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REALCOMPACTNESS AND MONOLITHITY ARE FINITELY ADDITIVE IN $C_P(X)$

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Abstract

In this paper we prove the finite additivity for realcompactness as well as for paracompactness, Dieudonné-completeness and monolithity in the class of the spaces $C_p(X)$. It was proved in [Tka83] that X is discrete in case that $C_p(X)$ is homeomorphic to $\mathbb{R}^\tau$ for some cardinal τ. We generalize this result showing that X is discrete also when $C_p(X)$ is a finite union of subspaces homeomorphic to $\mathbb{R}^{\tau_i}$ for some cardinals τ_i. We also establish that if $C_p(X)$ is a countable union of Eberlein-Grothendieck subspaces then $C_p(X)$ is itself an Eberlein-Grothendieck space.

0. Introduction

The well-known Mrówka-Isbell space Ψ [GJ] is an example of a non-realcompact space wich is a union of two discrete and hence realcompact subspaces. Many known topological cardinal invariants and properties have the same behaviour as the realcompactness: they are non-additive.

V. V. Tkachuk proved in [Tka94] that several non-additive topological properties $\mathcal{P}$ are *countably additive in the class of space* $C_p(X)$ (that is, if

$$C_p(X) = \bigcup\{Y_n : n < \omega\},$$

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and each Y_n has the property $\mathcal{P}$, then $C_p(X)$ itself has the property $\mathcal{P}$). For instance, countable additivity was established for any $\mathcal{P} \in \{$ metrizability, Čech-completeness, Fréchet-Uryshon property, i-weight $\leq \tau$, tightness $\leq \tau\}$. However, in case of realcompactness, the countable additivity was only proved for closed subspaces (see [Tka94], Corollary 1.5). In view this result it is natural to ask: *If* $C_p(X) = \bigcup\{C_n : n \in \omega\}$*, where all the* C_n *are realcompact, must* $C_p(X)$ *be realcompact?* In this paper we study this question. We prove that for the case of finite decomposition the answer is "yes".

In the paper [Tka83] the discreteness of X was proved under the assumption that there was a homeomorphism of $C_p(X)$ onto $\mathbb{R}^\tau$. As an application of our result on finite additivity for realcompactness, we show that the same turns out to be true in the case when $C_p(X)$ is a finite union of subspaces homeomorphic to $\mathbb{R}^{\tau_i}$ for some cardinals τ_i and that the Dieudonné completeness is also a finitely additive property in $C_p(X)$.

We also establish that if $C_p(X)$ is a countable union of Eberlein-Grothendieck subspaces then $C_p(X)$ is itself an Eberlein-Grothendieck space.

1. Notation and Terminology

We denote respectively by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{I}$, the reals, the naturals and the unit interval $[0,1]$. Throughout this paper "a space" means "a Tychonoff space". If X is a space then $\mathcal{T}(X)$ is its topology and $\mathcal{T}^*(X) = \mathcal{T}(X) \setminus \{\emptyset\}$. If X and Y are spaces, the symbol $X \simeq Y$ will mean that they are homeomorphic.

For space X and $A \subset X$, we denote by $\overline{A}$ the closure of A in X. If it might not be clear in which space the closure is taken, then we write $\mathrm{cl}_X(A)$ for the closure of A in X. If $f : X \to Y$ is a map, and $A \subset X$, then $f{\restriction}_A$ is the restriction of f to A.

A cardinal number τ is identified with the smallest ordinal number having power τ. We use $nw(X)$, $s(X)$, $hL(X)$ and $hd(X)$ to denote the network weight, spread, hereditary Lindelöf number and hereditary density of X.

The symbol $C(X,Y)$ denotes the set of all continuous mappings of a space X into a space Y. We consider the topology of pointwise convergence and uniform convergence on $C(X,Y)$. The topological spaces obtained thereby will be denoted by $C_p(X,Y)$ and $C_u(X,Y)$, respectively. The space $C_p(X,\mathbb{R})$ (respectively, $C_u(X,\mathbb{R})$) of all real-valued continuous function on X will be denoted simply by $C_p(X)$ (respectively, $C_u(X)$). For $A \subset X$, let $\pi_A : C_p(X) \to C_p(A)$ denote the restriction map defined by $\pi_A(f) = f\restriction_A$ for all $f \in C_p(X)$. If $Z \subset C_p(X)$ then $W \subset Z$ is a standard open subset of Z iff $W = [x_1, \ldots, x_n; U_1 \ldots, U_n] = \{g \in Z : g(x_i) \in U_i \text{ for } i = 1, \ldots, n\}$ for some natural $n \geq 1$ and $x_1 \ldots, x_n \in X$, $U_1, \ldots, U_n \in \mathcal{T}^*(\mathbb{R})$ and each U_i is an interval in $\mathbb{R}$.

We are going to make use of some notions and results of C_p-theory. Let us introduce and formulate them shortly. The reader can find the proofs and detailed discussions in [Arh92]. All other notions are standard and can be found in [Eng].

2. Some Properties Finitely Additive in $C_p(X)$

Let us start with the following technical lemma, which will be useful for our further analysis of finite additivity of realcompactness in $C_p(X)$.

Lemma 2.1. *Let $1 \leq n < \omega$. Suppose that $\mathcal{P}$ is a topological property which is hereditary with respect to any F_σ subspace. If $C_p(X) = \bigcup\{C_i : i < n\}$, where all the subspaces C_i have the property $\mathcal{P}$, then there are subspaces $B_0, B_1, \ldots, B_{m-1}$ ($m \leq n$) such that:*

(1) *B_i has the property $\mathcal{P}$ and $\overline{B_i} = C_p(X)$ for every $i < m$;*

(2) *$C_p(X) = \bigcup\{B_i : i < m\}$.*

Proof. We will proceed by induction on n. For $n = 1$ our statement is trivially true. Suppose that $2 \leq n < \omega$ and that the result is true for every decomposition of at most $n-1$ subspaces.

Suppose that $C_p(X) = \bigcup\{C_i : i < n\}$, where all the subspaces C_i have the property $\mathcal{P}$. Evidently, if $\overline{C_i} = C_p(X)$ for every $i < n$ then there is nothing to prove. So, suppose that there is an $i < n$ such that $\overline{C_i} \neq C_p(X)$. Then there is a standard open set W in $C_p(X)$ such that $W \subset C_p(X) \setminus C_i \subset \bigcup_{j \neq i} C_j$. Define $D_j = W \cap C_j$ $(j \neq i)$. Observe that, being an F_σ subset of C_j, each D_j has the property $\mathcal{P}$. Now, since $C_p(X)$ is homeomorphic to W, we can choose a homeomorphism h from W onto $C_p(X)$. Then, we have that $C_p(X) = \bigcup_{j \neq i} h(D_j)$, where each subset $h(D_j)$ has the property $\mathcal{P}$. Now we can apply the induction hypothesis to find subspaces $B_0, B_1, \dots, B_{m-1}$ $(m \leq n-1)$ such that properties (1) and (2) hold. □

Note that the Mrówka-Isbell space Ψ is a union of two paracompact subspaces but Ψ is non-paracompact. Therefore, the paracompactness is a non-additive topological property in the class of Tychonoff spaces. Using Lemma 2.1 we can prove that the paracompactness is a finitely additive property in the class of spaces $C_p(X)$.

Theorem 2.2. *Let $1 \leq n < \omega$. If $C_p(X) = \bigcup\{C_i : i < n\}$ where C_i is paracompact for every $i < n$, then $C_p(X)$ is paracompact.*

Proof. In view of Lemma 2.1, we can assume that $\overline{C_i} = C_p(X)$ for every $i < n$. Then each subspace C_i has the Souslin property (because $C_p(X)$ has this property). Recall that every paracompact space with the Souslin property is Lindelöf and that the Lindelöf property is countably additive, so $C_p(X)$ is Lindelöf. Hence, $C_p(X)$ is paracompact. □

For the sake of completeness, we present here the following well-known notions and results.

Let τ be an infinite cardinal, and let Y be a subspace of a space X. Y is said to be *τ-placed* in X if for each $x \in X \setminus Y$ there is a set P of type G_τ in X such that $x \in P \subset X \setminus Y$. The Hewitt-Nachbin number of X is

$$q(X) = min\{\tau \geq \omega : X \text{ is } \tau\text{-placed in } \beta X\}.$$

It is well-known that $q(X) \leq \omega$ if and only if X is realcompact. On the other hand, a space X is called *m_τ-space* if for each open set U and for each $x \in \overline{U}$, there is a set P of type G_τ in X such that $x \in P \subset \overline{U}$. The minimal cardinal $\lambda \geq \omega$ such that X is an m_λ-space is denote $m(X)$. It is well-known that $m(C_p(X)) \leq \omega$ for any X. The following result belongs to A. Ch. Chigogidze (see [Arh92], Chapter 2).

Proposition 2.3. *If $Y \subset X = \overline{Y}$, $q(Y) \leq \tau$ and $m(X) \leq \tau$ then Y is τ-placed in X.*

We recall that a real-valued function f on X is said to be *strictly τ-continuous* if for any set $A \subset X$ of cardinality at most τ the restriction to A of f coincides with the restriction to A of some real-valued function continuous on the whole space X. The *minitightness* of a space X does not exceed τ ($t_m(X) \leq \tau$) if each strictly τ-continuous function on X is continuous. A. V. Arhangel'skiĭ [Arh83] had shown that $q(C_p(X)) = t_m(X)$ for any X.

The following lemma plays a very important role in the proof that the realcompactness is a finitely additive property in the class of spaces $C_p(X)$. Recall that $[X]^{\leq\tau}$ is the family of all subsets of X of cardinality $\leq \tau$.

Lemma 2.4. *Let $\tau \geq \omega$. Then the following conditions are equivalent for any space X:*

(1) $q(C_p(X)) \leq \tau$;

(2) *There are $\phi \in C_p(X)$ and $Y \in [X]^{\leq\tau}$ such that*

$$q(\{f \in C_p(X) : f\restriction_Y \equiv \phi\restriction_Y\}) \leq \tau.$$

Proof. (1) $\Longrightarrow$ (2). Choose arbitrarily a $\phi \in C_p(X)$ and $Y \in [X]^{\leq\tau}$. Consider the restriction map $\pi_Y : C_p(X) \to C_p(Y)$. Then $\{f \in C_p(X) : f\restriction_Y \equiv \phi\restriction_Y\} = \pi_Y^{-1}(\phi\restriction_Y)$ is a closed subset in $C_p(X)$. Since the Hewitt-Nachbin number is not increased on a closed subset, we have $q(\{f \in C_p(X) : f\restriction_Y \equiv \phi\restriction_Y\}) \leq \tau$.

(2) $\Longrightarrow$ (1). Let us first note that due to the homogeneity of $C_p(X)$, we can assume that $q(\{f \in C_p(X) : f\restriction_Y \equiv 0\}) \le \tau$. Note also that

$$G = \pi_Y^{-1}(0) = \{f \in C_p(X) : f\restriction_Y \equiv 0\} = \{f \in C_p(X) : f\restriction_{\overline{Y}} \equiv 0\}.$$

Consider the restriction map $\pi_{X\setminus\overline{Y}} : C_p(X) \to C_p(X \setminus \overline{Y})$. It is a routine verification that $\pi_{X\setminus\overline{Y}}\restriction_G : G \to \pi_{X\setminus\overline{Y}}(G)$ is a homeomorphism. Therefore, we have $q(\pi_{X\setminus\overline{Y}}(G)) \le \tau$. We claim that $\pi_{X\setminus\overline{Y}}(G)$ is a dense subset in $\mathbb{R}^{X\setminus\overline{Y}}$. Indeed, let $W = [x_1, \dots, x_k; W_1, \dots, W_k]$ be a standard open subset of $\mathbb{R}^{X\setminus\overline{Y}}$. Pick $h \in W$. Since X is Tychonoff, there is $g \in C(X)$ such that $g\restriction_{\overline{Y}} \equiv 0$ and $g(x_j) = h(x_j)$ for all $j = 1, \dots, k$. Then $g \in G$ and $\pi_{X\setminus\overline{Y}}(g) \in W$. Our claim is thus proved. Now applying 2.3, we can conclude that $\pi_{X\setminus\overline{Y}}(G)$ is τ-placed in $\mathbb{R}^{X\setminus\overline{Y}}$.

To finish the proof of the lemma, suppose that $t_m(X) = q(C_p(X)) > \tau$. Then there is strictly τ-continuous function $\psi \in \mathbb{R}^X \setminus C_p(X)$. Since $Y \in [X]^{\le\tau}$, there exists $f \in C_p(X)$ such that $f\restriction_Y = \psi\restriction_Y$. Let $\eta = \psi - f$. The function η is discontinuous and $\eta\restriction_Y \equiv 0$. Observe that η is strictly τ-continuous. We claim that

$$\eta\restriction_{X\setminus\overline{Y}} \in \mathbb{R}^{X\setminus\overline{Y}} \setminus \pi_{X\setminus\overline{Y}}(G).$$

Indeed, if it was not so, then a function h can be chosen in G such that $\eta\restriction_{X\setminus\overline{Y}} = h\restriction_{X\setminus\overline{Y}}$. Therefore $\eta = h \in C_p(X)$ (note that $\eta\restriction_{\overline{Y}} \equiv 0$, because η is strictly τ-continuous), which is a contradiction. Thus, $\eta\restriction_{X\setminus\overline{Y}} \in \mathbb{R}^{X\setminus\overline{Y}} \setminus \pi_{X\setminus\overline{Y}}(G)$.

Now, since $\pi_{X\setminus\overline{Y}}(G)$ is τ-placed in $\mathbb{R}^{X\setminus\overline{Y}}$, there is $Z \in [X \setminus \overline{Y}]^{\le\tau}$ such that

$$\pi_{X\setminus\overline{Y}}(\eta)\restriction_Z \in \{f \in \mathbb{R}^{X\setminus\overline{Y}} : f\restriction_Z \equiv \pi_{X\setminus\overline{Y}}(\eta)\restriction_Z\} \subset \mathbb{R}^{X\setminus\overline{Y}} \setminus \pi_{X\setminus\overline{Y}}(G).$$

But η is strictly τ-continuous and $Z \cup Y \in [X]^{\le\tau}$, so that there exists an $h \in C_p(X)$ such that $h\restriction_{Z\cup Y} = \eta\restriction_{Z\cup Y}$. Since $\eta\restriction_Y \equiv 0$, we have $h\restriction_Y \equiv 0$ and $h\restriction_Z = \eta\restriction_Z$. Now $h\restriction_{X\setminus\overline{Y}} \in \pi_{X\setminus\overline{Y}}(G)$ and

$$h\restriction_{X\setminus\overline{Y}} \in \{f \in \mathbb{R}^{X\setminus\overline{Y}} : f\restriction_Z \equiv \pi_{X\setminus\overline{Y}}(\eta)\restriction_Z\} \subset \mathbb{R}^{X\setminus\overline{Y}} \setminus \pi_{X\setminus\overline{Y}}(G),$$

which is a contradiction. □

Now we are ready to prove one of our main results.

Theorem 2.5. *Let $1 \leq n < \omega \leq \tau$. If $C_p(X) = \bigcup\{C_i : i < n\}$ and $q(C_i) \leq \tau$ for every $i < n$, then $q(C_p(X)) \leq \tau$. In particular, the realcompactness is finitely additive in spaces of the type $C_p(X)$.*

Proof. We will proceed by induction on n. For $n = 1$ our statement is trivially true. Suppose $2 \leq n < \omega$ and that we are done for $n - 1$.

Let $C_p(X) = \bigcup\{C_i : i < n\}$, with $q(C_i) \leq \tau$ for every $i < n$. Applying Lemma 2.1, we can assume that $\overline{C_i} = C_p(X)$ for every i. Since $m(C_p(X)) \leq \tau$ and $q(C_i) \leq \tau$ for every $i < n$, we have that each C_i is τ-placed in $C_p(X)$ (see 2.3).

Now, if $C_p(X) \setminus \bigcup_{i<n-1} C_i = \emptyset$ then $q(C_p(X)) \leq \tau$ (by the induction hypothesis). Otherwise, choose arbitrarily a $\phi \in C_p(X) \setminus \bigcup_{i<n-1} C_i$. Due to the fact that each C_i is τ-placed in $C_p(X)$, there is a subset G_i of type G_τ of $C_p(X)$ such that $\phi \in G_i \subset C_p(X) \setminus C_i$, for every $i < n - 1$. A standard argument shows that there exists $Y_i \in [X]^{\leq\tau}$ such that $\phi \in \{f \in C_p(X) : f\restriction_{Y_i} \equiv \phi\restriction_{Y_i}\} \subset G_i$, for each $i < n - 1$.

Let $Y = \bigcup_{i<n-1} Y_i$. Then $Y \in [X]^{\leq\tau}$ and

$$\phi \in \{h \in C_p(X) : h\restriction_Y \equiv \phi\restriction_Y\} \subset \bigcap_{i<n-1} G_i \subset C_p(X) \setminus \bigcup_{i<n-1} C_i \subset C_{n-1}.$$

Since $\{h \in C_p(X) : h\restriction_Y \equiv \phi\restriction_Y\} \subset C_{n-1}$ is closed, we have

$$q(\{h \in C_p(X) : h\restriction_Y \equiv \phi\restriction_Y\}) \leq \tau.$$

By Lemma 2.4, we conclude that $q(C_p(X)) \leq \tau$. □

The last theorem has some interesting consequences. Indeed, suppose that $1 \leq n < \omega$ and $C_p(X) = \bigcup\{C_i : i < n\}$ where each C_i is Dieudonné complete. Using Lemma 2.1, we can assume that $\overline{C_i} = C_p(X)$ for every $i < n$. Then each C_i has the Souslin property. Since every Dieudonné complete space with countable Souslin number is realcompact [Eng], each C_i is realcompact. By the previous theorem, $C_p(X)$ is realcompact, hence it is Dieudonné complete. Thus, we have

Theorem 2.6. *The Dieudonné completeness is a finitely additive property in the spaces $C_p(X)$.*

It is shown in [Tka83] that a space X is discrete in case that $C_p(X)$ is homeomorphic to $\mathbb{R}^\tau$ for some cardinal τ. Using our result on realcompactness, we can generalize this theorem.

Theorem 2.7. *Let $1 \leq n < \omega$. Suppose that τ_i is a cardinal number for each $i < n$. If $C_p(X) = \bigcup\{C_i : i < n\}$ and C_i is homeomorphic to $\mathbb{R}^{\tau_i}$ for every $i < n$, then X is discrete.*

Proof. The space $C_p(X)$ is realcompact, because each subspace C_i is realcompact. But also each C_i is pseudocomplete. Then, $C_p(X)$ is pseudocomplete (it is easy to see that the pseudocompleteness is a finitely additive topological property). Finally, any $C_p(X)$ that is realcompact and pseudocomplete coincide with $\mathbb{R}^X$ (see [Tka85] and [Tka86]) so our theorem is proved. □

Recall that a space X is *τ-monolithic* ($\tau \geq \omega$) [Arh92] if $nw(\overline{A}) \leq \tau$ for all $A \subset X$ such that $|A| \leq \tau$. A space X is *monolithic* if X is τ-monolithic for every $\tau \geq \omega$. It is well-known that there are spaces X which are non-monolithic being a finite union of monolithic subspaces (for instance, the Niemytzki plane works to prove this). We finish this section establishing that such examples of $C_p(X)$ do not exist. To prove this fact we need the following lemma (compare with 2.1).

Lemma 2.8. *Let $1 \leq n < \omega$. Suppose that $\mathcal{P}$ is a hereditary topological property. If $C_p(X) = \bigcup\{C_i : i < n\}$, where all subspaces C_i have the property $\mathcal{P}$, then there are subspaces $B_0, B_1, \ldots, B_{m-1}$ ($m \leq n$) such that:*

(1) *B_i has the property $\mathcal{P}$ and $cl_{C_u(X)}(B_i) = C_u(X)$ for every $i < m$;*

(2) $C_p(X) = \bigcup\{B_i : i < m\}$.

Proof. The proof is by induction on n. For $n = 1$, the statement is trivially true. Suppose it is already proved for every $k < n$. In case that each subspace C_i is dense in $C_u(X)$, evidently there is nothing to prove. So, suppose that there is an $i < n$ such that $\overline{C_i} \neq C_u(X)$. Then there are $g \in C(X)$ and $\epsilon > 0$ such that $B_\epsilon(g) = \{h \in C(X) : |h(x) - g(x)| < \epsilon \text{ for all } x \in X\} \subset C_u(X) \setminus C_i \subset \bigcup_{j \neq i} C_j$. Due to the homogeneity of $C_u(X)$, we may assume that $g \equiv 0$.

Define $D_j = B_\epsilon(g) \cap C_j$, for every $j \neq i$. Since $C_p(X) \simeq C_p(X, (-\epsilon, \epsilon)) = B_\epsilon(g)$, we have $C_p(X) = \bigcup_{j \neq i} D'_j$ where each D'_j has the property $\mathcal{P}$. By induction, there are subspaces $B_0, \ldots, B_{m-1}$ (for some $m \leq n - 1$) such that the properties **(1)** and **(2)** hold, and the lemma is proved. □

Following Tkachuk [Tka91], given $\tau \geq \omega$ and a cardinal function η, we call a space X η_τ-monolithic if for all $A \subset X$ such that $|A| \leq \tau$ we have $\eta(\overline{A}) \leq \tau$ (note that the nw_τ-monolithity is equivalent to τ-monolithity). X is called η-monolithic if it is η_τ-monolithic for every cardinal $\tau \geq \omega$.

Theorem 2.9. *Let $\tau \geq \omega$. Suppose that η is a finitely additive hereditary cardinal function. If $C_p(X) = \bigcup\{D_i : i < n\}$ where all subspaces D_i are η_τ-monolithic, then $C_p(X)$ is η_τ-monolithic.*

Proof. Observe firstly that η_τ-monolithity is an hereditary property if η is hereditary.

Applying Lemma 2.8, we can assume that every subspace D_i is a dense subset of $C_u(X)$.

Take any $A \subset C_p(X)$ with $|A| \leq \tau$. Define $A_j = A \cap D_j$ for every $j < n$. Then $\overline{A} = \bigcup_{j<n} \overline{A_j}$. Since η is finitely additive, to prove that $\eta(\overline{A}) \leq \tau$ it is sufficient to prove that $\eta(\overline{A_j}) \leq \tau$ for every $j < n$. Fix a $k < n$. Now, observe that $A_k \subset D_k$ and $\overline{A_k} = \bigcup_{j<n}(\overline{A_k} \cap D_j)$. Again, due to the fact that η is finitely additive it is sufficient to prove that $\eta(\overline{A_k} \cap D_j) \leq \tau$ for each $j < n$. Since $|A_k| \leq \tau$ and D_k is η_τ-monolithic, we have $\eta(\overline{A_k} \cap D_k) = \eta(\mathrm{cl}_{D_k}(A_k)) \leq \tau$. Now, fix $j < n$ with $j \neq k$.

Since $C_u(X)$ is Fréchet-Uryshon and D_j is dense in $C_u(X)$, for every $a \in A_k$ there exits $\{f_n^a\}_{n<\omega} \subset D_j$ such that f_n^a converges to a. Let $B = \{f_n^a : a \in A, n < \omega\}$. Then $\overline{A_k} \subset \overline{B}$. Therefore, we have $\overline{A_k} \cap D_j \subset \overline{B} \cap D_j$. Since D_j is η_τ-monolithic, $B \subset D_j$, and $|B| \leq \tau$, we have $\eta(\overline{B} \cap D_j) = \eta(\mathrm{cl}_{D_j}(B)) \leq \tau$. Hence $\eta(\overline{A_k} \cap D_j) \leq \tau$, and the theorem is proved. □

Corollary 2.10. *Let $\eta \in \{nw, s, hd, hL\}$. If $C_p(X) = \bigcup\{C_i : i < n\}$ where all subspaces C_i are η-monolithic, then $C_p(X)$ is η-monolithic.*

Corollary 2.11. *Let τ be an infinite cardinal. If $C_p(X)$ is a finite union of τ-monolithic subspaces, then $C_p(X)$ is τ-monolithic.*

3. Eberlein-Grothendieck Property is Countably Additive in $C_p(X)$

It is a well-known fact of C_p-theory that every space with countable base can be embedded into $C_p(K)$ for some compact space K (see [Arh92]). In fact, spaces with this property have a special name: a space X is called an *Eberlein-Grothendieck* space (or EG-space) if it can be embedded into $C_p(K)$ for some compact space K. The property of being an EG-space is non-additive in the class of Tychonoff spaces: if $p \in \beta\mathbb{N} \setminus \mathbb{N}$ then $\mathbb{N} \cup \{p\}$ cannot be embedded in a C_p over a compact space [Arh92]. So, it is natural to ask it is a finitely additive property in the class of spaces of type $C_p(X)$. The answer is "yes". We will prove that this property is even countably additive in the class of spaces $C_p(X)$. We need to recall the method used by O. G. Okunev to prove the following result (see [Arh92], Chapter 3).

Theorem 3.1 [Oku]. *A space X is σ-compact if and only if $C_p(X)$ is an Eberlein-Grothendieck space.*

A set $\Phi \subset C(X, \mathbb{R})$ is called *$\mathcal{D}$-separating* if $0 \in \Phi$, $f(X) \subset [-1, 1]$ for all $f \in \Phi$, and the following condition holds: $\forall \epsilon > 0$, if $P \subset X$ is closed and $\{x_1, x_2, \dots, x_n\} \subset X \setminus P$, then there exists an $f \in \Phi$ such that $|f(x_i)| < \epsilon$ for $i = 1, 2, \dots, n$ and $|f(x)| \in [\frac{3}{4}, 1]$ for all $x \in P$.

Remark 3.2. Every dense subspace of $C_u(X)$ is (homeomorphic to) a $\mathcal{D}$-separating family of real-valued continuous functions. Indeed, let D be a dense subset of $C_u(X)$. Due to the homogeneity of $C_u(X)$, we can assume that $0 \in D$. Let $\epsilon > 0$, $\{x_1, \dots, x_n\} \subset X$, and P a closed subset of X such that $P \cap \{x_1, \dots, x_n\} = \emptyset$. Since X is Tychonoff, there is $g \in C(X)$ such that $g\restriction_{\{x_1,\dots,x_n\}} \equiv 0$ and $g(P) \subset \{\frac{7}{8}\}$. Let $\delta = \min\{\epsilon, \frac{1}{8}\}$. Since D is dense in $C_u(X)$, there is $d \in D$ such that $d \in B_\delta(g) = \{h \in C_u(X) : |h(x) - g(x)| < \delta \text{ for all } x \in X\}$. It is easy to check that $|d(x_i)| < \epsilon$ for $i = 1, 2, \dots, n$ and $|d(x)| \in [\frac{3}{4}, 1]$ for all $x \in P$. Thus, D is a $\mathcal{D}$-separating family.

Proposition 3.3 [Arh92]. *For a space X, let $\mathcal{K}(X)$ be the minimal class of spaces containing X and all compact spaces, which is closed with respect to finite products, free unions of countable families, closed subspaces and continuous images. If some $\mathcal{D}$-separating subspace $\Phi \subset C_p(X)$ is homeomorphic to a subspace of $C_p(Y)$, then $X \in \mathcal{K}(Y)$.*

Remark 3.4. It is well-known [Arh92] that if X is σ-compact (respectively, Lindelöf Σ-space, K-analytic space) then $\mathcal{K}(X)$ consists of all σ-compact spaces (respectively, Lindelöf Σ-spaces, K-analytic spaces). Recall that a space X is K-analytic (analytic) if it is a continuous image of a space of type $K_{\sigma\delta}$ (of the irrational numbers) [RJ].

Now we are ready to prove our last result.

Theorem 3.5. *Let X be a space. Suppose that $C_p(X) = \bigcup\{C_n : n \in \omega\}$. If each C_n can be embedded into $C_p(K_n)$ for some σ-compact space K_n (respectively, Lindelöf Σ-space K_n, K-analytic space K_n, and analytic space K_n) then X is a σ-compact space (respectively, Lindelöf Σ-space, K-analytic space, and analytic space).*

Proof. By Lemma 1.1 in [Tka94], there exist $f \in C_p(X)$, $\epsilon > 0$, and $m < \omega$ such that $\Phi = (C_m + f) \cap C(X, (-\epsilon, \epsilon))$ is dense in $C_u(X, (-\epsilon, \epsilon))$. Since $C_u(X, (-\epsilon, \epsilon))$ is homeomorphic to $C_u(X)$, the set Φ is homeomorphic to a dense subspace Φ_1 of $C_u(X)$. The subspace Φ_1 is a $\mathcal{D}$-separating family of real-valued continuous functions (see 3.2). Therefore, by 3.3 (evidently, Φ_1 can be embedded into $C_p(K_m)$), we have $X \in \mathcal{K}(K_m)$. So, applying 3.4, the result follows (in case of analytic space recall that every analytic space is K-analytic, the network weight is a countably additive cardinal invariant, and that a K-analytic space with countable network is an analytic space). □

4. Open Questions

Of course the most intriguing unsolved questions are the ones related to countable additivity of non-additive topological properties in the spaces $C_p(X)$. We present here the list of questions that we could not solve while working on this paper.

Question 4.1. *Suppose that $C_p(X) = \bigcup\{C_n : n \in \omega\}$, where all the C_n are realcompact. Must $C_p(X)$ be realcompact?*

Question 4.2. *Is the paracompactness a countably additive property in the spaces $C_p(X)$?*

Question 4.3. *Is the metacompactness a countably additive property in the spaces $C_p(X)$? Is it a finitely additive property?*

Question 4.4. *Is the monolithity a countably additive property in the spaces $C_p(X)$?*

Question 4.5. *Let $C_p(X) = A \cup B$, where A and B are normal subspaces of $C_p(X)$. Must $C_p(X)$ be normal? What if X is compact?*

Question 4.6. *Let $C_p(X) = A \cup B$, where A and B are perfectly normal subspaces of $C_p(X)$. Must $C_p(X)$ be perfectly normal?*

Question 4.7. *Let $C_p(X) = A \cup B$, where A and B are collectionwise normal subspaces of $C_p(X)$. Must $C_p(X)$ be collectionwise normal?*

Question 4.8. *Let $C_p(X) = A \cup B$, where A and B are hereditary normal subspaces of $C_p(X)$. Must $C_p(X)$ be hereditary normal?*

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