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ON THE POSET OF TOTALLY DENSE SUBGROUPS OF COMPACT GROUPS

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Abstract

A subgroup H of a topological group G is said to be *essential* [resp., *totally dense*] in G if $|H \cap N| > 1$ [resp., $H \cap N$ is dense in N] for every closed, non-trivial normal subgroup N of G. We study the poset $\mathcal{ED}(K)$ of all essentially dense subgroups of a compact (abelian) group K and its subposet $\mathcal{TD}(K)$ of totally dense subgroups. Specifically we show:

Theorem A. *For a compact abelian group K the following are equivalent:*

(a) K admits a smallest totally dense subgroup;

(b) $\mathcal{TD}(K)$ is a (complete) lattice;

(c) the torsion subgroup $t(K)$ of K is totally dense in K;

(d) $t(K)$ is essentially dense in K; and

(e) K contains copies of the group $\mathbb{Z}_p$ of p-adic integers for no prime p.

Theorem B. *For a compact abelian group K the following are equivalent:*

(a) K admits a smallest essentially dense subgroup;

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Key words: compact group, minimal group, essential subgroup, essentially dense subgroup, totally dense subgroup.

(b) $\mathcal{ED}(K)$ *is a (complete) lattice;*

(c) K *admits a smallest totally dense subgroup and* $soc(K)$ *is either dense or open in* K. We study also the class $\mathcal{C}$ of topological groups K such that every essentially dense subgroup of K is totally dense, and we describe the compact groups in $\mathcal{C}$ that are either abelian or connected.

1. Introduction

1.1. Summary of Principal Results.

A subgroup H of a topological group G is

- *essential* if $H \cap N \neq \{0\}$ for every non-trivial closed normal subgroup N of G.
- *essentially dense* in G if H is dense and essential in G.
- *totally dense* in G if $H \cap N$ is dense in N for every closed normal subgroup N of G.

The following three cardinal invariants of a topological group G are related to the above properties; the first two were introduced in [21] (see also [12]) for compact abelian groups and studied in [2] in the general case.

$ED(G) = \min\{\,|H| : H$ is an essentially dense subgroup of $G\}$; and

$TD(G) = \min\{\,|H| : H$ is a totally dense subgroup of $G\}$.

The poset of totally dense [resp., essentially dense] subgroups of a group G is denoted $\mathcal{TD}(G)$ [resp., $\mathcal{ED}(G)$]. Since $\mathcal{TD}(G) \subseteq \mathcal{ED}(G)$, the relations $ED(G) \leq TD(G)$ always hold. It was proved by Stoyanov [21] that $ED(G) = TD(G)$ for every compact abelian group. Boschi and the second-named author [2] recently proved that the invariants $TD(G)$ and $ED(G)$ coincide also for many other classes of topological groups G, including: LCA groups, compact connected groups, totally minimal abelian groups, free Abelian topological groups, topologically simple groups, and LCA groups with their Bohr topology. In a forthcoming paper [3] we have proved that the equality $ED(G) = TD(G)$ may fail even for minimal abelian groups. Thus the difference between total and essential density is plain at the quantitative level. In contrast to [2] and [3], we develop here a *qualitative* analysis and comparison of total and essential density by studying the poset $\mathcal{ED}(K)$ of all essentially dense subgroups of a compact (abelian) group K and its subposet $\mathcal{TD}(K)$ of all totally dense subgroups. To make clearly the point that these concepts are distinct, we characterize in Theorem 3.6 below those compact abelian groups G for which $\mathcal{ED}(G) = \mathcal{TD}(G)$.

In Section 4 we discuss the question of the existence of a totally dense nucleus (smallest totally dense subgroup) of a given topological group and we obtain a complete description of those compact abelian group that admit such a nucleus (Theorem 4.1). The analogous question of the existence of an essentially dense nucleus is faced in Section 5 with a similar result in the case of compact abelian groups (Theorem 5.1).

1.2. Notation and Terminology

The topological groups considered in this paper are assumed to satisfy the Hausdorff separation axiom; they are, then, Tychonoff spaces [13]. The class of all topological groups is denoted $\mathcal{G}$, and

$\mathcal{C} = \{G \in \mathcal{G} : \mathcal{ED}(G) = \mathcal{TD}(G)\}$.

We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive naturals and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ the unit circle group. The group of p-adic integers is denoted by $\mathbb{Z}_p$ ($p \in \mathbb{P}$), and $\mathbb{Z}(n)$ is the cyclic group of order $n > 1$. For an abelian group G we use the following notation:

$r(G)$: the torsion-free rank of G.

$t(G)$: the torsion subgroup of G ($= \{x \in G : x$ is a torsion element$\}$).

$t_p(G)$: the p-torsion subgroup of G.
$G[p]$: the p-socle of G (=$\{x \in G : p \cdot x = 0\}$).
$soc(G)$: the socle of G ($= \oplus_{p\in\mathbb{P}} G[p]$).

If the group G is not necessarily abelian, then the torsion part $t(G)$ is just a *subset*, not necessarily a subgroup of G. We denote by $\langle X\rangle$ the subgroup of G generated by a subset $X \subseteq G$.

For an LCA group we denote by $\widehat{G}$ its Pontryagin dual, for a subgroup H of G we write $A(H) = \{\chi \in \widehat{G} : \chi[G] = \{0\}\}$, and for a subgroup N of $\widehat{G}$ we identify $A(N) \subseteq \widehat{\widehat{G}}$ with the subgroup $\{x \in G : (\forall \chi \in N)\chi(x) = 0\}$.

3. Remark. Several of our results were presented at the 14th Summer Conference on General Topology and Its Applications (C. W. Post campus of Long Island University, USA, August, 1999). See in this connection the announcement,

http://at.yorku.ca/cgi-bin/amca/calc_19,

and the forthcoming paper [3].

2. Historical Background and Preliminaries

The posets $\mathcal{TD}(G)$ and $\mathcal{ED}(G)$ are *sup-complete* in the sense that each subfamily $\mathcal{A}$ has a least upper bound (given by $\vee\mathcal{A} = \langle\cup\mathcal{A}\rangle$). In contrast, neither of these posets is (in general) even a lattice: It will become clear in Theorem 4.1, for example, that for $p \in \mathbb{P}$ the compact group $\mathbb{Z}_p$ contains totally dense subgroups H_0 and H_1 such that $H_0 \cap H_1 = \{0\}$.

Definition 2.1. ([22]) *For a topological group G and $p \in \mathbb{P}$, an element $x \in G$ is a quasi-p-torsion element of G if the cyclic subgroup $\langle x\rangle$ with the topology inherited from G is either a finite p-group or is isomorphic to $\mathbb{Z}$ equipped with the p-adic topology.*

The set of quasi-p-torsion elements of G is denoted $td_p(G)$.

Properties of the sets $td_p(G)$ are explored in detail in [22] and [12](Chapter 4). We will use the following facts.

Theorem 2.2. ([12, Ch. 4]) *Let $p \in \mathbb{P}$ and let G and G_i ($i \in I$) be abelian topological groups. Then:*

(a) $td_p(G)$ is a subgroup of G;

(b) $td_p(H) = H \cap td_p(G)$ for each subgroup H of G;

(c) $td_p(\Pi_{i\in I} G_i) = \Pi_{i\in I} td_p(G_i)$;

(d) if G is compact then:

- *(d_1) the subgroup $td_p(G)$ of G has a natural structure of an abstract $\mathbb{Z}_p$-module defined by the isomorphism $td_p(G) \cong Hom(\widehat{G}, \mathbb{Z}(p^\infty))$;*
- *(d_2)* (local criterion for total and essential density [12, Theorem 4.3.7]) *a subgroup $H \le G$ is totally (essentially) dense in G if and only if the the subgroup $td_p(H)$ is totally (essentially) dense in $td_p(G)$ for every prime p; and*
- *(d_3) if G is totally disconnected, then for every $p \in \mathbb{P}$ the subgroup $td_p(G)$ of G is closed and G is topologically isomorphic to $\Pi_{p\in\mathbb{P}} td_p(G)$.*

For a proof of Theorem 2.2(d_3), see [12, 3.5.9, 4.1.3]. See also [13](25.22) and [14](8.8) for other treatments of this phenomenon.

The following simple fact about pushing total density along surjective homomorphisms has been noted independently by several authors. (Its counterpart concerning essential density fails [10].)

Lemma 2.3. *Let G_0 and G_1 be topological groups with a continuous, surjective homomorphism $\phi : G_0 \twoheadrightarrow G_1$. If H is a totally dense subgroup of G_0, then $\phi[H]$ is totally dense in G_1.*

Proof. It is enough to note that if N is a closed normal subgroup of G_1 then $\phi^{-1}(N)$ is a closed normal subgroup of G_0. ∎

Definition 2.4. *A continuous surjective homomorphism $f : G \twoheadrightarrow N$ is said to* preserve closed subgroups *(briefly, to be CSP) if it sends closed subgroups of G to closed subgroups of N.*

The exponential map $\mathbb{R} \twoheadrightarrow \mathbb{T}$ is not CSP. Every continuous open surjective homomorphism $f : G \twoheadrightarrow N$ with compact kernel is CSP (for $f(K) = f(K \cdot \ker f)$ for every subgroup K of G, and $K \cdot \ker f$ is closed if K is closed and $\ker f$ is compact).

Lemma 2.5. *Let $f : G \twoheadrightarrow N$ be a continuous surjective CSP homomorphism.*

(I) If H is a subgroup of G such that $\overline{H}$ contains $\ker f$ and $f[H]$ is dense in N, then H is dense in G.

(II) If H is an essential (totally dense) subgroup of in N, then the subgroup $H_1 = f^{-1}(H)$ of G has the same property.

Proof. (I) The group $L = \overline{H}$ is a closed subgroup containing $\ker f$, and $f[L]$ is closed N. Since $f[L]$ contains the dense subgroup $f[H]$, we conclude $f[L] = N$. Now from $\ker f \subseteq L$ we have $G = f^{-1}(N) = f^{-1}(f[L]) = L \cdot \ker f = L$ (see also [12](Lemma 6.1.10)).

(II) Suppose first that the subgroup H is essential, and let K be a nontrivial closed normal subgroup G. Then $f[K]$ is a closed normal subgroup of N. If $f[K] = 1$, then $K \subseteq \ker f \subseteq H_1$ and surely $H_1 \cap K \neq \{1\}$. If $f[K] \neq 1$, then there exists $y \in K \cap H_1$ such that $1 \neq x := f(y) \in f[K] \cap H$. Clearly $y \neq 1$.

Suppose next that H is totally dense in N and that K is a closed normal subgroup of G. Then $f[K] \cap H$ is dense in $f[K]$ so $K \cap H_1$ is dense in K by (I) (with $K, f|K, f[K]$ replacing (G, f, N)). This proves that the subgroup $H_1 = f^{-1}(H)$ of G is totally dense whenever the subgroup H of N has the same property. □

3. When Total Density and Essentiality Coincide

Total density implies essential density. In order to investigate those groups G for which also $\mathcal{ED}(G) \subseteq \mathcal{TD}(G)$ (that is, the groups $G \in \mathcal{C}$) we isolate also the following two subclasses of $\mathcal{G}$:

$\mathcal{D} := \{G \in \mathcal{G}\colon$ every dense subgroup of G is totally dense$\}$, and

$\mathcal{S} := \{G \in \mathcal{G}\colon soc(G)$ is totally dense in $G\}$.

Clearly $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{S} \subseteq \mathcal{C}$, and $\mathcal{D}$ contains all finite groups. Moreover, $\mathbb{Z}_p \in \mathcal{D}$ for every prime p. Indeed, it suffices to note that every non-trivial closed subgroup of $\mathbb{Z}_p$ is open, so every dense subgroup of $\mathbb{Z}_p$ is totally dense. According to the following theorem announced by T. Soundararajan [19] (for a proof see [12, Exer. 5.5.6]), these are the only infinite compact abelian groups in $\mathcal{D}$.

Theorem 3.1. *An infinite compact abelian group G belongs to $\mathcal{D}$ if and only if there exists $p \in \mathbb{P}$ such that $G \cong \mathbb{Z}_p$.*

Lemma 3.2. *Let G be a compact abelian group. Then $G \in \mathcal{S}$ if and only if*

$$G = \prod_{p\in A} \mathbb{Z}(p)^{\alpha_p} \qquad (1)$$

for cardinals $\alpha_p > 0$ and $A \subseteq \mathbb{P}$.

Proof. Obviously, a group G as in (1) has a totally dense socle. For the converse, let us prove first that every compact $G \in \mathcal{S}$ is totally disconnected. Assuming the contrary, there is a continuous homomorphism $f : G \twoheadrightarrow \mathbb{T}$, and since $soc(G)$ is totally dense in G we have from Lemma 2.3 that $f[soc(G)] \subseteq soc(\mathbb{T})$ is totally dense in $\mathbb{T}$. Then $soc(\mathbb{T})$ is totally dense in $\mathbb{T}$, a contradiction. This proves that G is totally disconnected. By Theorem 2.2(c) G has the form $G = \prod_p td_p(G)$. Since $td_p(G)$ is a continuous homomorphic image of G we have $td_p(G) \in \mathcal{S}$ as well, so $soc(td_p(G))$ is totally dense in $td_p(G)$. On the other hand, $soc(td_p(G))$ is closed in G, so $td_p(G) = soc(td_p(G)) = G[p] \cong \mathbb{Z}(p)^{\alpha_p}$. This proves (1). □

Let us note that a group as in (1) is torsion if and only if the "support set" A is finite. These groups will play in important role in the further description of $\mathcal{C}$.

Lemma 3.3. *Let $f : G \to N$ be a continuous surjective homomorphism that preserves closed subgroups. If $G \in \mathcal{C}$ then $N \in \mathcal{C}$.*

Proof. According to Lemma 2.5, the group $H_1 = f^{-1}(H)$ is essentially dense in G whenever H is essentially dense in N. Now our hypothesis yields that H_1 is totally dense in G, so $H = f[H_1]$ is totally dense in N by Lemma 2.3. □

Corollary 3.4. *Let $G \in \mathcal{C}$.*

(a) If $G = K \times N$ then $K, N \in \mathcal{C}$.

(b) If H is a compact, normal subgroup of G then $G/H \in \mathcal{C}$.

Proof. (a) If $H \in \mathcal{ED}(K)$, then $H \times N \in \mathcal{ED}(G) = \mathcal{TD}(G)$. Then Lemma 2.3 applied to the canonical homomorphism $\phi : K \times N \twoheadrightarrow K$ shows that $H = \phi[H \times N]$ is totally dense in K. This shows that $K \in \mathcal{C}$. Similarly, $N \in \mathcal{C}$.

(b) In this case the map $\phi : G \twoheadrightarrow G/H$ preserves closed subgroups, so Lemma 3.3 applies. □

3.1. Compact Abelian Groups where Essential Density Coincides with Total Density

Lemma 3.5. *Let $G = K \times L$, where $K \in \mathcal{D}$ and $L \in \mathcal{S}$ is torsion. Then $G \in \mathcal{C}$.*

Proof. For $H \in \mathcal{ED}(G)$ we have $L = soc(L) \subseteq soc(G) \subseteq H$, so $H = H_1 \times L$ for some subgroup $H_1 \subseteq K$. Since H is dense in G we see that H_1 is dense in $K \in \mathcal{D}$, so H_1 is totally dense in K. Then with $\phi : G \twoheadrightarrow K$ the usual projection, it follows from Lemma 2.5 that $H = \phi^{-1}(H_1)$ is totally dense in G. □

Now we prove that the groups described in Lemma 3.5, together with the groups of $\mathcal{S}$, are exactly the compact abelian groups in the class $\mathcal{C}$.

Theorem 3.6. *A compact abelian group $G \in \mathcal{C}$ if and only if one of the following three conditions holds:*

(1) $G \in \mathcal{S}$ (i.e., $G = \prod_{p \in A} \mathbb{Z}(p)^{\alpha_p}$);
(2) $G \cong F \times L$, where F is finite and $L \in \mathcal{S}$ is torsion;
(3) $G \cong \mathbb{Z}_q \times L$, where $L \in \mathcal{S}$ is torsion.

Proof. The sufficiency follows from the inclusion $\mathcal{S} \subseteq \mathcal{C}$ and Lemma 3.5.

We split the proof of the necessity into several steps.

(a) First, arguing as in the proof of Lemma 3.2, we note that every compact abelian group in $G \in \mathcal{C}$ is totally disconnected. For otherwise there is a continuous homomorphism from G onto $\mathbb{T}$, so $\mathbb{T} \in \mathcal{C}$ by Lemma 3.3. This is ridiculous, since $soc(\mathbb{T}) \in \mathcal{ED}(\mathbb{T}) \setminus \mathcal{TD}(\mathbb{T})$.

We assume from now on that $G \in \mathcal{C}$ is totally disconnected. Then $G = \prod_p G_p$, where $G_p = td_p(G) \in \mathcal{C}$ by Lemma 3.3. The following observations will help us to describe the subgroups G_p.

(b) $\mathbb{Z}_p \times \mathbb{Z}(q^2) \notin \mathcal{C}$ for every $q \in \mathbb{P}$ (both $p = q$ and $p \neq q$ are permitted). Let $\xi \in \mathbb{Z}_p$ be an irrational p-adic integer. Denote by c the generator of $\mathbb{Z}(q^2)$. Then the dense subgroup $H = \langle (0, qc), (1, 0), (\xi, c) \rangle$ of $\mathbb{Z}_p \times \mathbb{Z}(q^2)$ is essential but not totally dense (since it does not contain the element $(0, c)$).

(c) $\mathbb{Z}_p \times \mathbb{Z}_q \notin \mathcal{C}$ (both $p = q$ and $p \neq q$ are permitted). This follows directly from (b) and Corollary 3.4, since $\mathbb{Z}_p \times \mathbb{Z}(q^2)$ is a quotient of $\mathbb{Z}_p \times \mathbb{Z}_q$.

(d) $\mathbb{Z}(p^2)^\omega \notin \mathcal{C}$: The dense subgroup $H = \mathbb{Z}(p^2)^{(\omega)} + \mathbb{Z}(p)^\omega$ of $\mathbb{Z}(p^2)^\omega$ is essential but not totally dense.

Claim 1. *Let A be an infinite set of primes and let $\alpha_q > 0$ be cardinals for $q \in A$. Then*

(i) $\mathbb{Z}(p^2) \times \prod_{q \in A} \mathbb{Z}(q)^{\alpha_q} \notin \mathcal{C}$ for any $p \in \mathbb{P}$;

(ii) $\mathbb{Z}_p \times \prod_{q\in A} \mathbb{Z}(q)^{\alpha_q} \notin \mathcal{C}$ *for any* $p \in \mathbb{P}$*; and*

(iii) $\prod_{q\in A} \mathbb{Z}(q^2)^{\alpha_q} \notin \mathcal{C}$.

Proof of Claim 1. (i) According to Corollary 3.4 we can replace the group $\prod_{q\in A} \mathbb{Z}(q)^{\alpha_q}$ by the group $L = \prod_{q\in A} \mathbb{Z}(q)$. Denote by c the generator of $\mathbb{Z}(p^2)$ and let $b = (b_q) \in G$ be a topological generator of L. Then the dense subgroup

$$H = \langle (pc, 0)\rangle + \langle (c, b)\rangle + \{0\} \times soc(L)$$

of $\mathbb{Z}(p^2) \times L$ is essential but not totally dense. (To see that H is not totally dense note that $(c, 0) \notin H$. Density of H in $\mathbb{Z}(p^2) \times L$ follows from the fact that the closure of H properly contains the maximal proper subgroup $\langle pc\rangle \times L$ of $\mathbb{Z}(p^2) \times L$.)

(ii) In the notation of (i) replace $\prod_{q\in A} \mathbb{Z}(q)^{\alpha_q}$ by $L = \prod_{q\in A} \mathbb{Z}(q)$ and let $H = \langle (p, 0)\rangle + \langle (1, b)\rangle + \{0\} \times soc(L)$. Then H is a dense essential subgroup of $\mathbb{Z}_p \times L$ that is not totally dense. (To see that H is dense in $\mathbb{Z}_p \times L$ note that its closure properly contains the maximal proper subgroup $p\mathbb{Z}_p \times L$ of $\mathbb{Z}_p \times L$.)

(iii) This follows from (i), using Corollary 3.4(b).

The proof of Claim 1 is complete.

Next, applying (b)-(d) and Claim 1, we complete the proof that G has the form (1), (2) or (3). First we prove

Claim 2. *For each* $p \in \mathbb{P}$ *the compact abelian group* $G_p = td_p(G)$ *is either of the form* $G_p = F_p \times \mathbb{Z}(p)^\alpha$*, with* F_p *a finite* p*-group, or of the form* $G_p = \mathbb{Z}_p \times \mathbb{Z}(p)^\alpha$.

Proof of Claim 2. Assume first that G_p is torsion. Then there exists n such that $p^n G_p = \{0\}$. By Corollary 3.4 and item (d) above, $\mathbb{Z}(p^2)^\omega$ cannot be a quotient of G_p. This yields $G = F_p \times \mathbb{Z}(p)^\alpha$ with F_p a finite p-group, since G_p must be a product of cyclic p-groups.

In case G_p is not torsion, we can find a closed subgroup N of G_p such that $N \cong \mathbb{Z}_p^\sigma$ and G_p/N is a product of cyclic p-groups.

To see this, let X be the Pontryagin dual of G_p. Since X is a p-torsion abelian group (cf. [12]), there exists a subgroup B (a p-basic subgroup) of X such that:

(I) B is a direct sum of cyclic p-groups, say $B = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)^{(\alpha_n)}$;

(II) B is a pure subgroup, i.e., $B \cap p^n X = p^n B$ for every $n \in \mathbb{N}$; and

(III) X/B is divisible, i. e., $X = B + p^n X$ for every $n \in \mathbb{N}$.

Let $X/B \cong \mathbb{Z}(p^\infty)^{(\sigma)}$; then with $N := A(B) \cong \widehat{X/B}$ we have $N \cong \mathbb{Z}_p^\sigma$, and $G_p/N \cong \widehat{B} \cong \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)^{\alpha_n}$ is a product of cyclic p-groups, as asserted. We note also that G_p/N is a bounded group. (For if not, there would exist a continuous surjective homomorphism $G_p/N \to \mathbb{Z}(p^2)^\omega$, which is impossible in view of (d) and Corollary 3.4(b)). Then, since $G_p/N \cong \widehat{B}$ is a bounded group, B itself is bounded—i.e., there exists n such that $p^n B = \{0\}$. Now (II) and (III) yield $X = B \oplus p^n X$, i.e., the subgroup B of X splits off. Then also $G_p \cong N \times t(G)$. Now (c) yields $\sigma \leq 1$. The case $\sigma = 0$ we already discussed above, so we assume now that $\sigma = 1$. By item (i) of Claim 1, no cyclic group in the product $t(G)$ may have order $> p$. Thus $G_p = \mathbb{Z}_p \times \mathbb{Z}(p)^\alpha$ and the proof of Claim 2 is complete.

To finish the proof of Theorem 3.6 we note now that for every prime p there exists, by Claim 2, a cardinal α_p such that either $G_p = \mathbb{Z}_p \times \mathbb{Z}(p)^{\alpha_p}$ or $G_p = F_p \times \mathbb{Z}(p)^{\alpha_p}$ for some finite p-group F_p that has no direct summands of order p. By item (c) at most one $\mathbb{Z}_p$ may appear in the product $\prod_p G_p$, hence we are left with two cases.

Case 1. G has no direct summand isomorphic to $\mathbb{Z}_p$. If $A := \{p \in P : \alpha_p > 0\}$ is infinite, then all F_p vanish by Claim 1 and hence $G = \prod_{p\in A} \mathbb{Z}(p)^{\alpha_p}$, i.e., (1) holds. If A is finite, then by Claim 1 only finitely many of the groups F_p are non-trivial. Hence $F := \prod_p F_p$ is finite and $G = F \times \mathbb{Z}(p)^{\alpha_p}$, so (2) holds.

Case 2. There exists $q \in \mathbb{P}$ such that $G_q = \mathbb{Z}_q \times \mathbb{Z}(q)^{\alpha_q}$. Now (b) yields $F_p = \{0\}$ for all $p \neq q$, and A is finite by item (ii) of Claim 1, so (3) holds. □

The next corollary gives a new description of the compact abelian groups in the subclass $\mathcal{S} \subseteq \mathcal{C}$. Note that (b) is the definition of the class $\mathcal{S}$, while the condition in (a) is stronger than the condition $G \in \mathcal{C}$ since, in general, an essential subgroup need not be dense.

Corollary 3.7. *For a compact abelian group G, the following statements are equivalent.*

a) every essential subgroup of G is totally dense;

b) the socle of G is totally dense;

c) $G \in \mathcal{S}$.

3.2. Other Groups in $\mathcal{C}$

The class $\mathcal{C}$ contains a wealth of non-abelian groups. For example, every topologically simple group belongs to $\mathcal{C}$. The next example extends this observation to simple Lie groups; as is well known (cf. [14]), these have finite center.

Example 3.8. Let L be a simple compact connected Lie group. Then $L \in \mathcal{C}$ if and only if $Z(L) = soc(Z(L))$.

Proof. Assume that $Z(L) = soc(Z(L))$ and let H be a dense essential subgroup of L. Then $H \supseteq soc(Z(L)) = Z(L)$. Every closed normal subgroup N of L satisfies either $N \subseteq Z(L) \subseteq H$ or $N = L$, so surely $N \cap H$ is dense in N.

Now assume that $Z(L) \neq soc(Z(L))$. Fix a free dense 2-generated subgroup F of L (this is possible by Kuranishi's theorem [15]), and set $H := soc(Z(L)) \cdot F$. That H is dense and essential is checked as above; and if $z \in Z(L) \setminus soc(Z(L))$, then $z \notin H$ since $H \cong soc(Z(L)) \times F$, so H is not totally dense. □

Theorem 3.9. *Let G be a connected compact group. Then $G \in \mathcal{C}$ if and only if G is a simple compact connected Lie group such that $Z(G)$ coincides with its socle.*

Proof. The sufficiency was proved above. Now assume that $G \in \mathcal{C}$. By item (a) of the proof of Theorem 3.6 and Corollary 3.4, G admits no quotients isomorphic to $\mathbb{T}$. Hence $G = G'$. The quotient $G/Z(G)$ is isomorphic to a product of simple compact connected groups L_i. Let us show next that a product $L = L_1 \times L_2$ of two such groups never belongs to $\mathcal{C}$. Indeed, again by Kuranishi's theorem there is a dense subgroup F of L freely generated by two elements, say $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Choose an element $h_2 \in L_2$ such that the subgroup $\langle a_2, b_2, h_2 \rangle$ of L_2 is free, and take any non-trivial element $h_1 \in L_1$. Then the subgroup H of L generated by $a, b, (h_1, 1)$ and $(1, h_2)$ is dense and essential in L, but not totally dense since $H \cap (L_1 \times \{1\}) = \langle (h_1, 1) \rangle$ cannot be dense in L_1. □

The class $\mathcal{C}$ contains many non-compact abelian groups—for example, all discrete groups. This suggests the following quite general question.

Question 3.10. Describe the compact groups in $\mathcal{C}$ and the LCA groups in $\mathcal{C}$.

We are far from a general answer to Question 3.10. Indeed, we have not characterized the members of the subclass $\mathcal{D} \subseteq \mathcal{C}$. The following is a partial response to Question 3.10.

Proposition 3.11. *Every locally compact abelian group $G \in \mathcal{C}$ is totally disconnected.*

Proof. The group $\mathbb{R}$ has no proper totally dense subgroups, while many proper essentially dense subgroups are available. Hence $\mathbb{R} \notin \mathcal{C}$. Thus G does not admit a direct summand $\cong \mathbb{R}$ by Lemma 3.4. By the structure theory of LCA groups ([13]),

G contains a compact open subgroup K. Since the connected component $c(G)$ of G is obviously contained in K, it follows that $c(G)$ is compact. Suppose that G is not totally disconnected, i.e., $c(G) \neq \{0\}$. Then there exists a surjective continuous homomorphism $f : c(G) \to \mathbb{T}$ that can be extended to a continuous (open) surjective homomorphism $f_1 : K \to \mathbb{T}$ and then to a homomorphism $f_2 : G \to \mathbb{T}$ (which is necessarily continuous and open since K is open in G). Note that $N_2 = \ker f_2$ intersects K in $N_1 = \ker f_1$ – a compact group. Now consider the homomorphism $g = \langle f_2, h\rangle : G \to \mathbb{T} \times G/K$, where $h : G \to G/K$ is the canonical homomorphism to the (discrete) quotient group G/K. Then $\ker g = N_2 \cap K = N_1$ is compact and $g : G \twoheadrightarrow g[G]$ is open, so g is a CSP homomorphism and from Lemma 3.4 we have $g[G] \in \mathcal{C}$. Since $\mathbb{T} \times \{0\} = g[K]$ is an open divisible subgroup of $g[G]$, we have $g(G) \cong \mathbb{T} \times D$, where D is a discrete subgroup of $g[G]$, so again by Lemma 3.4 we conclude $\mathbb{T} \in \mathcal{C}$, a contradiction. This proves that G is totally disconnected. □

Remark 3.12. (a) By what we have proved, a locally compact abelian group $G \in \mathcal{C}$ has an open compact totally disconnected subgroup K. We do not know in this context whether of necessity $K \in \mathcal{C}$. Suppose that this is the case, so that K has the form (1), (2) or (3) as in Theorem 3.6. If $G \neq B(G) = \bigcup\{L : L \subseteq G \text{ compact}\}$ then no proper subgroup of G can be totally dense, while (at least in some cases) K itself may still have proper dense essential subgroups. For example, when $G = K \times \mathbb{Z}$, then $G \in \mathcal{C}$ if and only if K has the form (2). (For then, $soc(K)$ is open in K, hence open in G as well; therefore, every essential subgroup H of G must be open and consequently closed. Thus G has no proper essentially dense subgroups.) On the other hand, if K has form (1) or (3) then K admits a proper essentially dense subgroup H. Fix $z \in K \setminus H$ and set $H_1 := H + \langle(z, 1)\rangle \subseteq G$. Then H_1 is a proper essentially dense subgroup of G, so $G \notin \mathcal{C}$ (contrary to assumption).

In case $G = B(G)$, we conjecture that $G \in \mathcal{C}$ if and only if $K \in \mathcal{C}$. For example, $\mathbb{Q}_p \in \mathcal{C}$ as every non-trivial closed subgroup here is open (so that total density coincides now with density).

(b) According to Theorem 3.6 the compact abelian groups in $\mathcal{C}$ are profinite. It will be a challenging problem to describe the profinite groups in $\mathcal{C}$ in the non-abelian case (as a part of Question 3.10).

4. The Totally Dense Nucleus

Let us say that a subgroup K_t of a group K is the *totally dense nucleus* of K if K_t is (with respect to inclusion) the smallest totally dense subgroup of K. Clearly for general K, no such subgroup K_t exists. When a totally dense nucleus K_t does exist, the computation of the cardinal number $TD(K)$ becomes extremely simple:

$TD(K) = TD(G) = |K_t|$ for every totally dense subgroup G of K.

Let us discuss the existence of the totally dense nucleus K_t for K abelian. Clearly, K_t contains the torsion subgroup $t(K)$. Therefore it is obvious that $K_t = t(K)$ precisely when $t(K)$ is totally dense in K. It seems tempting to conjecture that the totally dense nucleus exists only in this circumstance, but the group $K = \mathbb{R}$ (with $K_t = K$, as $\mathbb{R}$ has no proper totally dense subgroups) shows that this may fail strongly even for such nice metrizable LCA groups as $\mathbb{R}$. This motivates us to restrict attention to the class of compact abelian groups K with totally dense nucleus K_t. We show that these are exactly the compact abelian groups K for which $t(K)$ is totally dense in K. These are the groups commonly known as *exotic tori*; their structure has been described in detail in [11].

Theorem 4.1. *Let K be a compact abelian group. Then the following conditions are equivalent.*

(a) K admits a totally dense nucleus;

(b) the intersection of all totally dense subgroups of K is totally dense in K;

(c) the poset of all totally dense subgroups of K is a (complete) lattice;

(d) the intersection of any two totally dense subgroups of K is essentially dense in K;

(e) K is an exotic torus, i.e., $t(K)$ is totally dense in K;

(f) the subgroup $soc(K)$ is essential in K;

(g) K contains copies of $\mathbb{Z}_p$ for no prime p;

(h) every closed non-trivial subgroup of K contains a minimal closed non-trivial subgroup of K;

(i) $n = \dim K$ is finite and K admits a short exact sequence of continuous homomorphisms

$$0 \longrightarrow \Pi_{p\in\mathbb{P}} B_p \longrightarrow K \longrightarrow \mathbb{T}^n \longrightarrow 0, \tag{1}$$

where each B_p is a compact abelian p-group;

(j) $n = \dim K$ is finite and $ker f \cong \Pi_{p\in\mathbb{P}} B_p$ for every continuous surjective homomorphism $f : K \to \mathbb{T}^n$, where each B_p is a compact abelian p-group.

Proof. The implications (e) $\Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (f) According to Theorem 2.2(d_2) it suffices to prove that each $td_p(K)$ is torsion. Suppose for some $p \in \mathbb{P}$ that $td_p(K)$ is not torsion. Since this subgroup carries a natural structure of an abstract $\mathbb{Z}_p$-module (Theorem 2.2(d$_1$)), one can fix a free $\mathbb{Z}_p$-submodule $F \cong \oplus_\rho \mathbb{Z}_p$ (where ρ is the $\mathbb{Z}_p$-rank of $td_p(K)$). Then the submodule $A_p = F \oplus t_p(K)$ is essential in $td_p(K)$, i.e., it meets non-trivially every closed subgroup of $td_p(K)$. From the assumption $F \neq \{0\}$, there is a non-trivial direct summand $N \cong \mathbb{Z}_p$ of F. Then $F = N \oplus L$ for some $\mathbb{Z}_p$-submodule L. Now pick two dense "disjoint" subgroups C_1 and C_2 of N, and set $H_i = C_i + L + t_p(K)$. Let us see that H_i is essential in $td_p(K)$ for $i = 1, 2$. Since essentiality is transitive, it suffices to see that H_i is essential in A_p. For this it suffices to check that whenever $M \cong \mathbb{Z}_p$ is an infinite closed monothetic subgroup of A_p, then $M \cap H_i \neq 0$. Let $z = (x, y) \in (N \times L) \cap M$ be non-zero (such an element exists since $M \cong \mathbb{Z}_p$). If $x = 0$, then $z \in H_i$ and we are done, so we assume $x \neq 0$. Then there exists $\xi \neq 0$ in $\mathbb{Z}_p$ such that $0 \neq \xi x \in C_i$. Then $(\xi x, \xi y) \in M \cap (C_i \times L)$. According to [2, Lemma 5.3] the subgroup (*saturation* of H_i) $R_i = \{x \in td_p(K) : n \cdot x \in H_i \text{ for some } n > 0\}$ is totally dense in $td_p(K)$ for $i=1, 2$ since the group $td_p(K)$ is covered by compact subgroups. Then the subgroups $S_i = (\bigoplus_{q \neq p} td_q(K)) \oplus R_i$, $i = 1, 2$, are totally dense in K by the local criterion of total density (Theorem 2.2(d$_2$)), so by assumption (d) the intersection $I = S_1 \cap S_2$ must be essentially dense in K. Again by the local criterion of total density (Theorem 2.2(d$_2$)), this entails that $td_p(I)$ is essential in $td_p(K)$. We show now that $td_p(I)$ cannot be essential in $td_p(K)$. Indeed, if $x \in I \cap N$, then there exists a non-zero $k \in \mathbb{N}$ such that $k \cdot x \in (C_1 + L) \cap (C_2 + L)$. Then there exist $g_i \in C_i$, $l_1, l_2 \in L$ such that $k \cdot x = g_1 + l_1 = g_2 + l_2$. Then $g_1 - g_2 = l_2 - l_1 \in N \cap L = \{0\}$, so $g_1 = g_2 \in C_1 \cap C_2 = \{0\}$. Hence $g_1 = g_2 = 0$ and consequently $k \cdot x \in L$. Since obviously $k \cdot x \in N$, we conclude that $k \cdot x = 0$ since $N \cap L = \{0\}$. Since $N \cong \mathbb{Z}_p$ is torsion-free, this yields $x = 0$. Therefore $N \cap I = \{0\}$, so that I cannot be essential in K. This contradiction completes the proof that (d) $\Rightarrow$ (f).

The equivalence of (e), (f), (g), (h), (i) and (j) is proved in [11], but the following remarks may be helpful. First, the implications

$$\text{(j)} \Rightarrow \text{(i)} \Rightarrow \text{(g) and (f)} \Rightarrow \text{(h)} \Rightarrow \text{(g)}$$

are routine. Next, note that property (i) is hereditary under taking closed subgroups, and that $t(K)$ is dense in K whenever (i) holds. Therefore, (i) implies (e). The last implication (g) $\Rightarrow$ (j) requires more effort. The argument given in [11] exploits the fact that (assuming (g)) the Pontryagin dual $X = \widehat{K}$ of K has no divisible quotients. From this one concludes that $n = r(X) < \infty$ and for every monomorphism $h : \mathbb{Z}^n \to X$ the cokernel $X/h(\mathbb{Z}^n)$ is isomorphic to $\bigoplus_{p \in \mathbb{P}} L_p$, where each L_p is a bounded p-group. This is the dual of the assertion in (j). □

Remark 4.2. (a) We say that a topological group G is *minimal* if its topology is minimal among all Hausdorff group topologies ([20]); and G is *totally minimal* if G/N is minimal for each closed, normal subgroup N of G ([10]). The relation between total minimality and total density is clarified by the following criterion due to Dikranjan and Prodanov [10]: *a dense subgroup H of a topological group G is totally minimal if and only if G is totally minimal and H is totally dense in G.* According to this fact, one can add to the equivalent conditions of Theorem 4.1 also the following one: *the poset of all dense totally minimal subgroups of K is a (complete) lattice.*

(b) Item (h) of the theorem gives a property of the lattice of *closed* subgroups of the group K (namely, it has sufficiently many atoms).

(c) When a compact abelian group K admits a totally dense nucleus K_t, i.e., when $t(K)$ is totally dense in K, its structure may be computed as follows. Algebraically

$$t(K) \cong (\mathbb{Q}/\mathbb{Z})^n \times (\oplus_p R_p),$$

where R_p is an appropriate finite-index subgroup of B_p for each p. If K is connected with $\dim K = n$ then $r_p(B_p) \leq n$ and B_p is finite for each p. Moreover in this case all the groups R_p vanish, i.e., $t(K) \cong (\mathbb{Q}/\mathbb{Z})^n$ for an n-dimensional connected exotic torus. This is in harmony with $t(\mathbb{T}^n) \cong (\mathbb{Q}/\mathbb{Z})^n$ for the usual n-dimensional torus. The motivation for introducing the exotic tori in [11] was the fact that they are precisely the completions of the torsion minimal abelian groups. For more details see [11] or [12].

(d) The compact connected groups K in which $t(K)$ is totally dense were described in [8]; these are the groups K such that $Z(K)$ is an exotic torus.

5. The Essentially Dense Nucleus

In parallel with the preceding discussion, let us say that a subgroup K_e of a group K is the *essentially dense nucleus* if K_e is (with respect to inclusion) the smallest essentially dense subgroup of K. As before we have $ED(K) = ED(G) = |K_e|$ for every essentially dense subgroup G of K. Clearly for general K, no such subgroup K_e exists. In the case of a compact abelian group K the existence of K_e implies that K is an exotic torus (see Theorem 5.1).

As in the case of the totally dense nucleus, K_e (when it exists) contains the socle $soc(K)$ when K is abelian. Thus it is obvious that $K_e = soc(K)$ precisely when $soc(K)$ is essentially dense in K. It is tempting to conjecture that the essentially dense nucleus exists only in this circumstance. Even in well-behaved cases, however, for example when K is a connected exotic torus, the socle $soc(K)$ may fail to be dense in K [12](Exercise 4.5(1)). The condition that a group K admits an essentially dense nucleus K_e, then, is more restrictive than the existence of the totally dense nucleus (see Theorem 5.1).

When K is a connected exotic torus such that K_e exists, however, $soc(K)$ is essentially dense. To see this, choose algebraically independent topological generators x and y of K and set $H_x := \langle x \rangle + soc(K)$ and $H_y := \langle y \rangle + soc(K)$. Then $soc(K) = H_x \cap H_y$ is essentially dense, as asserted.

In the case when K is not connected, however, $soc(K)$ may fail to be dense in K. Indeed, it may happen that $soc(K)$ is *closed* in K, so that the existence of K_e leads to the existence of a smallest dense subgroup of the quotient group $K/soc(K)$. By one of the principal theorems of [4] a compact abelian group has a smallest dense subgroup if and only if it is finite. This proves that when $soc(K)$ is closed (e.g., when K itself is torsion), then the existence of K_e need not yield density of $soc(K)$. In this case $K_e = K$ (i.e., K has no proper essentially dense subgroups). In the following theorem we summarize and extend these remarks.

Theorem 5.1. *Let K be a compact abelian group. Then the following are equivalent:*

(a) K admits an essentially dense nucleus K_e;

(b) the poset of all essentially dense subgroups of K is a complete lattice;

(c) the poset of all essentially dense subgroups of K is a lattice;

(d) K is an exotic torus (i.e., the poset of totally dense subgroups of K is a complete lattice) and one of the following conditions is fulfilled:

(d_1) $soc(K)$ is dense in K;

(d_2) $soc(K)$ is an open subgroup of K.

Moreover: if these conditions hold then $K_e = soc(K)$ when (d_1) holds, and $K_e = K$ when (d_2) holds.

Proof. Obviously (a) and (b) are equivalent, and (b) $\Rightarrow$ (c).

Assume that (c) holds. Since essential density is preserved by passing to bigger subgroups, (c) yields that the intersection of two essentially dense subgroups of K is essentially dense. In particular, (d) of Theorem 4.1 is fulfilled. Hence K is an exotic torus, so $soc(K)$ is essential in K. Let $L = \overline{soc(K)}$. We prove first that the closed subgroup L of K is open (and hence has a finite index). Indeed, assume that the quotient group K/L is infinite and let $f : K \to K/L$ be the quotient map. By [4] there exist dense subgroups D_1, D_2 of K/L such that $F = D_1 \cap D_2$ is a finite subgroup of K/L. Then $B_i = f^{-1}(D_i)$ $(i = 1, 2)$ are dense subgroups of K containing $soc(K)$. Thus both B_1 and B_2 are essentially dense. Now (c) yields that also $A := f^{-1}(F) = B_1 \cap B_2$ is essentially dense. On the other hand, the subgroup A obviously contains the closed subgroup L as a closed subgroup of finite index. Hence A is closed in K as a finite union of cosets of L. This yields $A = K$, and hence $L = K$, a contradiction. We have proved in this way that the subgroup L is open. Now we consider two cases depending on whether or not L is torsion.

If L is torsion then it is bounded torsion, so that $soc(K)$ is bounded torsion too. Then $soc(K)$ has finitely many non-trivial primary components $G[p_i]$ $(i = 1, \ldots, n)$, so it is a closed subgroup of K (as the set of solutions of the equation $(p_1 \cdot \ldots \cdot p_n) \cdot x = 0$ in K). Thus $L = soc(K)$ when L is torsion, $soc(K)$ is open in K, and (d_2) is verified. Hence to complete the proof of the implication (c) $\Rightarrow$ (d) it remains only to show that if L is not torsion, then $soc(K)$ is dense in K. Now $r(L) \geq 2^\omega$, since L is a non-torsion compact abelian group. We have already proved that K/L is finite, so there are $g_1, \ldots, g_n \in G$ such that $K = \langle g_1, \ldots, g_n\rangle + L$. Pick independent $x_1, \ldots x_n \in L$ such that $\langle x_1, \ldots, x_n\rangle \cap \langle g_1, \ldots, g_n\rangle = 0$ (this is possible since $r(L) \geq 2^\omega$), and set

$$H_1 = \langle g_1, \ldots, g_n\rangle + soc(K) \text{ and } H_2 = \langle g_1 + x_1, \ldots, g_n + x_n\rangle + soc(K).$$

From $\overline{H_i} \supseteq L$, $\overline{H_i} \supseteq \langle g_1, \ldots, g_n\rangle$ and $H_i \supseteq soc(K)$ we infer that

H_i is essentially dense in K for $i = 1, 2$. Then (c) yields that $H_1 \cap H_2$ is essentially dense in K. We prove now that $H_1 \cap H_2 = soc(K)$, so that (d_1) holds. Let $z \in H_1 \cap H_2$. Then there exist $k_1, \ldots, k_n \in \mathbb{Z}$ and $m_1, \ldots, m_n \in \mathbb{Z}$ and $t_1, t_2 \in soc(K)$ such that

$$z = \sum_{i=1}^{n} m_i g_i + t_1 = \sum_{i=1}^{n} k_i (g_i + x_i) + t_2.$$

After multiplication by an appropriate non-zero $m \in \mathbb{Z}$ we eliminate t_1, t_2 to get

$$m \sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} l_i g_i \in \langle x_1, \ldots, x_n \rangle \cap \langle g_1, \ldots, g_n \rangle = \{0\}.$$

Now $m \sum_{i=1}^{n} k_i x_i = 0$ along with $m \neq 0$ imply $k_1 = \ldots = k_n = 0$ by the choice of $x_1, \ldots x_n$, so $z = t_2 \in soc(K)$, as required.

If (d) holds with (d_1), then obviously $K_e = soc(K)$ is the essentially dense nucleus of K. If (d) holds with (d_2), then for every essentially dense subgroup H of K one has $soc(K) \subseteq H$, so H is open (hence closed) in K and $H = K$. Thus under (d_2) K has no proper essentially dense subgroups and hence $K_e = K$. This verifies the final assertion of the theorem, and proves also the implication (d) $\Rightarrow$ (a). □

Remark 5.2. Note that in the case (d_2) the compact group K is torsion with clopen subgroup $soc(K)$ of (necessarily) finite index. More generally, with K an exotic torus the subgroup $soc(K)$ of K is closed if and only if it is (bounded) torsion. In this case $soc(K)$ need not be open.

Corollary 5.3. *Let K be a compact abelian group that is not totally disconnected. Then K admits an essentially dense nucleus K_e if and only if K admits a totally dense nucleus K_t and $soc(K)$ is dense in K.*

We close the section with this general question.

Question 5.4. Which topological groups K admit a totally dense nucleus K_t [resp., an essentially dense nucleus K_e]?

Remark 5.5. Unlike the (connected) exotic tori, the non-abelian compact Lie groups need not admit a totally dense nucleus. Indeed, if L is a simple compact Lie group, then every dense subgroup of L is totally dense; so the choice of any pair of dense subgroups of L whose intersection is not dense then witnesses the failure of condition (b) of Theorem 4.1.

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