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SPLITTABILITY, CONSTRUCTIBLE SETS AND PARTIAL ORDERS

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Abstract

The power and utility of cleavability (also known as splittability) has been well established by A. V. Arhangel'skiĭ and his associates within a topological framework. The definition was transferred to partially ordered sets by D. J. Marron and T. B. M. McMaster. The connections between topology and order are here further exploited as we examine properties derived from splitting over simple finite partial orders. All spaces are assumed T_0. The minimal topologies for these properties are readily derived from splittability considerations.

Introduction

Given a topological space X it is well known that a quasi-order is generated under the relation $x \leq y$ if and only if $x \in \overline{\{y\}}$. Moreover, if X is a T_0 topology then $(X, \leq)$ is a partially ordered set (poset). This connection is highly suggestive of how splittability might be treated for posets. It should be noted that, for infinite posets at least, the correspondence between topologies and posets is many-to-one. We review some splittability results for posets and consider the topological implications. A key influence in our approach is a paper [1] by J-P. Allouche concerning constructible sets.

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In identifying those topological spaces that split over a given space (or its corresponding partial order, suitably 'topologised') we obtain a topological invariant. The invariants we consider are closely linked to constructible sets. Joint work with S. D. McCartan [4] has shown that related minimality results can be intuitively derived from a splittability approach. In this paper we present the preliminary material and show how the invariants arise naturally from partial orders.

Definition 1. [2] *A topological space X is* splittable *over a class $\mathcal{P}$ of topological spaces if for every subset A of X there exists a continuous map $f : X \to Y \in \mathcal{P}$ such that $f(A) \cap f(X \setminus A) = \emptyset$. We use the term* onto splittable *to mean that the* splitting maps *can be chosen to be onto. The space X is said to be* pointwise splittable *over $\mathcal{P}$ if splitting maps can be found for every singleton subset of X. When working with posets we need only replace the "continuous map f" concept with an* increasing *map (i.e. $x \leq y$ in X implies $f(x) \leq f(y)$). A class of spaces $\mathcal{P}$ is said to be a* splittability class *if X splittable over $\mathcal{P}$ implies $X \in \mathcal{P}$.*

For background information the reader is referred to [2, 3] for splittability over topological spaces and to [6, 7, 8] for splittability over partially ordered sets.

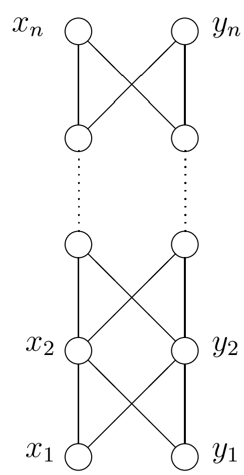
Definition 2. *Given a poset $(X, \leq)$, we use the notation $\uparrow x = \{y \in X : x \leq y\}$ and $\downarrow x = \{y \in X : y \leq x\}$. The* weak *topology, $\mathcal{W}$, is the topology on X whose proper closed subsets are generated by the family of subsets $\{\downarrow x : x \in X\}$. The* Alexandroff *topology, $\mathcal{A}$, is the topology on X whose open sets are the increasing subsets of X. We use $\mathbf{C_n}$ to denote (interchangeably) the poset on $\{1, 2, \ldots, n\}$ with the usual ordering and the corresponding topology obtained by imposing the weak topology. Thus $\mathbf{C_n}$ is the n-point T_0 nested space (where by nested we intend a space whose open subsets are linearly ordered by set-inclusion). Indeed, all posets in this paper are considered as topological spaces when endowed with the weak topology.*

Definition 3. *Let X be a poset that contains no infinite chains. By the* height of X *we shall mean the cardinality of the largest chain in X. The* height of a point $x \in X$ *is the height of the poset* $\downarrow x$.

The following key result of Marron's [6] identifies precisely which finite posets are splittable over the n-point chain.

Theorem 4. *Let X be a finite poset; then X is splittable over* $\mathbf{C_n}$ *if and only if*
(i) X contains no $n+1$ element chain, and
(ii) X contains no disjoint n element chains.

Clearly $\mathbf{C_n}$ will split over any poset of height n. Likewise, it can be shown that any poset of height n will split over the poset shown below. We will refer to this poset as *the ladder of height* n denoted by $\mathbf{L_n}$. Hence $\mathbf{C_n}$ and $\mathbf{L_n}$ represent the two extremes of splittability for posets of height n.



Definition 5. *Let X be a topological space. A subset of X is* locally closed *if it can be expressed as the intersection of an open set and a closed set. We will say that X is:*

– *door if every subset $A \subseteq X$ is either open or closed,*

- *submaximal if every subset* $A \subseteq X$ *is locally closed,*
- T_{ESC} *if every subset* $A \subseteq X$ *is a finite union of locally closed sets.*

Recall [1] *that the* constructible *sets in a topological space are the sets in the Boolean algebra generated by the closed and the open sets. It can be shown that the constructible sets are precisely those which are finite unions of locally closed sets. Hence a space is* T_{ESC} *if and only if* $\underline{e}$*very* $\underline{s}$*ubset is* $\underline{c}$*onstructible.*

Results

Observing that every finite T_0 space is T_D (that is, every singleton is locally closed) it is clear since every subset is a finite union of singletons that finite T_0 spaces are T_{ESC}. It is clear that every T_{ESC} space is T_D and therefore T_0. To see that there exist infinite T_{ESC} spaces one need only note that any included point topology is *door* and therefore T_{ESC}.

Lemma 6. *Every finite* T_0 *space is splittable over a finite* T_0 *nested space.*

Proof. Recall that minimal T_0 topologies are *nested* [5]. If X is an n-point T_0 space then X is stronger than a minimal T_0 topology in the lattice of topologies definable on the finite set X. Hence X is splittable over a finite T_0 *nested* space. □

Theorem 7. *If* X *is splittable over a class of* T_{ESC} *spaces then* X *is* T_{ESC}.

Proof. Let $\mathcal{P}$ be such a class of T_{ESC} spaces. Suppose that $A \subseteq X$ and choose continuous $f : X \to Y \in \mathcal{P}$ that splits X along A over Y. Since Y is T_{ESC} we can write:

$$f(A) = (B_1 \cap C_1) \cup (B_2 \cap C_2) \cup \ldots \cup (B_n \cap C_n),$$

where each B_i is open in Y and each C_i is closed in Y. Now

$$A = f^{-1}f(A) = \bigcup_{1}^{n} f^{-1}(B_i \cup C_i) = \bigcup_{1}^{n}(f^{-1}(B_i) \cap f^{-1}(C_i)).$$

Hence A is the union of finitely many locally closed sets. □

When concentrating solely on partially ordered sets we can easily describe the posets that are splittable over chains and ladders. The corresponding topological results are more difficult to formulate. Before making any attempt to do so we adopt the notation of Allouche [1].

Definition 8. *If X is a topological space and $A \subseteq X$ we recursively define:*

$$\overset{\vee}{A} = \overline{A} \cap X \setminus A, \qquad \overset{n\vee}{A} = \overset{\vee}{\left(\overset{(n-1)\vee}{A} \right)} \qquad \text{for } n > 1.$$

Theorem 9. [1] *Let X be a topological space with $A \subseteq X$.*

(i) $\left(\overline{\overset{n\vee}{A}}\right)$ *is a decreasing sequence of (closed) sets.*

(ii) A is the union of n locally closed sets if and only if $\overset{(2n)\vee}{A} = \emptyset$. Hence the smallest such n is the smallest integer n such that $\overset{(2n)\vee}{A} = \emptyset$. Furthermore the set A admits the following canonical decomposition into n disjoint locally closed sets:

$$A = \left(\overline{A} \setminus \overline{\overset{\vee}{A}}\right) \cup \left(\overline{\overset{2\vee}{A}} \setminus \overline{\overset{3\vee}{A}}\right) \cup \ldots \cup \left(\overline{\overset{(2n-2)\vee}{A}} \setminus \overline{\overset{(2n-1)\vee}{A}}\right).$$

(iii) The set A is the union of n locally closed sets and a closed set if and only if $\overset{(2n+1)\vee}{A} = \emptyset$. Hence the smallest such n is the smallest integer n such that $\overset{(2n+1)\vee}{A} = \emptyset$.

Theorem 10. *If X is T_{ESC} then X is splittable over the finite T_0 nested spaces.*

Proof. If A is a subset of X then A is a finite union of locally closed sets. Hence we can find $n \in \mathbb{N}$ such that $\overset{(2n)\vee}{A} = \emptyset$. We can write A in its canonical form as in Theorem 9 (ii) above and define a map $f : X \to \mathbf{C_{2n+1}}$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \overline{\overset{(2n-1)\vee}{A}} \\ 2 & \text{if } x \in \overline{\overset{(2n-2)\vee}{A}} \setminus \overline{\overset{(2n-1)\vee}{A}} \\ \vdots & \\ k & \text{if } x \in \overline{\overset{(2n-k)\vee}{A}} \setminus \overline{\overset{(2n-k+1)\vee}{A}} \\ \vdots & \\ 2n & \text{if } x \in \overline{A} \setminus \overline{\overset{\vee}{A}} \\ 2n+1 & \text{otherwise.} \end{cases}$$

Recall from Theorem 9 that $\left(\overline{\overset{n\vee}{A}}\right)$ is a decreasing sequence of closed sets so we see that the map f is continuous. Moreover we observe that $f(A) = \{2, 4, 6, \ldots, 2n\}$ while $f(X \setminus A) = \{1, 3, 5, \ldots, 2n+1\}$ so f splits X along A over $\mathbf{C_{2n+1}}$. □

Corollary 11. *A topological space X is T_{ESC} if and only if X is splittable over $\{\mathbf{C_n} : \mathbf{n} \in \mathbb{N}\}$.*

Lemma 12. [1] *Let A be a subset of a topological space X. For any integer n:*

$$\overset{(2n+1)\vee}{A} = \overline{\overset{(2n)\vee}{A}} \cap (X \setminus A) \quad \textit{and} \quad \overset{(2n+2)\vee}{A} = \overline{\overset{(2n+1)\vee}{A}} \cap A.$$

Lemma 13. *Let X be a topological space. Let $A \subseteq X$ and set $B = X \setminus A$. For all $n \in \mathbb{N}$ we have:*

$$\overline{\overset{n\vee}{A}} \cap \overline{\overset{n\vee}{B}} \subseteq \overset{(n+1)\vee}{A} \cup \overset{(n+1)\vee}{B}.$$

Proof. Using Lemma 12 we see that:

$$\begin{aligned}
\overset{\overline{n\vee}}{A} \cap \overset{\overline{n\vee}}{B} &= \Big(\overset{\overline{n\vee}}{A} \cap \overset{\overline{n\vee}}{B}\Big) \cap (A \cup B) \\
&= \Big(\overset{\overline{n\vee}}{A} \cap \overset{\overline{n\vee}}{B} \cap A\Big) \;\cup \Big(\overset{\overline{n\vee}}{A} \cap \overset{\overline{n\vee}}{B} \cap B\Big) \\
&= \begin{cases} \Big(\Big(\overset{\overline{n\vee}}{A} \cap A\Big) \cap \overset{\overline{n\vee}}{B}\Big) \cup \Big(\overset{\overline{n\vee}}{A} \cap \Big(\overset{\overline{n\vee}}{B} \cap B\Big)\Big) & n \text{ odd} \\ \Big(\overset{\overline{n\vee}}{A} \cap \Big(\overset{\overline{n\vee}}{B} \cap A\Big)\Big) \cup \Big(\Big(\overset{\overline{n\vee}}{A} \cap B\Big) \cap \overset{\overline{n\vee}}{B}\Big) & n \text{ even} \end{cases} \\
&= \begin{cases} \Big(\overset{(n+1)\vee}{A} \cap \overset{\overline{n\vee}}{B}\Big) \cup \Big(\overset{\overline{n\vee}}{A} \cap \overset{(n+1)\vee}{B}\Big) & n \text{ odd} \\ \Big(\overset{\overline{n\vee}}{A} \cap \overset{(n+1)\vee}{B}\Big) \cup \Big(\overset{(n+1)\vee}{A} \cap \overset{\overline{n\vee}}{B}\Big) & n \text{ even} \end{cases} \\
&\subseteq \overset{(n+1)\vee}{B} \cup \overset{(n+1)\vee}{A}.
\end{aligned}$$

□

Theorem 14. *Let X be a topological space. Each subset $A \subseteq X$ can be expressed as the union of n locally closed sets if and only if X is splittable over the poset $\mathbf{L_{2n}}$.*

Proof. If f is a map that splits X along A over $\mathbf{L_{2n}}$ we can assume that

$$f(A) \subseteq \{x_1, x_2, \ldots, x_{2n}\} \text{ and } f(X \setminus A) \subseteq \{y_1, y_2, \ldots, y_{2n}\}.$$

Hence

$$\begin{aligned}
A = f^{-1}f(A) &= f^{-1}(\{x_1, x_2, \ldots, x_{2n}\}) \\
&= \bigcup_1^n f^{-1}(\{x_{2i-1}, x_{2i}\}) \\
&= \bigcup_1^n f^{-1}(\uparrow (x_{2i-1}) \cap \downarrow (x_{2i})) \\
&= \bigcup_1^n \Big(f^{-1}(\uparrow (x_{2i-1})) \cap f^{-1}(\downarrow (x_{2i}))\Big).
\end{aligned}$$

We see that A is the union of n locally closed sets.

Conversely suppose that each subset $A \subseteq X$ has the stated property. Set $B = X \setminus A$. Since both A and B are unions of n locally closed sets, from Theorem 9 we have:

$$A = \left(\overline{A} \setminus \overline{\overset{\vee}{A}}\right) \cup \left(\overline{\overset{2\vee}{A}} \setminus \overline{\overset{3\vee}{A}}\right) \cup \ldots \cup \left(\overline{\overset{(2n-2)\vee}{A}} \setminus \overline{\overset{(2n-1)\vee}{A}}\right),$$

$$B = \left(\overline{B} \setminus \overline{\overset{\vee}{B}}\right) \cup \left(\overline{\overset{2\vee}{B}} \setminus \overline{\overset{3\vee}{B}}\right) \cup \ldots \cup \left(\overline{\overset{(2n-2)\vee}{B}} \setminus \overline{\overset{(2n-1)\vee}{B}}\right).$$

Define a map $f : X \to \mathbf{L_{2n}}$ given by:

$$f(z) = \begin{cases} x_1 \ \text{ if } z \in \overline{\overset{(2n-1)\vee}{A}} \\ y_1 \ \text{ if } z \in \overline{\overset{(2n-1)\vee}{B}} \\ x_i \ \text{ if } z \in \overline{\overset{(2n-i)\vee}{A}} - \left(\overline{\overset{(2n-i+1)\vee}{A}} \cup \overline{\overset{(2n-i+1)\vee}{B}}\right) \\ y_i \ \text{ if } z \in \overline{\overset{(2n-i)\vee}{B}} - \left(\overline{\overset{(2n-i+1)\vee}{A}} \cup \overline{\overset{(2n-i+1)\vee}{B}}\right). \end{cases}$$

Note that if $z \in \overline{\overset{(2n-1)\vee}{A}} \cap \overline{\overset{(2n-1)\vee}{B}}$ then, by Lemma 12, $z \in \overset{(2n)\vee}{A} \cup \overset{(2n)\vee}{B} = \emptyset$ using Theorem 9 (ii). Likewise if $z \in \overline{\overset{i\vee}{A}} \cap \overline{\overset{i\vee}{B}}$ then, by Lemma 12, $z \in \overset{(i+1)\vee}{A} \cup \overset{(i+1)\vee}{B}$. Clearly f is a well defined function, which is also continuous since

$$\begin{aligned} f^{-1}(\downarrow (x_i)) &= \overline{\overset{(2n-i)\vee}{A}} \cup \overline{\overset{(2n-i+1)\vee}{B}} \quad \text{and} \\ f^{-1}(\downarrow (y_i)) &= \overline{\overset{(2n-i)\vee}{B}} \cup \overline{\overset{(2n-i+1)\vee}{A}}. \end{aligned}$$

By inspection we can see that f splits X along A over $\mathbf{L_{2n}}$, completing the proof. □

Theorem 15. *Let X be a topological space. Each subset $A \subseteq X$ can be expressed as the union of a closed set and n locally closed sets if and only if X is splittable over the poset $\mathbf{L_{2n+1}}$.*

Proof. Similar to proof of Theorem 14. □

We consider now precisely what topological spaces are splittable over an n-point T_0 *nested* space. We already know that such a space X must have the property that every subset is constructible. It follows that for each $A \subseteq X$ we can find a least $p \in \mathbb{N}$ such that $\overset{p\vee}{A} = \emptyset$. The obvious question arises: how is p related to n?

Lemma 16. *If X is a topological space that is splittable over $\mathbf{C_k}$ then $\overset{k\vee}{A} = \emptyset$ for all $A \subseteq X$.*

Proof. We consider only the case when k is even. Suppose that X splits along A over $\mathbf{C_{2n}}$ via a map f, but fails to split over $\mathbf{C_{2n-1}}$. We assume that f is onto and that $f(A) = \{1, 3, \ldots, 2n-1\}$ while $f(X \setminus A) = \{2, 4, 6, \ldots, 2n\}$. Hence we can express A and $X \setminus A$ as follows:

$$\begin{aligned} A = f^{-1}f(A) &= f^{-1}(\{1, 3, \ldots, 2n-1\}) \\ &= f^{-1}(1) \cup \bigcup_{2}^{n} f^{-1}(\downarrow (2k-1) \cap \uparrow (2k-1)), \end{aligned}$$

$$\begin{aligned} X \setminus A = f^{-1}f(X \setminus A) &= f^{-1}(\{2, 4, \ldots, 2n\}) \\ &= \bigcup_{1}^{n} f^{-1}(\downarrow (2k) \cap \uparrow (2k)). \end{aligned}$$

We observe that A is the union of a closed set and $(n-1)$ locally closed sets while $X \setminus A$ is the union of n locally closed sets. Theorem 9 gives $\overset{(2n-1)\vee}{A} = \overset{(2n)\vee}{X \setminus A} = \emptyset$. □

Theorem 17. *A topological space X is splittable over $\mathbf{C_m}$ if and only if for each $A \subseteq X$:*

(*i*) $\overset{m\vee}{A} = \emptyset$ *and,*

(*ii*) $\overset{(m-1)\vee}{A} \neq \emptyset$ *implies* $\overset{(m-1)\vee}{X \setminus A} = \emptyset$.

Proof. We consider only the case $m = 2n$. If X splits over $\mathbf{C_n}$ then, by Lemma 16, we have $\overset{(2n)\vee}{A} = \emptyset$. If $\overset{(2n-1)\vee}{A} \neq \emptyset$ then A cannot be expressed as the union of a closed set and $n-1$ locally closed sets. In the proof of Lemma 16 we therefore have $f(A) = \{2, 4, \ldots, 2n\}$ and $f(X \setminus A) = \{1, 3, \ldots, 2n-1\}$. Hence $X \setminus A$ is the union of a closed set and $n-1$ locally closed ones, so by Theorem 9 $\overset{(2n-1)\vee}{X \setminus A} = \emptyset$.

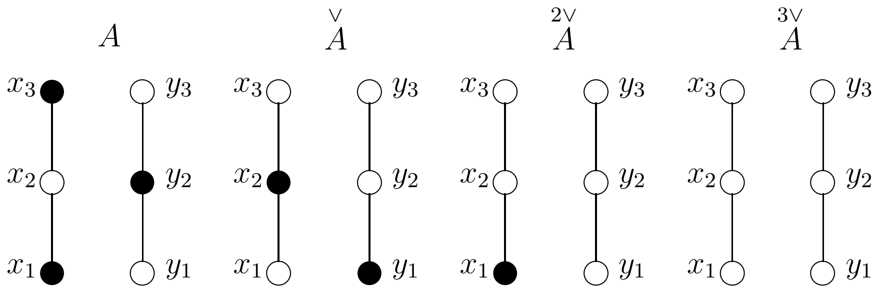
Conversely suppose that each subset of X satisfies conditions (i) and (ii). Let A be a subset of X and set $B = X \setminus A$. We must have $\overset{(2n-1)\vee}{A} = \emptyset$ or $\overset{(2n-1)\vee}{B} = \emptyset$. By reworking the proof of Theorem 10 we see that X splits along A over $\mathbf{C_{2n}}$. $\square$

It is clear that the complement of a constructible subset is also constructible. Splittability arguments, similar to those used previously, yield the following less intuitive result.

Theorem 18. *Let X be T_{ESC} and let $A \subseteq X$ such that $\overset{n\vee}{A} = \emptyset$ but $\overset{(n-1)\vee}{A} \neq \emptyset$. If k is the least integer such that $\overset{k\vee}{X \setminus A} = \emptyset$ then $k = n-1, n$ or $n+1$.*

It is not hard to find subsets of ladders and chains to show that all three cases in Theorem 18 are possible. We can now interpret Allouche's operator as a measure of the number of steps required to construct a given constructible set. Hence a locally closed set can be constructed in two steps by intersecting an open set with a closed set. The previous result says that the complement of a constructible set A can be constructed in the same number of steps as A, give or take one.

Example 19. *Let $X = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ be the space as shown below. Let A be the subset $\{x_1, x_3, y_2\}$. We examine the sequence $\left(\overset{n\vee}{A}\right)$, shown shaded in each successive iteration.*



$$\overset{\vee}{A} = \overline{\{x_1, x_3, y_2\}} \setminus \{x_1, x_3, y_2\} = \{x_2, y_1\}$$
$$\overset{2\vee}{A} = \overline{\{x_2, y_1\}} \setminus \{x_2, y_1\} = \{x_1\}$$
$$\overset{3\vee}{A} = \overline{\{x_1\}} \setminus \{x_1\} = \emptyset$$

Both A and $X \setminus A$ can be expressed as the union of a closed set and a locally closed set. It is therefore consistent with Allouche that $\overset{3\vee}{A} = \overset{3\vee}{X \setminus A} = \emptyset$. *We observe also that X will not split along A over $\mathbf{C_3}$. This shows the necessity of condition* (ii) *in Theorem 17.*

We are now in a position to name the properties identified and summarize our findings.

Property	$\mathcal{Q}_n$	$\mathcal{R}_n$	$\mathcal{P}_n$	$\mathcal{S}_n$
$\forall A \subseteq X$	A is the union of n locally closed sets	A or $X \setminus A$ is the union of n locally closed sets	A is the union of n locally closed sets and a closed set	A or $X \setminus A$ is the union of n locally closed sets and a closed set
$\forall A \subseteq X$	$\overset{2n\vee}{A} = \emptyset$	$\overset{2n\vee}{A} = \emptyset$ or $\overset{2n\vee}{X \setminus A} = \emptyset$	$\overset{(2n+1)\vee}{A} = \emptyset$	$\overset{(2n+1)\vee}{A} = \emptyset$ or $\overset{(2n+1)\vee}{X \setminus A} = \emptyset$
Splittability equivalent	$S(\mathbf{L_{2n}})$	$S(\mathbf{C_{2n+1}})$	$S(\mathbf{L_{2n+1}})$	$S(\mathbf{C_{2n+2}})$

Note that $\mathcal{S}_0$ corresponds to the invariant *door* and $\mathcal{Q}_1$ corresponds to *submaximal*. It is clear that $\mathcal{Q}_n \Rightarrow \mathcal{R}_n \Rightarrow \mathcal{P}_n \Rightarrow \mathcal{S}_n \Rightarrow \mathcal{Q}_{n+1} \Rightarrow T_{ESC}$ for all $n \geq 1$. As an example of the type of minimality results that can now be obtained we present the following theorem.

Theorem 20. *Let X be a set such that $x \notin A \subseteq X$. The topology $\mathcal{M}(A) \cap \mathcal{E}(x)$ is minimal $\mathcal{R}_1$. Recall that $\mathcal{M}(A) = \{G \subseteq X : G \subseteq A$ or $A \subseteq G\}$ and $\mathcal{E}(x) = \{G \subseteq X : x \notin G\}$.*

When working exclusively with Alexandroff topologies it is useful to note that the increasing maps correspond exactly with the continuous maps. Moreover, Alexandroff topologies are clearly principal. These observations prove invaluable in identifying principal minimal topologies.

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