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ON A CONJECTURE OF CHOU AND APPLICATIONS

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Abstract

Let S be an infinite, discrete, cancellative semigroup and let βS be the Stone-Čech compactification of S. Then βS is a semigroup with an operation which extends that of S and which is continuous only in one variable. We start with a subset V of S such that $sV \cap tV$ is finite for all $s, t \in S$, $s \neq t$. We show that if x_1, $x_2 \in \overline{V}$, $\|x_1\| = \|x_2\|$ and $x_1 \neq x_2$ then $\beta S x_1 \cap \beta S x_2 = \emptyset$. This result is a partial positive answer to a conjecture given by C. Chou about thirty years ago, and it implies the known result that the number of these ideals is $2^{2^{|S|}}$. We also show that any point in $\overline{V}$ is right cancellative in βS. This result improves our earlier result where V was countable. With these two theorems, we deduce that the dimension of any non-zero right ideal of $\ell_\infty(S)^*$ when equipped with the first Arens product is $2^{2^{|S|}}$. This latter result was known only for the radical of $\ell_\infty(S)^*$ when S is an amenable group.

1. Introduction and Notations

Let S be an infinite discrete semigroup written multiplicatively, $\ell_\infty(S)$ be the space of bounded complex-valued functions on S

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with the supremum norm and $\ell_\infty(S)^*$ be its Banach dual. As is well-known, $\ell_\infty(S)^*$ may be equipped with an Arens product to become a Banach algebra. Recall that this product is defined as follows. For all $f \in \ell_\infty(S)$, $s \in S$ and $\mu \in \ell_\infty(S)^*$, let f_s and f_μ be defined on S by

$$f_s(t) = f(st) \quad \text{and} \quad f_\mu(t) = \mu(f_t).$$

Then the first Arens product of μ and ν in $\ell_\infty(S)^*$ is given by

$$(\mu\,\nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in \ell_\infty(S).$$

We may regard the Stone-Čech compactification βS of S as the spectrum of the $\ell_\infty(S)$, i.e.,

$$\beta S = \{x \in \ell_\infty(S)^* : x \neq 0 \text{ and } x(fg) = x(f)x(g) \text{ for all } f, g \in \ell_\infty(S)\}$$

equipped with the relative weak*-topology inherited from $\ell_\infty(S)^*$. βS also inherits the Arens product from $\ell_\infty(S)^*$, and so it becomes a compact right topological semigroup (which is also written multiplicatively), i.e., the mappings

$$x \mapsto x\,y \;:\; \beta S \to \beta S$$

are continuous for each $y \in \beta S$, and with the additional property that the mappings

$$x \mapsto s\,x \;:\; \beta S \to \beta S$$

are continuous for each $s \in S$. The closure in βS of a subset A of βS is denoted by $\overline{A}$. The set

$$\{\overline{A} \;:\; A \subset S\}$$

forms a basis for the topology on βS. For $A \subseteq S$, $|A|$ denotes the cardinality of A, and for $x \in \beta S$, we let the *norm* of x be

$$\|x\| = \min\{|A| \;:\; x \in \overline{A}\}.$$

Note that $\|sx\| \leq \|x\|$ for all $s \in S$ and $x \in \beta S$. For these notions of norms of elements in βS and much more on βS, the reader is directed to [7].

Following [1], we say that a subset V of S is *thin* if it is infinite and $|sV \cap tV| < \omega$ whenever $s \neq t$ in S. An example of such a set is $V = \{2^n : n \in \mathbb{N}\}$ in $(\mathbb{N}, +)$. Many other examples of thin sets were given and used by many authors, but we are concerned with the one constructed by transfinite induction in [1, Proposition 4.1]. This construction shows in fact that every infinite subset (instead of subsemigroup) A of a group contains a thin set V with $|V| = |A|$ (just start the proof of the proposition with an infinite subset instead of a subsemigroup). C. Chou went on to conjecture that if V is a thin set in a cancellative semigroup and if $x_1, x_2 \in \overline{V}$ with $x_1 \neq x_2$ then $\beta S x_1 \cap \beta S x_2 = \emptyset$. In [1, Proposition 5.1], he proved the conjecture only when $\|x_1\| = \|x_2\| = \omega$. In Theorem 1 of this paper, we show that Chou's conjecture is true independently of the norms of x_1 and x_2 but we keep $\|x_1\| = \|x_2\|$. So this gives a partial positive answer. The second theorem in this section shows that every $x \in \overline{V}$ is right cancellative in βS, i.e., $yx \neq zx$ whenever $y \neq z$ in βS. In our earlier papers [3] and [4], these points were produced only when V is countable.

In Section 3, we apply these two theorems to study $\ell_\infty(S)^*$. Recall that a *mean* is a positive element of norm 1 in $\ell_\infty(S)^*$. A mean μ is *left invariant* if $\mu(f_s) = \mu(f)$ for all $f \in \ell_\infty(S)$ and $s \in S$. When a left invariant mean exists in $\ell_\infty(S)^*$, we say that S is *amenable*. We see first that there are $2^{2^{|S|}}$ left invariant means in $\ell_\infty(S)^*$ when S is an amenable cancellative semigroup. This is a well known result when $S = G$ is an amenable group, it was proved with a different method by Chou ([2]) for G discrete, and then by Lau and Paterson ([8]) in $L_\infty(G)^*$ for any locally compact group. See also [9]. As noted by most of these authors, this implies that the dimension of the radical of the Banach algebra $\ell_\infty(S)^*$ is $2^{2^{|S|}}$. The results of Section 2 combined with the techniques used recently in [5] yield a much stronger result. In fact, we see in our last theorem that the dimension of any non-zero right ideal in $\ell_\infty(S)^*$ is $2^{2^{|S|}}$.

2. Chou's Conjecture

Lemma 1. *Let S be cancellative, V be a thin set in S and let x_1, $x_2 \in \overline{V}$ with $\|x_1\| = \|x_2\|$. Then*

(1) *$S\,x_1 \cup S\,x_2$ is a discrete subspace of βS.*

(2) *$S\,x_1 \cup S\,x_2$ is C^*−embedded in βS.*

Proof. If x_1 and x_2 belong to V, then this lemma is trivial. So suppose that x_1 and x_2 are in $\overline{V} \setminus V$. Since every subset of V is also a thin set, we may assume with no loss of generality that $\|x_1\| = \|x_2\| = |V|$. Let V_1 and V_2 be subsets of V with $x_1 \in \overline{V}_1$ and $x_2 \in \overline{V}_2$. If $x_1 = x_2$, then we take $V_1 = V_2 = V$. Otherwise, V_1 and V_2 are taken as a partition of V. Let $s \in S$ and $i = 1$ or 2. Clearly, $\overline{s\,V_i}$ is a neighbourhood of $s\,x_i$ in βS and $s\,x_i \in S\,x_i \cap \overline{s\,V_i}$. Suppose that there is $t \in S$, $t \neq s$ and $t\,x_i \in \overline{s\,V_i}$. Then

$$t\,x_i \in \overline{t\,V_i} \cap \overline{s\,V_i} = \overline{t\,V_i \cap s\,V_i},$$

which is clearly not possible since $|t\,V_i \cap s\,V_i| < \omega$, and $\|t\,x_i\| = \|x_i\| = |V|$. Therefore

$$\{s\,x_i\} = S\,x_i \cap \overline{s\,V_i}.$$

In other words, $S\,x_i$ is a discrete subspace of βS, which proves statement (1) when $x_1 = x_2$. So suppose that $x_1 \neq x_2$. Then $\overline{sV_1} \cap Sx_2 = \emptyset$. Otherwise, $sV_1 \cap tV_2$ is infinite for some $t \in S$. But this is not possible since $sV_1 \cap tV_2 = \emptyset$ for $s = t$ since S is left cancellative, and $sV_1 \cap tV_2$ is finite for $s \neq t$ since V is a thin set. Therefore

$$\overline{sV_1} \cap (S\,x_1 \cup S\,x_2) = \{sx_1\}.$$

The same argument shows that $\{sx_2\} = \overline{sV_2} \cap (S\,x_1 \cup S\,x_2)$. Hence $S\,x_1 \cup S\,x_2$ is a discrete subspace of βS.

Next we show that each bounded real function f on $S\,x_1 \cup S\,x_2$ has a continuous extension to βS, i.e., $S\,x_1 \cup S\,x_2$ is C$^*-$embedded in βS. Let f be such a function. We shall define g on S whose continuous extension $\tilde{g}$ to βS agrees with f on $Sx_1 \cup Sx_2$. We start with the following equivalence relation: $s \sim t$ if and only if there exist $t_1, t_2, \ldots, t_m \in S$ such that

$$t_1 = s,\ t_m = t \ \text{ and } t_i\,V \cap t_{i+1}\,V \neq \emptyset \text{ for each } i = 1,\ 2,\ \ldots,\ m-1.$$

We partition S into equivalence classes. Let $\tilde{s}$ be any class, and $\{s_\alpha\}_{\alpha<\kappa}$ be a well ordering of $\tilde{s}$ (note that $\kappa = |\tilde{s}| \leq |V|$). We start by defining g on $s_0 V$ by

$$g(s_0 v) = \begin{cases} f(s_0 x_1) & \text{if } s_0 v \in s_0 V_1 \\ f(s_0 x_2) & \text{if } s_0 v \in s_0 V_2 \setminus s_0 V_1. \end{cases}$$

(Note that in the case of $x_1 = x_2$, we have $g(s_0\,v) = f(s_0\,x)$ for all $v \in V$.) Let $\alpha < \kappa$, and suppose that $g(s_\beta\,v)$ has been defined for all $v \in V$ and $\beta < \alpha$. Then we let

$$g(s_\alpha v) = \begin{cases} g(s_\beta v') & \text{if } s_\alpha v = s_\beta v' \text{ for some } v' \in V \text{ and } \beta < \alpha \\ f(s_\alpha x_1) & \text{if } s_\alpha v \in s_\alpha V_1 \setminus \bigcup_{\beta<\alpha} s_\beta V \\ f(s_\alpha x_2) & \text{if } s_\alpha v \in s_\alpha V_2 \setminus (\bigcup_{\beta<\alpha} s_\beta V \cup s_\alpha V_1). \end{cases}$$

In such a way g is defined on $\tilde{s}V$ for each class $\tilde{s}$. Finally, any fixed value may be given to g on $S \setminus SV$ if this set is not empty. The function g is well defined since $s\,V \cap t\,V = \emptyset$ when s and t are in different classes and it is clearly bounded on S. So let $\tilde{g}$ be the extension of g to βS.

We check that $\tilde{g}(s\,x_1) = f(s\,x_1)$ for every $s \in S$. Let $s \in S$, then s is in some class, and so $s = s_\alpha$ for some $\alpha < \kappa$ (remember that $\kappa = |\tilde{s}| \leq |V|$). Then, for $i = 1,\ 2$, we have two cases.

Case 1: If $|V| = \omega$, then

$$|\bigcup_{\beta<\alpha}(s_\alpha\,V_i \cap s_\beta\,V)| \leq |\alpha| \sup\{|s_\alpha\,V_i \cap s_\beta\,V| \ :\ \beta < \alpha\} < \omega = |V|.$$

Case 2: If $|V| > \omega$, then

$$|\bigcup_{\beta<\alpha}(s_\alpha V_i \cap s_\beta V)| \leq |\alpha| \sup\{|s_\alpha V_i \cap s_\beta V| : \beta < \alpha\} \leq |\alpha|\omega < |V|.$$

Since $\|s_\alpha x_1\| = \|x_1\| = |V|$, this implies that

$$s_\alpha x_1 \in \overline{s_\alpha V_1 \setminus \bigcup_{\beta<\alpha} s_\beta V},$$

and accordingly, $\tilde{g}(s_\alpha x_1) = f(s_\alpha x_1)$. Similarly, $\tilde{g}(s_\alpha x_2) = f(s_\alpha x_2)$ follows from

$$\begin{aligned} |s_\alpha V_2 \cap (\bigcup_{\beta<\alpha} s_\beta V \cup s_\alpha V_1)| &= |\bigcup_{\beta<\alpha}(s_\alpha V_2 \cap s_\beta V) \cup (s_\alpha V_2 \cap s_\alpha V_1)| \\ &= |\bigcup_{\beta<\alpha} s_\alpha V_2 \cap s_\beta V| < |V| \end{aligned}$$

and from $\|s_\alpha x_2\| = \|x_2\| = |V|$. □

Our first corollary is, as promised by the title, about a conjecture of C. Chou [1, page 202].

Theorem 1. *Let V be a thin subset of a cancellative semigroup S. Then x_1, $x_2 \in \overline{V} \setminus V$, $x_1 \neq x_2$ and $\|x_1\| = \|x_2\|$ implies that $(\beta S)x_1 \cap (\beta S)x_2 = \emptyset$.*

Proof. Let $D = S\,x_1 \cup S\,x_2$. As already noted in the proof of the first statement of the lemma above, $sV_1 \cap sV_2 = \emptyset$ since S is left cancellative, and $sV_1 \cap tV_2$ is finite for $s \neq t$ since V is a thin set. Accordingly, $(S\,x_1) \cap (S\,x_2) = \emptyset$, and so

$$cl_{\beta D}(Sx_1) \cap cl_{\beta D}(Sx_2) = \emptyset$$

since D is discrete. Since D is C$^*-$embedded in βS, this is precisely

$$\overline{S\,x_1} \cap \overline{S\,x_2} = (\beta S)\,x_1 \cap (\beta S)\,x_2 = \emptyset,$$

as required. □

In [3] and [4], all the right cancellative points found in βS had norm equal to ω. Since thin set can have arbitrary sizes, the following theorem shows that right cancellative points in βS may have arbitrary norm as well.

Theorem 2. *Let V be a thin set of a cancellative semigroup S. Then every x in $\overline{V}$ is right cancellative in βS, i.e., $yx \neq zx$ whenever $y \neq z$ in βS.*

Proof. Let $x \in \overline{V}$, y and z be two distinct points in βS, and let Y and Z be a partition of S with $y \in \overline{Y}$ and $z \in \overline{Z}$. Then, as seen in Lemma 1, $sx \neq tx$ whenever s and t are distinct in S since $sV \cap tV$ is finite. Hence $Yx \cap Zx = \emptyset$, and so

$$cl_{Sx}(Yx) \cap cl_{Sx}(Zx) = \emptyset$$

since Sx is discrete by Lemma 1. As in Theorem 1, this is precisely

$$\overline{Yx} \cap \overline{Zx} = \emptyset,$$

and accordingly $yx \neq zx$ since $yx \in \overline{Yx}$ and $zx \in \overline{Zx}$. □

3. Application to the Study of $\ell_\infty(S)^*$

Let $C(\beta S)$ be the space of continuous functions on βS. It is well known that with the Gelfand mapping $f \mapsto \tilde{f}$, where

$$\tilde{f}(x) = x(f) \; for \; x \in \beta S \text{ and } f \in \ell_\infty(S),$$

the Banach spaces $\ell_\infty(S)$ and $C(\beta S)$ may be identified. So $\ell_\infty(S)^*$ and $C(\beta S)^*$ may also be identified by the mapping $\mu \mapsto \tilde{\mu}$, where

$$\tilde{\mu}(\tilde{f}) = \mu(f) \quad for \quad f \in \ell_\infty(S).$$

Now with the Riesz representation theorem ([6, Theorem 14.10]), we may regard every $\mu \in \ell_\infty(S)^*$ as a Borel measure on βS, and so we may talk about its total variation $|\mu|$ and its support $supp(\mu)$.

Lemma 2. *Let S be a cancellative semigroup, x be a right cancellative point in βS and μ a non-zero element in $\ell_\infty(S)^*$. Then*

(1) $|\mu x| = |\mu| x$ *(and so $\|\mu x\| = \|\mu\|$).*

(2) $supp(\mu x) = supp(\mu)x$.

Proof. Similar to [5, Lemma 2]. □

Corollary 1. *When S is amenable there are $2^{2^{|S|}}$ left invariant means in $\ell_\infty(S)^*$.*

Proof. Let μ be a left invariant mean in $\ell_\infty(S)^*$. By [1], let V be a thin set of S with $|V| = |S|$. Then since there are $2^{2^{|S|}}$ points x in $\overline{V}$ with $\|x\| = |S|$ (see [7]), it follows from Theorem 1 that there are $2^{2^{|S|}}$ pairwise disjoint left ideals in βS. Now it is easy to see that μx is a nonzero left invariant mean for each $x \in \beta S$, and so the lemma above and Theorem 1 complete the proof. □

Corollary 2. *Let S be an infinite subsemigroup of a group. Then every non-zero right ideal in $\ell_\infty(S)^*$ has dimension $2^{2^{|S|}}$.*

Proof. Let R be a non-zero right ideal in $\ell_\infty(S)^*$, and let $\mu \in R$, $\mu \neq 0$. Again by [1], let V be a thin set of S with $|V| = |S|$. Then, by Theorem 2 and Lemma 2, $\mu x \neq 0$ for each $x \in \overline{V}$. Let $x_1, x_2, ..., x_n \in \overline{V} \setminus V$ such that $\|x_1\| = \|x_2\| = ... = \|x_n\|$. Then, by Theorem 1 and Lemma 2, the identity

$$a_1 \mu x_1 + a_2 \mu x_2 + ... + a_n \mu x_n = 0$$

is possible only when $a_1 = a_2 = ... = a_n = 0$. Therefore the set

$$\{\mu x \ : \ x \in \overline{V} \text{ and } \|x\| = |S|\}$$

contains $2^{2^{|S|}}$ linearly independent non-zero elements of R, as required. □

Remark. It is worthwhile to note that Corollary 2 implies that the dimension of the radical of $\ell_\infty(S)^*$ is either 0 or $2^{2^{|S|}}$ without assuming that S is amenable.

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