. Now define the one-sided inverses $s_\alpha : X \to X$ of the shift map by

$$s_\alpha(x)_i = \begin{cases} \alpha & i = 1 \\ x_{i-1} & i > 1, \end{cases}$$

(1.21)

for $\alpha = 0, 1$. From (1.21) we have $\mu(U) = \frac{a}{3^n}$ with $U = s_0(U_0) \cup s_1(U_1)$. □

Thus, while $S(\mu)$ is a topological invariant it is not a complete invariant. We will be able to obtain a complete invariant when we introduce additional structure, namely ordering.

This invariant does suffice to show that there are uncountably many nonhomeomorphic measures, recovering Theorems 3.1 and 3.2 of [7].

Proposition 1.8. *For each r in $(0, 1)$ there are only countably many s in $(0, 1)$ such that $\beta(r, 1-r)$ is homeomorphic to $\beta(s, 1-s)$. Thus, among the measures $\{\beta(r, 1-r) : 0 < r < 1\}$ the set of homeomorphism equivalence classes has the cardinality of the continuum.*

Proof. Since $s \in S(\beta(s, 1-s))$, $\beta(r, 1-r)$ homeomorphic to $\beta(s, 1-s)$ requires that s is in the countable set $S(\beta(r, 1-r)) = S(\beta(s, 1-s))$. As each equivalence class is countable the set of equivalence classes has the cardinality of $(0, 1)$ itself. $\square$

2. Ordered Cantor Spaces

By an *order* $\leq$ on set X we will mean a reflexive, antisymmetric, transitive relation on X which is also complete, i.e. either $x_1 \leq x_2$ or $x_2 \leq x_1$ for every pair $x_1, x_2 \in X$. As usual, we will write $x_1 < x_2$ for $x_1 \leq x_2$ and $x_1 \neq x_2$. We will adopt the usual interval notation so that, for example,

$$(x_1, x_2] = \{x \in X : x_1 < x \leq x_2\}. \tag{2.1}$$

As a relation on X, $\leq$ is a set of ordered pairs, i.e. a subset of $X \times X$. Hence, if X is a space we can define $\leq$ to be a *closed order* if it is an order which is closed as a subset of $X \times X$. An *order space* is a pair $(X, \leq)$ where X is a space and $\leq$ is a closed order on X. An example is $(I, \leq)$ where I is the unit interval and $\leq$ is the usual order.

If $(X_\alpha, \leq_\alpha)$ $(\alpha = 1, 2)$ are order spaces then $h : (X_1, \leq_1) \to (X_2, \leq_2)$ is an *order space map* when $h : X_1 \to X_2$ is a continuous map which preserves order, i.e.

$$x \leq_1 y \Rightarrow h(x) \leq_2 h(y). \tag{2.2}$$

Proposition 2.1. *Let $(X, \leq)$ be an order space.*

(a) There exists an injective order space map $L : (X, \leq) \to (I, \leq)$ so that L is a homeomorphism of X onto a closed subset of the interval I. We will call such a map a Lyapunov function *for the order $\leq$.*

(b) Every nonempty closed subset of X contains a maximum and a minimum. In particular, we will denote by 1 and 0 the maximum and minimum elements of X.

(c) The topology on X is the order topology induced by $\leq$. That is, the collection of all intervals of the form $(x, 1]$ and $[0, x)$ for $x \in X$ is a subbase for the topology.

Proof. The result in (a) is a special case of the existence theorem of complete Lyapunov functions for closed relations. We refer to [1] Theorem 2.12 for a proof of this dynamics result. The results in (b) and (c) are obvious for closed subsets of the interval whence they lift to an arbitrary $(X, \leq)$ by using a Lyapunov function. □

Remark. (a) By using an affine change if necessary we can adjust the Lyapunov function so that it satisfies

$$L(0) = 0 \text{ and } L(1) = 1. \tag{2.3}$$

We will assume these equalities for any Lyapunov function we use.

(b) It is an easy exercise to show that an order $\leq$ on a space X is closed iff the intervals $[0, x]$ and $[x, 1]$ are closed for all $x \in X$ (see [1] Exercise 2.19 for a hint).

Corollary 2.2. *Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be order spaces. If $h : X_1 \to X_2$ is an order preserving bijection then it is a homeomorphism and so $h : (X_1, \leq_1) \to (X_2, \leq_2)$ is an order space isomorphism.*

Proof. Since $h^{-1}([h(x), h(y)]) = [x, y]$ when h is bijective and order preserving, continuity of h follows from part (c). □

Let $\varphi : (I, \leq) \to (I, \leq)$ be an order space isomorphism. Let $l \geq 1$ be a real number. We call φ a *Lipschitz isomorphism* with constant l if $x_1 < x_2$ in I implies

$$l^{-1} \leq \tfrac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq l. \tag{2.4}$$

It is easy to check that if φ is piece-wise linear, i.e. the graph consists of a finite number of line segments, then (2.4) holds iff each segment has slope between l and l^{-1}.

Lemma 2.3. *Let D_1 and D_2 be countable dense subsets of I with $\{0,1\} \subset D_1 \cap D_2$. For any $l > 1$ there exists a Lipschitz isomorphism $\varphi : (I, \leq) \to (I. \leq)$ with constant l such that*

$$\varphi(D_1) = D_2. \tag{2.5}$$

Proof. We use some care with a classic construction. Enumerate so that $D_1 = \{0, 1, a_1, a_2, a_3, \ldots\}$ and $D_2 = \{0, 1, b_1, b_2, \ldots\}$. Define $A_0 = B_0 = \{0, 1\}$ and $\varphi_0(x) = x$ for all $x \in I$. Proceed inductively. Assume that A_n and B_n are finite sets with

$$\{0, 1, a_1, \ldots, a_n\} \subset A_n \subset D_1$$

$$\{0, 1, b_1, \ldots, b_n\} \subset B_n \subset D_2 \tag{2.6}$$

and that $\varphi_n : (I, \leq) \to (I, \leq)$ is an isomorphism with

$$\varphi_n(A_n) = B_n. \tag{2.7}$$

$I \setminus A_n$ and $I \setminus B_n$ are finite unions of open intervals bijectively associated by φ_n, which is assumed to be linear on each of these intervals with slope *strictly* between l^{-1} and l.

For the next step choose α to be the first element of $D_1 \setminus A_n$ in the enumeration. Choose $\tilde{\alpha}$ in $D_2 \setminus B_n$ close to $\varphi_n(\alpha)$. Define $\varphi_{n+.5}(x) = \varphi_n(x)$ for $x \in A_n$ and $\varphi_{n+.5}(\alpha) = \tilde{\alpha}$. Connect the dots, i.e. interpolate linearly. The point α lies in one of the intervals of $I \setminus A_n$. We are replacing the point $(\alpha, \varphi_n(\alpha))$ on the segment by the point $(\alpha, \tilde{\alpha})$ and thus breaking this particular segment in two. Since $D_2 \setminus B_n$ is dense we can choose $\tilde{\alpha}$ close enough that the slopes of the two new pieces still lie strictly between l^{-1} and l. Since the slope of $\varphi_{n+.5}$ is everywhere positive it is an increasing function and so is an isomorphism on $(I, \leq)$.

Similarly, choose β to be the first enumerated element of $D_2 \setminus (B_n \cup \{\tilde{\alpha}\})$ and choose $\tilde{\beta}$ in $D_1 \setminus (A_n \cup \{\alpha\})$ close to $(\varphi_{n+.5})^{-1}(\beta)$. Define $\varphi_{n+1}(x) = \varphi_{n+.5}(x)$ for x in $A_n \cup \{\alpha\}$ and $\varphi_{n+1}(\tilde{\beta}) = \beta$. Extend linearly. Again we can make the choices so that the slopes lie strictly between l^{-1} and l.

Define $A_{n+1} = A_n \cup \{\alpha, \tilde{\beta}\}$ and $B_{n+1} = B_n \cup \{\tilde{\alpha}, \beta\}$. Observe that either $a_{n+1} \in A_n$ or $\alpha = a_{n+1}$. Similarly, $b_{n+1} \in B_{n+1}$.

By the Arzela-Ascoli Theorem the set of functions $\varphi : I \to I$ with Lipschitz constant $\leq l$ is compact in the topology of uniform convergence. So the sequence $\{\varphi_n\}$ has a limit point $\varphi : I \to I$ with Lipschitz constant at most l. The limit point is unique since for each $x \in D_1$, the sequence $\{\varphi_n(x)\}$ is eventually constant and so eventually equal to $\varphi(x)$. Similarly, $\{\varphi_n^{-1}\}$ converges uniformly to φ^{-1} with Lipschitz constant $\leq l$. The equation (2.5) is clear from the construction. □

In an order space $(X, \leq)$ a point x is called a *left endpoint* (or a *right endpoint*) if the closed interval $[0, x]$ (resp. $[x, 1]$) in X is clopen. Thus, 0 is a right endpoint and 1 is a left endpoint. Except for these two, the left and right endpoints can be matched up. If $x_- \neq 1$ is a left endpoint then $(x_-, 1]$ is a nonempty closed interval and so it equals $[x_+, 1]$ where x_+ is its minimum. Thus, x_+ is a right endpoint with $x_- < x_+$ and the open interval (x_-, x_+) is empty. Notice that if a point is both a left and right endpoint then $\{x\} = [0, x] \cap [x, 1]$ is open and so x is an isolated point. So if X is perfect then the set of left endpoints other than 1 and the set of right endpoints other than 0 are disjoint sets bijectively associated by this pairing. We call these the *endpoint pairs* of $(X, \leq)$. If L is a Lyapunov function for $(X, \leq)$ satisfying (2.3) then the image of the endpoint pairs are exactly the endpoints of the disjoint open intervals whose union is $I \setminus L(X)$.

We will call $(X, \leq)$ an *ordered Cantor space* if $(X, \leq)$ is an order space with X a Cantor space. In that case, $L(X)$ is a Cantor set and so the left endpoints and right endpoints are each countable dense subsets of X.

Lemma 2.4. *Let $(X, \leq)$ be an ordered Cantor space.*

(a) A closed interval $[x_1, x_2]$ with $x_1 \leq x_2$ is clopen iff x_1 is a right endpoint and x_2 is a left endpoint. The set of clopen intervals is a base for the topology of X.

(b) If U is a nonempty clopen subset of X then U contains finitely many maximal clopen subintervals and U is the disjoint union of them. That is, there is a unique increasing sequence $x_1 < x_2 < \ldots < x_{2k}$ for some positive integer k such that each x_i with i odd (or even) is a right (resp. left) endpoint,

$$U = \cup_{j=1}^{k} [x_{2j-1}, x_{2j}], \tag{2.8}$$

and for $j = 1, \ldots, k-1$ the pair $\{x_{2j}, x_{2j+1}\}$ is not *an endpoint pair, i.e.*

$$(x_{2j}, x_{2j+1}) \neq \emptyset \quad j = 1, \ldots, k-1. \tag{2.9}$$

Proof. (a) Since $[x_1, x_2] = [0, x_2] \cap [x_1, 1]$ the interval is clopen if x_1 is a right and x_2 is a left endpoint. The converse follows from the easily verified fact that if U is a nonempty clopen set then max U is a left endpoint and min U is a right endpoint. Since the left and right endpoints are each dense in X every point in X can be contained in clopen intervals of arbitrarily small diameter. Hence the clopen intervals form a base.

(b) Applying Lemma 1.1 to (a) we can express U as a finite union of clopen intervals in U. If $[x_1, x_2]$ and $[x_3, x_4]$ are subintervals of U which intersect or which satisfy $x_2 < x_3$ and $\{x_2, x_3\}$ is an endpoint pair, then the union is a subinterval of U. Concatenating intervals successively until we can no longer do so we arrive at a sequence $x_1 < \ldots < x_{2k}$ satisfying (2.8) and (2.9). Clearly, the $[x_{2j-1}, x_{2j}]$ for $j = 1, \ldots, k$ are the maximal subintervals of U by (2.9). Uniqueness of the decomposition follows. □

If $(X, \leq)$ is an order space and μ is a measure on X we define the *cumulative distribution function*, written CDF, of μ to be

$$F_\mu : X \longrightarrow I$$

$$F_\mu(x) = \mu([0, x]). \tag{2.10}$$

Proposition 2.5. *With (X, μ) an order space and μ a measure on X, let F_μ be the CDF of μ.*

(a) The function F_μ is order preserving, continuous from the right with respect to $\leq$ and satisfies $F_\mu(1) = 1$.

(b) The function F_μ is continuous iff the atoms of the measure μ, if any, are right endpoints of (X, μ).

(c) The measure μ is nonatomic iff F_μ is continuous and surjective, i.e. $F_\mu(X) = I$.

(d) Assume now that X is perfect, i.e. there are no isolated points. The measure μ is full iff for any three points $x_1, y, x_2 \in X$

$$x_1 < y < x_2 \Rightarrow F_\mu(x_1) < F_\mu(x_2). \tag{2.11}$$

This is equivalent to saying that $F_\mu(x_1) = F_\mu(x_2)$ for a pair x_1, x_2 of distinct points implies that x_1, x_2 is an endpoint pair.

Proof. (a) $F_\mu(1) = \mu(X) = 1$. If $x_1 < x_2$ then

$$F_\mu(x_2) - F_\mu(x_1) = \mu((x_1, x_2]) \geq 0. \tag{2.12}$$

So F_μ is order preserving. If $\{x_n\}$ is a decreasing sequence in X converging to x then $[0, x] = \cap_n \{[0, x_n]\}$ implies

$$F_\mu(x) = \mathrm{Lim}_{n\to\infty} F_\mu(x_n). \tag{2.13}$$

(b) If $\{x_n\}$ is a (strictly) increasing sequence converging to x then $[0, x) = \cup_n \{[0, x_n]\}$ implies

$$F_\mu(x) - \mu(\{x\}) = \mathrm{Lim}_{n\to\infty} F_\mu(x_n). \tag{2.14}$$

Hence, continuity from the left at x is equivalent to $\mu(\{x\}) = 0$. Continuity is equivalent to left and right continuity and so requires $\mu(\{x\}) = 0$ for every point x which is a limit point of some increasing sequence, i.e. x is not a right endpoint.

(c) If $\mu(\{p\}) > 0$ then

$$\begin{aligned} x \geq p &\Rightarrow F_\mu(x) \geq F_\mu(p) \geq \mu(\{p\}). \\ x < p &\Rightarrow F_\mu(x) \leq F_\mu(p) - \mu(\{p\}). \end{aligned} \tag{2.15}$$

If p is the minimum point 0, then the first inequality says $F_\mu(x) \geq F_\mu(0) > 0$ for all x. In any case, the numbers in the open subinterval of I: $(F_\mu(p) - \mu(\{p\}), F_\mu(p))$ are not in the image of F_μ.

Conversely, if μ is a nonatomic then by (b) F_μ is continuous and so $F_\mu \circ (L)^{-1}$ is continuous on the closed subset $L(X)$ of I where L Is a Lyapunov function. If $x_- < x_+$ is an endpoint pair then because x_+ is not an atom

$$F_\mu(x_-) = F_\mu(x_+). \tag{2.16}$$

Hence, $F_\mu \circ (L)^{-1}$ can be extended to a continuous map from $G^L_\mu : I \to I$ which is constant on each open subinterval of $I \setminus L(X)$. Since 0 is not an atom $G^L_\mu(0) = F_\mu(0) = 0$ (we are assuming (2.3) here). Hence, $F_\mu(X) = G^L_\mu(I) = I$ by the Intermediate Value Theorem.

(d) The inequalities $x_1 < y < x_2$ imply that the open interval (x_1, x_2) is nonempty. If μ is full then $F_\mu(x_1) < F_\mu(x_2)$ follows from (2.12). In general, if $x_1 < x_2$ then the interval (x_1, x_2) is empty only when $\{x_1, x_2\}$ is an endpoint pair. This proves the equivalent statement made after (2.11).

If X is perfect then any nonempty open set U is infinite and so contains some triple $x_1 < y < x_3$. Hence, (2.11) implies $\mu(U) > 0$. Thus, assumption (2.11) implies μ is full. □

Remark. An isolated point is an atom for any full measure. So a full, nonatomic measure exists on a space X only when X is perfect.

We will need the converse of part (a):

Proposition 2.6. *Let (X, μ) be an order space. If $F : X \to I$ is an order preserving, right continuous function satisfying $F(1) = 1$, then there is a unique measure μ on X such that $F = F_\mu$.*

Proof. This is the familiar Lebesgue-Stieltjes construction which begins by defining μ on the semiring of half open intervals by

$$\mu((x_1, x_2]) = F(x_2) - F(x_1). \tag{2.17}$$

See, e.g. [4] Chapter II. □

If $h : (X_1, \leq_1) \to (X_2, \leq_2)$ is an order space map then for every $y_1 \leq y_2$ in X_2:

$$\begin{aligned} &h^{-1}([y_1, y_2]) = [x_1, x_2], \text{ with} \\ &x_1 = \min h^{-1}([y_1, y_2]) \text{ and } x_2 = \max h^{-1}([y_1, y_2]). \end{aligned} \tag{2.18}$$

Furthermore, if $y_1, y_2 \in h(X_1)$ then

$$x_1 = \min h^{-1}(\{y_1\}) \text{ and } x_2 = \max h^{-1}(\{y_2\}). \tag{2.19}$$

Lemma 2.7. *For $\alpha = 1, 2$ let $(X_\alpha, \leq_\alpha)$ be an order space and μ_α be a nonatomic measure on X_α. If $h : (X_1, \leq_1) \to (X_2, \leq_2)$ is a surjective order space map then*

$$h_*\mu_1 = \mu_2 \Leftrightarrow F_{\mu_1} = F_{\mu_2} \circ h. \tag{2.20}$$

Proof. Assume $h_*\mu_1 = \mu_2$. Since h is surjective $0_1 = \min X_1 = \min h^{-1}(0_2)$ and so $h(0_1) = 0_2$. By (2.18) and (2.19) $F_{\mu_2}(y) = F_{\mu_1}(x)$ where $x = \max h^{-1}(\{y\})$. For any $x_1 = h^{-1}(\{y\})$ we have

$$0 \leq \mu_1((x_1, x]) \leq \mu_1(h^{-1}(\{y\})) = \mu_2(\{y\}) = 0 \tag{2.21}$$

because μ_2 is nonatomic. By (2.12) $F_{\mu_1}(x_1) = F_{\mu_1}(x) = F_{\mu_2}(y) = F_{\mu_2}(h(x_1))$.

Assume instead that $F_{\mu_1} = F_{\mu_2} \circ h$. For $y \in X_2$ let $x = \max h^{-1}(\{y\})$. By (2.18) and (2.19):

$$\mu_2([0, y]) = F_{\mu_2}(h(x)) = F_{\mu_1}(x) = h_*\mu_1([0, y]). \tag{2.22}$$

Because these intervals generate the σ-algebra of Borel subsets of X_2 we have $\mu_2 = h_*\mu_1$. □

Corollary 2.8. *Let* $(X, \leq)$ *be an order space and* μ *be a full nonatomic measure on* X *with CDF* $F_\mu : X \to I$. *With* λ *Lebesgue measure on* I,

$$F_{\mu *}\mu = \lambda. \tag{2.23}$$

Proof. Clearly, $F_\lambda : I \to I$ is the identity map. So with $h = F_\mu$ we have $F_\lambda \circ h = F_\mu$. Equation (2.23) follows from (2.20). □

For a measure μ on an order space $(X, \leq)$ we can define an invariant closely related to $S(\mu)$ of (1.10), the *special clopen values set*:

$$\tilde{S}(\mu) = \{F_\mu(x) : x \text{ a left endpoint of } (X, \leq)\} \cup \{0\}. \tag{2.24}$$

Recall that x is a left endpoint exactly when $[0, x]$ is clopen and then $F_\mu(x) = \mu([0, x])$. It clearly follows that

$$\tilde{S}(\mu) \subset S(\mu). \tag{2.25}$$

In particular, $\tilde{S}(\mu)$ is a countable subset of I. Since the maximum point 1 of X is always a left-endpoint we always have

$$1 \in \tilde{S}(\mu). \tag{2.26}$$

Our remaining results are all obtained from the following *Lifting Lemma*:

Lemma 2.9. *For* $\alpha = 1, 2$ *let* $(X_\alpha, \leq_\alpha)$ *be an order space and* μ_α *be a full, nonatomic measure on* X_α. *An order isomorphism* $\varphi : (I, \leq) \to (I, \leq)$ *satisfies the condition*

$$\varphi(\tilde{S}(\mu_1)) \supset \tilde{S}(\mu_2) \tag{2.27}$$

iff there exists a continuous map $h : X_1 \to X_2$ *such that the following diagram commutes.*

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad h \quad} & X_2 \\ {\scriptstyle F_{\mu_1}}\downarrow & & \downarrow{\scriptstyle F_{\mu_2}} \\ I & \xrightarrow[\quad \varphi \quad]{} & I \end{array}$$

(2.28)

When such a lifting h exists it is unique and $h : (X_1, \leq_1) \to (X_2, \leq_2)$ is a surjective order map. Furthermore, h is an order isomorphism iff equality holds in (2.27).

Proof. By Proposition 2.5 each $F_{\mu_\alpha} : (X_\alpha, \leq_\alpha) \to (I, \leq)$ is a surjective order map with X_α a perfect space. Furthermore, the map is almost one-to-one. That is, by (2.16) and part (d) of Proposition 2.5 a pair of distinct points in X_α is mapped by F_{μ_α} to the same point in I iff the pair is an endpoint pair. The set of all endpoints in X_α is a countable subset. We will denote by E_α the set of endpoints in X_α. By (2.16) and (2.24) we have

$$F_{\mu_\alpha}(E_\alpha) = \tilde{S}(\mu_\alpha). \tag{2.29}$$

In particular, for every point x of X_1 which does not lie in the countable exceptional set

$$\tilde{S}_{12} \equiv (\varphi \circ F_{\mu_1})^{-1}(\tilde{S}(\mu_2)), \tag{2.30}$$

there is a unique point $h(x)$ of X_2 such that $F_{\mu_2}(h(x)) = \varphi(F_{\mu_1}(x))$. The function $h : X_1 \backslash \tilde{S}_{12} \to X_2$ is order preserving. In fact, since φ is an order isomorphism, Proposition 2.5d implies that for $x_1, x_2 \in X_1 \backslash \tilde{S}_{12}$:

$$x_1 < x_2 \Rightarrow h(x_1) \leq h(x_2), \tag{2.31}$$

with equality iff $\{x_1, x_2\}$ is an endpoint pair for μ_1. Because X_1 is perfect and $\tilde{S}_{12}$ is countable, $X_1 \backslash \tilde{S}_{12}$ is dense in X_1 and the set of pairs (x_1, x_2) with $x_1, x_2 \in X_1 \backslash \tilde{S}_{12}$ and $x_1 < x_2$ is dense in the closed order $\leq_1 \subset X_1 \times X_1$.

Consequently, if h extends to a continuous function from X_1 to X_2 then the extension is uniquely defined and is order preserving. Similarly, the equality $F_{\mu_2} \circ h = \varphi \circ F_{\mu_1}$ extends to all of X_1. Because

$$h(X_1 \backslash \tilde{S}_{12}) = X_2 \backslash E_2 \tag{2.32}$$

and E_2 is a countable subset of the perfect space X_2, it follows that the continuous extension is surjective.

Now we derive condition (2.27). Clearly, $h(1_1) = 1_2$ as $\{1_\alpha\} = (F_{\mu_\alpha})^{-1}(1)$ for $\alpha = 1, 2$. Similarly for 0. Any other point of $\tilde{S}(\mu_2)$ is $F_{\mu_2}(y_-) = F_{\mu_2}(y_+)$ for some endpoint pair y_+, y_- in X_2. Because h is a surjective function there exist distinct points $x_+, x_- \in X_1$ such that

$$h(x_\pm) = y_\pm. \tag{2.33}$$

The commutative diagram (2.28) then implies that $\varphi(F_{\mu_1}(x_+))$ and $\varphi(F_{\mu_2}(x_-))$ are both the common value $F_{\mu_2}(y_\pm)$. Because φ is an isomorphism we have $F_{\mu_1}(x_+) = F_{\mu_2}(x_-)$. Because $x_+ \neq x_-$ they must be an endpoint pair of X_1. So $F_{\mu_1}(x_\pm)$ is a point of $\tilde{S}(\mu_1)$ mapped by φ to the point $F_{\mu_2}(y_\pm)$.

Now assume that (2.27) holds for φ. It says that $\tilde{S}_{12} \subset E_1$. That is, for every endpoint pair y_-, y_+ in X_2 there is an endpoint pair x_-, x_+ such that $\varphi(F_{\mu_1}(x_\pm)) = F_{\mu_2}(y_\pm)$. We extend the definition of h to all of X_1 by mapping the left endpoint x_- to the left endpoint y_- and the right to the right. This is exactly the choice so that the extension $h : X_1 \to X_2$ is order preserving. It is then easy to check that if $y_1 < y_2 \in X_2$ then

$$(x_1, x_2) = h^{-1}((y_1, y_2)) \tag{2.34}$$

where x_1 is the maximum of the one or two elements in the set $h^{-1}(y_1)$ and x_2 is the minimum of the set $h^{-1}(y_2)$. Continuity of h follows from Proposition 2.1c.

Finally, by reversing the order and applying the result to φ^{-1} we see that $h : X_1 \to X_2$ is a homeomorphism iff equality holds in (2.27). □

Remark. When equality does not hold in (2.27), $h(x_1) = h(x_2)$ iff $\{x_1, x_2\}$ is an endpoint pair in

$$E_1 \backslash \tilde{S}_{12} = F_{\mu_1}^{-1}(\tilde{S}(\mu_1) \backslash \varphi^{-1}(\tilde{S}(\mu_2))).$$

Our first application says that $\tilde{S}(\mu)$ is a complete invariant for order isomorphisms between full nonatomic measures.

Theorem 2.10. *For $\alpha = 1, 2$ let $(X_\alpha, \leq_\alpha)$ be an order space and μ_α be a full, nonatomic measure on X_α. There exists an order space isomorphism $h : (X_1, \leq_1) \to (X_2, \leq_2)$ such that $h_* \mu_1 = \mu_2$ iff $\tilde{S}(\mu_1) = \tilde{S}(\mu_2)$.*

Proof. By Lemma 2.9 with φ the identity on I, an order isomorphism $h : (X_1, \leq_1) \to (X_2, \leq_2)$ exists with $F_{\mu_1} = F_{\mu_2} \circ h$ iff $\tilde{S}(\mu_1) = \tilde{S}(\mu_2)$. By Lemma 2.7, $F_{\mu_1} = F_{\mu_2} \circ h$ iff $h_* \mu_1 = \mu_2$. □

In particular, if $(X, \leq)$, is $(I, \leq)$ then 1 is the only left endpoint and so $\tilde{S}(\mu) = \{0, 1\}$ for any measure μ. There is thus no obstruction finding an order isomorphism between any two full nonatomic measures. This, however, is the classic result, captured in Corollary 2.8, that for any such measure μ its CDF F_μ is an order isomorphism mapping μ to Lebesgue measure. Of course, our real interest is in Cantor spaces.

Proposition 2.11. *If $(X, \leq)$ is an ordered Cantor space and μ is a nonatomic measure on X then $\tilde{S}(\mu)$ is a countable dense subset of I containing $0, 1$.*

Proof. F_μ is a continuous surjection by Proposition 2.5c. In a Cantor space the left endpoints form a dense set. Because the continuous image of a dense set is dense, $\tilde{S}(\mu)$ is dense in I by definition (2.24). □

We already know that there are nonhomeomorphic measures on a Cantor space. The following describes how close we can come.

Theorem 2.12. *For $\alpha = 1, 2$ let $(X_\alpha, \leq_\alpha)$ be an ordered Cantor space and μ_α be a full, nonatomic measure on X_α. Let $l > 1$ be a real number. There exists an order space isomorphism $h : (X_1, \leq_1) \to (X_2, \leq_2)$ such that the Radon-Nikodym derivative $dh_*\mu_1/d\mu_2$ exists and satisfies:*

$$l^{-1} \leq \frac{dh_*\mu_1}{d\mu_2} \leq l \tag{2.35}$$

everywhere on X_2.

Proof. Define the countable dense set $D_\alpha = \tilde{S}(\mu_\alpha)$ $(\alpha = 1, 2)$. By Lemma 2.3 we can choose a Lipschitz isomorphism $\varphi : (I, \leq) \to (I, \leq)$ with constant l so that $\varphi(D_1) = D_2$. Because φ satisfies (2.27) with equality, Lemma 2.9 implies there exists an order space isomorphism $h : (X_1, \leq_1) \to (X_2, \leq_2)$ such that $F_{\mu_2} \circ h = \varphi \circ F_{\mu_1}$. If we let $\nu = h_*\mu_1$ then by Lemma 2.7, $F_\nu \circ h = F_{\mu_1}$ or, equivalently, $F_\nu = F_{\mu_1} \circ h^{-1}$. Consequently

$$F_{\mu_2} = \varphi \circ F_\nu. \tag{2.36}$$

It follows from (the reciprocal of) (2.4) that if $x_1 < x_2$ in X_2 and the interval (x_1, x_2) is nonempty then:

$$\begin{aligned} l^{-1} &\leq \nu((x_1, x_2])/\mu_2((x_1, x_2]) \\ &= (F_\nu(x_2) - F_\nu(x_1))/(\varphi(F_\nu(x_2)) - \varphi(F_\nu(x_1))) \leq l. \end{aligned} \tag{2.37}$$

From this it follows that for every Borel subset B of X_2

$$l^{-1}\mu_2(B) \le \nu(B) \le l\mu_2(B). \tag{2.38}$$

It follows that the Radon-Nikodyn derivative $d\nu/d\mu_2$ exists and is bounded by l (almost everywhere but we can choose a version bounded everywhere) and its reciprocal $d\mu_2/d\nu$ is also bounded by l. (See, e.g [4] Chapter VI, Section 31). □

As a corollary we obtain *The Uniqueness of Cantor, II.*

Corollary 2.13. *Any two ordered Cantor spaces are order isomorphic. In particular, any two nonempty clopen subsets in an ordered Cantor space are order isomorphic.*

Proof. Given ordered Cantor space $(X_1, \le_1)$ and $(X_2, \le_2)$ choose full, nonatomic measures μ_α on X_α $(\alpha = 1, 2)$. For example, use Uniqueness of Cantor to move a Bernoulli measure to each X_α. Apply Theorem 2.12 and forget the measures. □

3. Many Rigid Measures on Cantor Space

We begin by reinterpreting some results from the previous section.

Definition 3.1. *(a) Let $\mathcal{G}$ denote the group (under composition) of order isomorphisms of $(I, \le)$. Thus, $\varphi \in \mathcal{G}$ iff φ is an increasing continuous, real valued function on the unit interval with $\varphi(0) = 0$ and $\varphi(1) = 1$.*

(b) Let $\mathcal{D}$ denote the set of countable, dense subsets of $[0, 1]$ which contain $0, 1$. Thus, $D \in \mathcal{D}$ iff D is a countable dense, subset of the unit interval with $\{0, 1\} \subset D$.

(c) For an ordered Cantor space $(X, \le)$ let $\mathcal{M}_X$ denote the set of full, nonatomic measures on X.

For example, let $\tilde{Q} \equiv Q \cap [0, 1]$, where Q is the field of rational numbers. $\tilde{Q}$ is an element of $\mathcal{D}$.

For the remainder of the section we will fix $(X, \leq)$ and drop the subscript writing $\mathcal{M}$ for $\mathcal{M}_X$.

Theorem 3.2. *(a) The group $\mathcal{G}$ acts on the set $\mathcal{M}$. For $(\varphi, \mu_1) \in \mathcal{G} \times \mathcal{M}$ and $\mu_2 \in \mathcal{M}$:*

$$\mu_2 = \varphi\mu_1 \Leftrightarrow F_{\mu_2} = \varphi \circ F_{\mu_1}. \tag{3.1}$$

The action is transitive and free, i.e. for any $\mu_1 \in \mathcal{M}$ the map $\varphi \mapsto \varphi\mu_1$ is a bijection from $\mathcal{G}$ onto $\mathcal{M}$.

(b) The group $\mathcal{G}$ acts on the set $\mathcal{D}$. For $(\varphi, D_1) \in \mathcal{G} \times \mathcal{D}$ and $D_2 \in \mathcal{D}$:

$$D_2 = \varphi D_1 \Leftrightarrow D_2 = \varphi(D_1). \tag{3.2}$$

The action is transitive, i.e. for any $D_1 \in \mathcal{D}$ the map $\varphi \mapsto \varphi D_1$ is a surjection from $\mathcal{G}$ onto $\mathcal{D}$.

(c) $\tilde{S} : \mathcal{M} \to \mathcal{D}$ defined by $\mu \mapsto \tilde{S}(\mu)$ is a $\mathcal{G}$ equivariant surjection, i.e.

$$\tilde{S}(\varphi\mu) = \varphi\tilde{S}(\mu). \tag{3.3}$$

Proof. (a) For $\mu_1 \in \mathcal{M}$, $F_{\mu_1} : (X, \leq) \to (I, \leq)$ is a surjective order map by Proposition 2.5c. As $\varphi \circ F_{\mu_1}$ is also a surjective order map it equals F_{μ_2} for a unique measure μ_2 by Proposition 2.6. By Proposition 2.5 c,d μ_2 is also full and nonatomic, i.e. $\mu_2 \in \mathcal{M}$. This clearly defines an action of $\mathcal{G}$ on $\mathcal{M}$. Furthermore, $\mu_2 = \mu_1$ iff $F_{\mu_1} = \varphi \circ F_{\mu_1}$ iff φ is the identity since F_{μ_1} is surjective. Hence the action is free.

On the other hand, given any pair $\mu_1, \mu_2 \in \mathcal{M}$ the function $\varphi = F_{\mu_2} \circ (F_{\mu_1})^{-1}$ is well defined on I, because for any $y \in I$, $(F_{\mu_1})^{-1}(y)$ is either a singleton or an endpoint pair, by Proposition 2.5d. In the latter case, F_{μ_2} maps the two points to the same value by (2.16). By compactness the surjections F_{μ_1} and F_{μ_2} are quotient maps and so $\varphi \circ F_{\mu_1} = F_{\mu_2}$ implies that φ is continuous. Clearly, $y_1 < y_2$ in I implies $\varphi(y_1) < \varphi(y_2)$ and so φ is an order isomorphism, i.e. $\varphi \in \mathcal{G}$. Hence, $\mathcal{G}$ acts transitively.

(b) Equivalence (3.2) clearly defines a $\mathcal{G}$ action on $\mathcal{D}$. The action is transitive by Lemma 2.3.

(c) $\tilde{S}(\mu) \in \mathcal{D}$ by Proposition 2.11. If $\mu_2 = \varphi\mu_1$ then $F_{\mu_2} = \varphi \circ F_{\mu_1}$ and so $F_{\mu_2}(x) = \varphi(F_{\mu_1}(x))$ for every left endpoint x. This implies $\tilde{S}(\mu_2) = \varphi(\tilde{S}(\mu_1))$ which says that the map $\tilde{S}$ is $\mathcal{G}$ equivariant. It follows that $\tilde{S}$ is surjective because the action of $\mathcal{G}$ on $\mathcal{D}$ is transitive. □

Corollary 3.3. *The map $\tilde{S} : \mathcal{M} \to \mathcal{D}$ induces a bijection from the order isomorphism classes of full, nonatomic measures on X onto the set of countable dense subsets of (0,1). For any $\mu \in \mathcal{M}$ the order isomorphism class of μ admits a bijection onto*

$$Iso_{\tilde{Q}} = \{\varphi \in \mathcal{G} : \varphi(\tilde{Q}) = \tilde{Q}\}. \tag{3.4}$$

Proof. By Theorem 2.10, $\mu_1, \mu_2 \in \mathcal{M}$ are order isomorphic measures, i.e. there exists h an order isomorphism of $(X, \leq)$ such that $h_*\mu_1 = \mu_2$, iff $\tilde{S}(\mu_1) = \tilde{S}(\mu_2)$. Hence, $\tilde{S}$ induces a bijection from order isomorphism classes onto $\mathcal{D}$. Then $D \mapsto D\backslash\{0, 1\}$ induces a bijection from $\mathcal{D}$ onto the set of countable dense subsets of $(0, 1)$.

For $\mu \in \mathcal{M}$ and $D = \tilde{S}(\mu)$ the order isomorphism class of μ is $\tilde{S}^{-1}(D)$. Because the action of $\mathcal{G}$ on $\mathcal{M}$ is free and transitive the inverse of the bijection $\varphi \mapsto \varphi\mu$ restricts to a bijection of $\tilde{S}^{-1}(D)$ onto the *isotropy subgroup* of D (also called the *stabilizer* of D):

$$\mathrm{Iso}_D = \{\varphi \in \mathcal{G} : \varphi D = D\}. \tag{3.5}$$

Now choose $\mu_0 \in \mathcal{M}$ such that $\tilde{S}(\mu_0) = \tilde{Q}$. (Recall that $\tilde{S} : \mathcal{M} \to \mathcal{D}$ is surjective) and let φ_0 be the element of $\mathcal{G}$ such that $\mu = \varphi_0\mu_0$. Since $\tilde{S}$ is equivariant $\varphi_0\tilde{Q} = D$. Hence:

$$\mathrm{Iso}_{\tilde{Q}} = \varphi_0^{-1}\ \mathrm{Iso}_D\varphi_0. \tag{3.6}$$

The isotropy subgroups are conjugate. The inner automorphism $\varphi \mapsto \varphi_0^{-1}\varphi\varphi_0$ provides the required bijection from Iso_D to $\mathrm{Iso}_{\tilde{Q}}$. □

It is easy to construct many elements of $\mathrm{Iso}_{\tilde{Q}}$. Use piecewise linear maps on I made up of segments whose lines have positive rational slope and rational intercepts.

Corollary 3.3 shows that there are uncountably many order isomorphism classes of measures on $\mathcal{M}$. Of course, two nonisomorphic measures might still be homeomorphic. However, we can use these results together with a bit of algebra to construct the uncountably many nonhomeomorphic measures, described in the Introduction. To do this we relate the invariants $\tilde{S}(\mu)$ and $S(\mu)$.

For $\mu \in \mathcal{M}$ and U a nonempty clopen subset of X we define the *conditional measure*, μ_U, a full nonatomic measure on U by

$$\mu_U(B) = \mu(B)/\mu(U) \tag{3.7}$$

for every Borel subset B of U.

Define the U *part of* $\tilde{S}(\mu)$*:*

$$\tilde{S}(\mu, U) = \{F_\mu(x) : x \in U \text{ is a left or right endpoint of } (X, \leq)\}. \tag{3.8}$$

Since the left endpoints are dense, this set is nonempty. In fact, $\max U$ is a left endpoint in U since U is clopen. Similarly, $\min U$ is a right endpoint in U. Because μ is nonatomic, (2.16) implies that every nonzero right endpoint value is also a left endpoint value. Consequently, as expected from the notation

$$\tilde{S}(\mu, U) \subset \tilde{S}(\mu). \tag{3.9}$$

We included right endpoints in the definition because, for example, we need to include the F_μ value of $\min U$ even though the corresponding left endpoint does not lie in U.

Now for any subset A of $\mathbf{R}$ we denote by $Q[A]$ the *extension field* obtained by adjoining the points of A to the rationals. That is, $Q[A]$ is the smallest subfield of $\mathbf{R}$ which contains A. By analogy, we will let $G[A]$ denote the additive subgroup of $\mathbf{R}$ generated by A, i.e. the sums of differences of elements of A. In particular, $Q[\emptyset] = Q$ and by convention $G[\emptyset] = \{0\}$.

Lemma 2.3. *if U is a clopen subset of X, then*

$$\mu(U) \in G[\tilde{S}(\mu, U)] \tag{3.10}$$

and

$$S(\mu_U) \subset Q[\tilde{S}(\mu, U)]. \tag{3.11}$$

Proof. By Lemma 2.4, U can be expressed as a finite disjoint union of intervals $[x_1, x_2]$ with x_1 a right endpoint of $(X, \leq)$ and x_2 a left endpoint. Of course, $x_1, x_2 \in U$. Since $\mu([x_1, x_2]) = F_\mu(x_2) - F_\mu(x_1)$ (μ is nonatomic) which is $F_\mu(x_2)$ if $x_1 = 0$, we see that $\mu([x_1, x_2])$ is an element of $\tilde{S}(\mu, U)$ or else a difference between two such elements. Since $\mu(U)$ is a finite sum of such $\mu([x_1, x_2])$'s we have (3.10).

If V is a clopen subset of U then clearly,

$$\tilde{S}(\mu, V) \subset \tilde{S}(\mu, U) \tag{3.12}$$

and so both $\mu(V)$ and $\mu(U)$ lie in $G(\tilde{S}(\mu, U))$. Hence their ratio lies in the field. $\square$

Remark. It follows from (3.9) and (3.10) that

$$S(\mu) \subset G[\tilde{S}(\mu)]. \tag{3.13}$$

For our construction we recall that a set A of irrationals is called *algebraically independent* if no $\alpha \in A$ is algebraic with respect to $Q[A \backslash \{\alpha\}]$. and so, a fortiori,

$$\alpha \notin Q[A \backslash \{\alpha\}]. \tag{3.14}$$

Just as with linear independence we can choose A^* a maximal algebraically independent subset of $\mathbf{R}$. Such a set is called a *transcendence base* for $\mathbf{R}$ over Q and maximality implies that

$$\mathbf{R} = Q[A^*]. \tag{3.15}$$

See, e.g. [6] Section X.1.

For an infinite set A, the field $Q[A]$ has the same cardinality as A. Hence, we can choose a bijection $\alpha : \mathbf{R} \times N \to A^*$. Furthermore, if we multiply $\alpha(t,n)$ by a nonzero rational number the conditions (3.14) and (3.15) remain unaffected. Thus, we can assume that A^* and α satisfy

$$0 < \alpha(t,n) < 1 \quad \text{and} \quad |\alpha(t,n) - r_n| < n^{-1} \tag{3.16}$$

where $\{r_1, r_2 \dots\}$ is some enumeration of the rational numbers in $(0,1)$. Now define for $t \in \mathbf{R}$

$$D_t = \{0,1\} \cup \{\alpha(t,n) : n \in N\}. \tag{3.17}$$

From condition (3.16) and injectivity of α we have

$$D_t \in \mathcal{D} \text{ for all } t \in \mathbf{R}. \tag{3.18}$$

$$D_{t_1} \cap D_{t_2} = \{0,1\} \text{ for } t_1 \neq t_2 \in \mathbf{R}. \tag{3.19}$$

$$(\cup_{t \in \mathbf{R}} D_t) \setminus \{0,1\} \text{ is algebraically independent.} \tag{3.20}$$

Use Theorem 3.2c to choose for each $t \in \mathbf{R}$ a measure $\mu_t \in \mathcal{M}$ such that

$$\tilde{S}(\mu_t) = D_t \text{ for all } t \in \mathbf{R}. \tag{3.21}$$

Theorem 3.5. *The family of measures $\mathcal{M}^* = \{\mu_t : t \in \mathbf{R}\}$ satisfies the following property:*

For $t_1, t_2 \in \mathbf{R}$ and proper clopen subsets U_1, U_2 of X:

$$\mu_{t_1}(U_1) = \mu_{t_2}(U_2) \Leftrightarrow t_1 = t_2 \text{ and } U_1 = U_2. \tag{3.22}$$

In particular, for $t_1 \neq t_2$:

$$S(\mu_{t_1}) \cap S(\mu_{t_2}) = \{0,1\}. \tag{3.23}$$

Proof. Write μ_α and D_α for μ_{t_α} and D_{t_α} $(\alpha = 1, 2)$.

Suppose that U is a proper clopen subset of X. Apply Lemma 2.4 to write U as the union of its maximal clopen subintervals, i.e.

$$U = \bigcup_{j=1}^{k} [x_{2j-1}, x_{2j}] \tag{3.24}$$

where $x_1 < x_2 < \ldots x_{2k}$ in U is a sequence of endpoints alternately right and left, and the open intervals (x_{2j}, x_{2j+1}) are nonempty for $j = 1, \ldots, k-1$. Note that for any μ in $\mathcal{M}$ this implies

$$F_\mu(x_1) < F_\mu(x_2) < \ldots < F_\mu(x_{2k}) \tag{3.25}$$

in I. Because μ in $\mathcal{M}$ is nonatomic

$$\mu(U) = \sum_{j=1}^{k} (F_\mu(x_{2j}) - F_\mu(x_{2j-1})). \tag{3.26}$$

Because U is not $\emptyset$ or X, some $x_i \neq 0, 1$ in X. If $\mu = \mu_1$ each of the $2k$ summards in (3.26) is in D_1. So algebraic independence, i.e. (3.14), yields

$$x_i \notin \{0, 1\} \Rightarrow \mu_1(U) \notin Q[A^* \backslash \{F_{\mu_1}(x_i)\}]. \tag{3.27}$$

Assume that $t_1 \neq t_2$. By (3.13) $S(\mu_2) \subset Q[D_2]$. With $x_i \neq 0, 1$, $F_{\mu_1}(x_i) \in D_1 \backslash \{0, 1\}$ which is disjoint from D_2 by the construction, (3.17). Hence, $D_2 \subset A^* \backslash \{F_{\mu_1}(x_i)\}$. So (3.27) implies that for U a proper clopen subset of X and $t_1 \neq t_2$:

$$\mu_1(U) \notin S(\mu_2). \tag{3.28}$$

This completes the proof of (3.23) and implies that when $t_1 \neq t_2$, $\mu_1(U_1) \neq \mu_2(U_2)$ for any proper clopen subsets U_1 and U_2.

Now assume that $t_1 = t_2$ and that U_1, U_2 are clopen subsets such that $\mu_1(U_1) = \mu_1(U_2)$. We must prove that $U_1 = U_2$. By removing $U_1 \cap U_2$ from both we can assume that they are disjoint and so the common value is at most $\frac{1}{2}$. If the common value is 0 then $U_1 = U_2 = \emptyset$.

We are left with deriving a contradiction from the assumption that U_1 and U_2 are disjoint proper clopen subsets with the same μ_1 measure. Applying (3.27) with $U = U_1$ or U_2 we see that the common value cannot be rational. In particular it is not equal to $\frac{1}{2}$ so $U = U_1 \cup U_2$ is a proper clopen subset.

Decompose U as in (3.24). Suppose that $x_i \neq 0, 1$ is an endpoint on the list for U. Assume $x_i \in U_1$. Since U_2 is disjoint from U_1, $x_i \notin U_2$. The endpoint paired with x_i does not even lie in U and so is not in U_2. Hence, $F_{\mu_1}(x_i) \notin \tilde{S}(\mu_1, U_2)$. Equivalently this says

$$\tilde{S}(\mu_1, U_2) \subset \{0, 1\} \cup A^* \backslash \{F_{\mu_1}(x_2)\}. \tag{3.29}$$

By (3.27) this implies

$$\mu_1(U) \notin Q[\tilde{S}(\mu_1, U_2)]. \tag{3.30}$$

But $\mu_1(U) = 2\mu_1(U_2)$ and so this contradicts (3.10). $\square$

This result did not require algebraic independence and a transcendence base. Linear independence over Q and a Hamel base could have been used instead. The fields are needed for the following.

Theorem 3.6. *Let $\mathcal{U}$ be the countable family of nonempty clopen subsets of X. The family of measure $\mathcal{M}^* = \{\mu_t : t \in \mathbf{R}\}$ constructed above satisfies the following property.*

For $t_1, t_2 \in \mathbf{R}$ and $U_1, U_2 \in \mathcal{U}$

$$S(\mu_{t_1 U_1}) = S(\mu_{t_2 U_2}) \Leftrightarrow t_1 = t_2 \text{ and } U_1 = U_2. \tag{3.31}$$

Hence, for the uncountable family of measures $\{\mu_{tU}\}$ indexed by $(t, U) \in \mathbf{R} \times \mathcal{U}$ no two distinct members are homeomorphic.

Proof. Assume that U_1, U_2 are distinct elements of $\mathcal{U}$. Without loss of generality we can assume that $U = U_1 \backslash U_2$ is nonempty. Partition U as in (3.24) and choose y a left endpoint of X in the open interval (x_1, x_2). Let V be the clopen interval $[x_1, y]$. Observe that (with $\mu_{t_\alpha} = \mu_\alpha$ and $D_{t_\alpha} = D_\alpha$, $\alpha = 1, 2$):

$$\mu_{1U_1}(V) = (F_{\mu_1}(y) - F_{\mu_1}(x_1))/(\mu_1(U) + \mu_1(U_1 \cap U_2)). \tag{3.32}$$

Notice that the right endpoint associated with y also lies in (x_1, x_2) and so

$$F_{\mu_1}(y) \notin \tilde{A}, \text{ where} \tag{3.33}$$
$$\tilde{A} = \{F_{\mu_1}(x_i) : i = 1, \ldots, 2k\} \cup \tilde{S}(\mu_1, U_2) \cup \tilde{S}(\mu_2, U_2).$$

The remaining terms in the numerator and denominator of (3.32) do lie in $Q[\tilde{A}]$. So algebraic independence implies

$$\mu_{1U_1}(V) \notin Q[\tilde{A}]. \tag{3.34}$$

On the other hand, by (3.11)

$$S(\mu_{2U_2}) \subset Q[\tilde{A}]. \tag{3.35}$$

Since $\mu_{1U_1}(V) \in S(\mu_{1U_1})$ we see that

$$S(\mu_{1U_1}) \neq S(\mu_{2U_2}). \tag{3.36}$$

For the remaining case, where $U_1 = U_2$ and $t_1 \neq t_2$ we proceed as above letting $U = U_1 = U_2$, decomposing and defining $V = [x_1, y]$ with $y \in (x_1, x_2)$ just as before. We leave the final details to the reader. □

References

[1] E. Akin, *The General topology of dynamical systems,* Amer. Math. Soc., Providence, RI (1993).

[2] S. Alpern and V.S. Prasad, *Typical properties of volume preserving homeomorphisms,* to appear.

[3] P. Billingsley, *Ergodic theory and information,* Wiley, New York, NY (1965).

[4] P. Halmos, *Measure theory,* Van Nostrand, Princeton, NJ (1950).

[5] H. Kato, K. Kawamura, H.M. Tuncali, and E.D. Tymchatyn, *Measures and topological dynamics on Menger manifolds*, Topology and its Applications **103** (2000), 249-282.

[6] S. Lang, *Algebra*, Addison-Wesley, Reading, MA (1965).

[7] F.J. Navarro-Bermúdez, *Topologically equivalent measures in the Cantor space*, Proc. Amer. Math. Soc. **77** (1979), 229-236.

[8] F. J. Navarro-Bermúdez and J. Oxtoby, *Four topologically equivalent measures in the Cantor space*, Proc. Amer. Math. Soc. **104** (1988), 859-860.

[9] J. Oxtoby and S.M. Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math (2) **42** (1941), 874-920.

Mathematics Department, The City College, 137 Street and Convent Avenue, New York, N. Y. 10031

E-mail address: `etkcc@cunyvm.cuny.edu`