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SOME NEW RESULTS ON ČECH-COMPLETE SPACES

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Abstract

We prove that every topological group which is a finite union of Čech-complete spaces is Čech-complete. We also show that x is Čech-complete if the product x^{ω} is a finite union of Čech-complete subspaces. In case of finite products, we describe a model of ZFC with a non-Čech-complete space x such that x^{n} is the union of two Čech-complete subspaces for each $n\in\omega$.

0. Introduction

It is well known [En, Theorem 3.9.6] that the Čech-completeness is hereditary with respect to closed subsets and G_{δ}-subsets. The finite union of Čech-complete spaces is not necessarily Čech-complete; consider for example, the Lindelödification of a discrete, uncountable space. In this paper we show that Čech-completeness is finitely additive in topological groups [Theorem 3.2]. An example of a non-Čech-complete space with a point-finite open cover of Čech-complete subspaces is given in [ChČoNa]. Z. Frolík [Fro] proved that if X has a locally finite cover with its open Čech-complete subspaces, then X is Čech-complete. We establish that if X has a locally finite cover

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with its Čech-complete subspaces, then the space X contains a dense open Čech-complete subspace [Corollary 2.5].

The discrete sum, as well the countable product, of Čech-complete spaces is itself Čech-complete [En, Theorems 3.9.7 y 3.9.8]. The perfect (direct or inverse) image, of a Čech-complete space is a Čech-complete space [En, Teorema 3.9.10]. The space X is almost Čech-complete if it has a dense Čech-complete subspace. In [AaLu, Theorem 4.3.4.1] it was proved that almost Čech-completeness is preserved under open maps. It is known [En, 3.12.19(d)] that Čech-completeness might be destroyed by an open map. We show in Theorem 3.3 that in the case of topological groups the Čech-completeness is invariant under open maps.

M. G. Tkačenko was the first to consider the additivity of topological properties in countable powers. He proved, that if X is countably compact and X^ω is a countable union of metrizable subspaces then X is a metrizable space. V.V. Tkachuk [Tk] proved that metrizability is finitely additive in countable powers. In [OhYa] analogous properties were proved for paracompactness. We also establish that Čech-completeness is finitelly additive in countable powers [Corollary 2.11]. We also show that there are models of ZFC in which Čech-completeness is not 2-additive in finite powers [Example 3.1].

1. Notation and Terminology

Throughout this paper all spaces are Tychonoff. Given a space X, we denote its topology by $\mathcal{T}(X)$ and

$$\mathcal{T}^*(X) = \{U \in \mathcal{T}(X) : U \neq \emptyset\}.$$

If $A \subset X$ then $\mathcal{T}(A, X) = \{U \in \mathcal{T}(X) : A \subset U\}$ and $\mathcal{T}(x, X) = \mathcal{T}(\{x\}, X)$. The space X is Čech-complete if it is a G_δ-set in every compactification cX and is locally Čech-complete if every point of X has a Čech-complete neighborhood. The space $L(X)$ is the Lindelödification of the discrete uncountable space X, that is $L(X) = X \cup \{*\}$ where $* \notin X$, $X \subset \mathcal{T}(L(X))$ and $\mathcal{T}(*, L(X)) = \{\{*\} \cup (X - A) : |A| \leq \omega\}$. Given an $A \subset X$, we denote its closure in βX by $\overline{A}^{\beta X}$. A collection $\mathcal{P} \subset \mathcal{T}^*(X)$ is a π-base in X if given a $U \in \mathcal{T}^*(X)$, there exists a $V \in \mathcal{P}$ such that $V \subset U$. All maps considered here will be continuous and onto. Let $\{X_t\}_{t\in T}$ be a family of topological spaces. Given $n \in \mathbf{N} = \omega - \{0\}$, if $\alpha_1, \alpha_2, \ldots, \alpha_n \in T$ and $B_i \in \mathcal{T}(X_{\alpha_i})$, $i = 1, 2, \ldots, n$, let $< \alpha_1, \alpha_2, \ldots, \alpha_n; B_1, B_2, \ldots, B_n >$

$$= \{x \in \Pi_{t\in T} X_t : x(\alpha_i) \in B_i, i = 1, 2, \ldots, n\}.$$

If $x \in X$ then $x^n = (x_1, x_2, \ldots, x_n)$ where $x_i = x$ for all i.

2. Finite Additivity of Čech-completeness

We are going to prove some results on finite additivity of Čech-completeness in finite powers. We also prove that Čech-completeness is finitely additive in X^ω.

Lemma 2.1. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$ where A_i is Čech-complete for all $i \in \{1, \ldots, n\}$ then $\bigcap_{i=1}^n \overline{A_i}$ is Čech-complete.*

Proof. Let us prove that $\bigcap_{i=1}^n \overline{A_i}$ is a G_δ-subset of the compact set $P = \bigcap_{i=1}^n \overline{A_i}^{\beta X}$. Since each subspace A_i is Čech-complete for $1 \le i \le n$, we have

$$\overline{A_i}^{\beta X} - A_i = \bigcup_{j\in\mathbf{N}} F_j \quad \text{where } F_j \text{ is compact for every } j \in \mathbf{N}.$$

Let $A_i' = A_i \cap P$. We have

$$P - A_i' = P - A_i = P \bigcap (\overline{A_i}^{\beta X} - A_i) = P \bigcap (\bigcup_{j\in\mathbf{N}} F_j) = \bigcup_{j\in\mathbf{N}} (F_j \bigcap P).$$

As a consequence, $P - A_i'$ is an F_σ-subset of P; thus A_i' is a G_δ-subset of P. Since

$$\bigcup_{i=1}^n A_i' = \bigcup_{i=1}^n (A_i \bigcap P) = P \bigcap (\bigcup_{i=1}^n A_i) = X \bigcap P$$
$$= X \bigcap (\bigcap_{i=1}^n \overline{A_i}^{\beta X}) = \bigcap_{i=1}^n \overline{A_i},$$

the set $\bigcap_{i=1}^n \overline{A_i}$ is a finite union of G_δ-subsets of P and hence it is a G_δ-subset of P. □

Corollary 2.2. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is a Čech-complete and dense subspace of X for all $i \le n$, then X is a Čech-complete space.*

Theorem 2.3. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is locally Čech-complete for $i \leq n$, then there exists an open non-empty Čech-complete subspace in X.*

Proof. We shall consider two cases:

i) There exists an $i_0 \in \{1, 2, \ldots, n\}$ such that $X \neq \overline{A_{i_0}}$;
ii) $X = \overline{A_i}$ for all $i = 1, 2, \ldots, n$.

Case **i)**. We will use induction on n. Without loss of generality we may suppose that $i_0 = 1$. If $n = 2$, let $V = X - \overline{A_1} \in \mathcal{T}^*(X)$. Then $V \subset A_2$ and, since A_2 is locally Čech-complete, the subspace V is also locally Čech-complete. Therefore there exists an open non-empty Čech-complete subspace of X. Assume that we have proved the theorem for all $k \leq n-1$. Consider the open non-empty set $V = X - \overline{A_1} = A'_2 \cup A'_3 \cup \ldots \cup A'_n$ where $A'_i = V \cap A_i$ for $i = 2, 3 \ldots, n$. It is clear that A'_i is locally Čech-complete and by induction there is an open non-empty Čech-complete subspace in V and hence in X.

Case **ii)**. Pick an $x_1 \in A_1$. There exists a $U_1 \in \mathcal{T}(x_1, X)$ such that $U_1 \cap A_1$ is Čech-complete. Suppose that $k < n$ and we have constructed $U_1, \ldots, U_k \in \mathcal{T}^*(X)$ such that $U_1 \supset U_2 \supset \ldots \supset U_{k-1} \supset U_k$ and $U_i \cap A_j$ is Čech-complete for any $j \leq i$. Since A_{k+1} is dense in X we have $A_{k+1} \cap U_k \neq \emptyset$. Take a point $x_{k+1} \in A_{k+1} \cap U_k$. Being the space A_{k+1} locally Čech-complete, there exists an $U_{k+1} \in \mathcal{T}(x_{k+1}, X)$ such that $U_{k+1} \subset U_k$ and $U_{k+1} \cap A_{k+1}$ is Čech-complete. Now $U_{k+1} \cap A_j$ is an open subspace of $U_k \cap A_j$ for any $j \leq k$. Therefore $U_{k+1} \cap A_j$ is Čech-complete and our inductive construction can go on until $k = n$, so we have a subset $U_n \in \mathcal{T}^*(X)$ such that $U_n \cap A_i$ is Čech-complete for all $i \leq n$.

Applying Corollary 2.2 we can conclude that U_n is an open Čech-complete subspace, since

$$U_n = (U_n \cap A_1) \cup (U_n \cap A_2) \cup \cdots \cup (U_n \cap A_n) \quad \text{and}$$

$$\overline{U_n \cap A_i} = \overline{U_n} \quad \text{for} \quad i \leq n. \qquad \square$$

Corollary 2.4. *Under the hypothesis of Theorem 2.3, the space X has a dense open Čech-complete subspace.*

Proof. Let $U \in \mathcal{T}^*(X)$, so that $U = (A_1 \cap U) \cup \ldots \cup (A_n \cap U)$ where $A_i \cap U$ is locally Čech-complete for all $i \leq n$. There exists a $V \in \mathcal{T}^*(X)$, such that $V \subset U$ and V is Čech-complete. Therefore the family

$$\mathcal{P} = \{V \in \mathcal{T}^*(X) : V \text{ is Čech-complete}\}$$

is a π-base in X. Let μ be a maximal pairwise disjoint subfamily of $\mathcal{P}$. We consider the open set $G = \cup\mu$, which is Čech-complete, being homeomorphic to the discrete sum of elements of μ. It is clear that $\overline{G} = X$. □

Corollary 2.5. *If a space X is the union of a locally finite family of Čech-complete subspaces, then there exists a dense open Čech-complete subspace of X.*

Corollary 2.6. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is Čech-complete for $1 \leq i \leq n$, then there exists an open non-empty Čech-complete subspace in X.*

Corollary 2.7. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is almost Čech-complete for $1 \leq i \leq n$, then X is almost Čech-complete.*

Proof. For $i = 1, 2, \ldots, n$ denote by B_i the Čech-complete subspaces such that $B_i \subset A_i$ and $\overline{B_i} = \overline{A_i}$. Then $Y = \bigcup_{i=1}^n B_i$ is a dense subspace of X. Corollary 2.4 implies that there exists a Čech-complete dense subspace Z in Y. It is evident that Z is also dense in X. □

Lemma 2.8. *If X^ω has an open non-empty Čech-complete subspace, then X^ω is Čech-complete.*

Proof. If $U \in \mathcal{T}^*(X^\omega)$ is Čech-complete, let

$$B = < \alpha_1, \alpha_2, \ldots, \alpha_n; B_1, B_2, ..., B_n >$$

be an element of the standard base in X^ω such that $B \subset U$. For $A = \{\alpha_1, \ldots, \alpha_n\}$ take an $x \in X^A$ such that $x(\alpha_i) \in B_i$ for all $i = 1, \ldots, n$. Then $Y = \{x\} \times X^{\omega - A}$ is a closed subspace of X^ω such that $Y \subset B \subset U$. Hence, Y is closed in U and, being U Čech-complete, the subspace Y is also Čech-complete. Since Y is homeomorphic to X^ω, the latter is Čech-complete. □

Corollary 2.9. *If X^ω is locally Čech-complete then X^ω is Čech-complete.*

From 2.3 and 2.8 we have

Theorem 2.10. *If $X^\omega = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is locally Čech-complete for all $i \leq n$, then X is Čech-complete.*

Corollary 2.11. *If $X^\omega = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is Čech-complete for $1 \leq i \leq n$, then X is Čech-complete.*

Theorem 2.12. *If $X^n = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is a Čech-complete subspace for $1 \leq i \leq n$, then X is a locally Čech-complete space.*

Proof. Take an $x_0 \in X$. Let $Y = \{(x_0^{n-1}, y) : y \in X\}$ and $F = \bigcap_{i=1}^n \overline{A_i}$. Note that F is Čech-complete by Lemma 2.1. If $Y \subset F$, then Y is Čech-complete. Hence the space X is locally Čech-complete, because X and Y are homeomorphic.

In case $Y - F \neq \emptyset$, we proceed by induction. The case of $n = 1$ is clear. Suppose that our theorem is true for every $m < n$. Take $(x_0^{n-1}, y) \in Y - F$. There exist $U \in \mathcal{T}(x_0, X)$, $V \in \mathcal{T}(y, X)$ and $i_0 \in \{1, 2, \ldots, n\}$ such that $(U^{n-1} \times V) \bigcap A_{i_0} = \emptyset$. Without loss of generality, we may suppose that $i_0 = 1$. Since $U^{n-1} \times \{y\} \subset U^{n-1} \times V$ and $U^{n-1} \times \{y\}$ is homeomorphic to U^{n-1}, we have $U^{n-1} = A'_2 \cup A'_3 \cup \ldots, \cup A'_n$ where A'_j is Čech-complete for $j \leq n$. By induction $U \ni x_0$ is locally Čech-complete at x_0. Being x_0 arbitrary, the space X is locally Čech-complete. □

3. Non-additivity of Čech-completeness in Finite Powers

Recall that the sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X is called complete if every centered family γ of closed subspaces of X such that for every $n \in \mathbf{N}$ there are $H \in \gamma$, $U \in \mathcal{U}_n$ with $H \subset U$, has non-empty intersection. It is known [En, Theorem 3.9.2.] that X is Čech-complete if and only if X has a complete sequence of open covers. The set $C \subset \omega_1$ is a club if it is closed and unbounded in ω_1. The set $S \subset \omega_1$ is stationary if $S \cap C \neq \emptyset$ for every club C.

We also recall the axiom $\diamondsuit^+$ which is consistent with ZFC, see [Ku, Definition 7.9]:
there is a sequence $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ where $\mathcal{A}_\alpha$ is a countable family of subsets of α such that for any $A \subset \omega_1$, there exists a club $C \subset \omega_1$ such that for all $\alpha \in C$ we have $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$.

Example 3.1. *Under $\diamondsuit^+$ there exists a non-Čech-complete space X such that for every $n \in \omega$ we have $X^n = A_0 \cup A_1$, where A_0 and A_1 are Čech-complete subspaces of X^n.*

We will need the following result.

Lemma 3.2. *There are $E, G \subset L = \{\alpha \in \omega_1 : \alpha = \lim \alpha\}$ such that*
a) *E and G are stationary;*
b) *$E \cap G = \emptyset$;*
c) *For every $\alpha \in G$ there exists a sequence $\{\alpha_n\} \subset E$ such that $\alpha_{n+1} > \alpha_n$ for all n and $\alpha_n \longrightarrow \alpha$.*

Proof. It is known that there are stationary disjoint sets E, $G' \subset L$. Let

$$P = \{\alpha \in \omega_1 : \text{ there is a sequence } \{\alpha_n\} \subset E \text{ with } \alpha_{n+1} > \alpha_n \text{ and } \alpha_n \longrightarrow \alpha\}.$$

Let us prove that P is a club. Given an $\alpha \in \omega_1$, there is an increasing sequence $\{\alpha_n\}_{n\in\omega} \subset E$ with $\alpha_n > \alpha$ for all $n \in \omega$, because E is unbounded. The sequence $\{\alpha_n\}_{n\in\omega}$ converges, say, to $\beta < \omega_1$. It is clear that $\beta > \alpha$ and $\beta \in P$. Hence P is unbounded. To show that P is closed, take $\{\alpha_n\}_{n\in\omega} \subset P$ where $\alpha_{n+1} > \alpha_n$ and $\alpha_n \longrightarrow \beta$. Then, for each $n \in \omega$ there is $\{\beta_i^n\}_{i\in\omega} \subset E$ with $\beta_i^n \longrightarrow \alpha_n$. Let $\gamma_n \in (\alpha_{n-1}, \alpha_n) \cap \{\beta_i^n\}_{i\in\omega}$. Then $\{\gamma_n\}_{n\in\omega} \subset E$, $\gamma_{n+1} > \gamma_n$ and $\gamma_n \longrightarrow \beta$, so that $\beta \in P$. Finally, the set $G = G' \cap P$ is stationary, so that E and G satisfy a), b) and c). □

We now construct a set $A_\alpha \subset \alpha - G$ for every $\alpha \in G$. There is an increasing sequence $\{\beta_n\}_{n\in\omega} \subset \omega_1 - L$ converging to α. Let $\mathcal{A}_\alpha = \{A_\alpha^n\}_{n\in\omega}$ be the family given by $\diamondsuit^+$.

We define a sequence s_n^α in the following way: If $A_\alpha^n \cap E$ is cofinal in α, then s_n^α is a sequence of elements of $A_\alpha^n \cap E \cap (\beta_n, \alpha)$ converging to α; otherwise, take $s_n^\alpha = \emptyset$. Let $s(\alpha) = \cup s_n^\alpha$. Note that, if $s(\alpha) \neq \emptyset$ then $s(\alpha) \longrightarrow \alpha$. Indeed, let $\beta < \alpha$. We may suppose that $\beta = \beta_n$ since $\beta_n > \beta$, for some $n \in \omega$. By construction $s_i^\alpha \subset (\beta_i, \alpha]$. As $\{\beta_i\}_{i\in\omega}$ is increasing, we have $s_i^\alpha \subset (\beta_i, \alpha] \subset (\beta, \alpha]$ for any $i > n$. Therefore all s_j^α are contained in $(\beta_n, \alpha]$ except for possibly those given by $0 \leq i \leq n$. Since each sequence s_i^α converges to α or is empty, each sequence, and hence its union, can have at most a finite number of elements in $\omega_1 - (\beta_n, \alpha]$. Hence $s(\alpha) \longrightarrow \alpha$. So if $s(\alpha) \neq \emptyset$ let $s(\alpha) = \{s_k : k \in \omega, s_{k+1} > s_k\}$.

Given a $\beta \in E$, we take the sequence

$$\gamma(\beta) = \{\gamma_n(\beta) : n \in \omega\} \subset \omega_1 - L,$$

with $\gamma_{n+1}(\beta) > \gamma_n(\beta)$ and $\gamma_n(\beta) \longrightarrow \beta$. As $s(\alpha) \subset E$, for every $k \in \omega$ there is an $m_k \in \omega$ such that $\gamma_l(s_k) \in (s_{k-1}, s_k)$ for all $l \geq m_k$. Again, if $s(\alpha) \neq \emptyset$ we define

$$A_\alpha = \bigcup_{k\in\omega} \{\gamma_l(s_k) : l \geq m_k\}. \tag{2}$$

Let X be the ladder system space on the stationary set G of ω_1, whose "ladder" for $\alpha \in G$ is the set A_α given in (2), see [BaGrTk]. We shall establish that X is a Tychonoff space showing that every basic neighborhood $V = ((\beta, \alpha] \cap A_\alpha) \cup \{\alpha\}$ of $\alpha \in G$ is closed.

Take a $\delta \in \omega_1 - V$. We consider two cases: $\delta > \alpha$ and $\delta < \alpha$. In case $\delta > \alpha$, the ordinal δ belongs to $U = (\alpha, \delta]$ and $U \cap V = \emptyset$. In case $\delta < \alpha$, if $\delta \notin G$ then $\{\delta\} \cap V = \emptyset$; if $\delta \in G$, it follows that $\delta \notin E$, so that $\delta \notin s(\alpha)$. Let $s_{-1} = 0$ and choose a $k \in \omega$ such that $\delta \in (s_{k-1}, s_k)$. If $k = 0$ we take $U = (0, \delta]$; then $U \cap V = \emptyset$. If $k \geq 1$, Let $B = \{j \in \omega : s_{k-1} < \gamma_j(s_k) < \delta\}$. Note that $|B| < \omega$. If $B = \emptyset$, we take $U = (s_{k-1}, \delta]$; otherwise, let $U = (\gamma_{j_0}(s_k), \delta]$, where $j_0 = \max B$. Both possibilities give $U \cap V = \emptyset$. Hence V is closed. Clearly, X is a Hausdorff space, and thus it is Tychonoff.

Suppose that X is Čech-complete. Then, there is a complete sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X. Without loss of generality, we may suppose that each $U \in \mathcal{U}_n$ is $\{\alpha\}$ if $\alpha \notin G$ or $((\beta, \alpha) \cap A_\alpha) \cup \{\alpha\}$ if $\alpha \in G$. Furthemore, since for each $\alpha \in X$ there is a $U \in \mathcal{U}_n$ with $\alpha \in U$, we may suppose that $\mathcal{U}_n = \{U^n_\alpha : \alpha \in \omega_1\}$ where $U^n_\alpha = \{\alpha\}$ or $U^n_\alpha = ((\beta^\alpha_n, \alpha] \cap A_\alpha) \cup \{\alpha\}$ with $\beta^\alpha_{n+1} > \beta^\alpha_n$.

Definition 3.3. *Let $\alpha \in G$ and let U be a basic neighborhood of α. We say that $\beta \in E$ almost belongs to U if there exists a $\gamma < \alpha$ such that $U = ((\gamma, \alpha] \cap A_\alpha) \cup \{\alpha\}$ and $\beta \in s(\alpha) \cap (\gamma, \alpha)$.*

Claim. *There are $\beta \in E$ and $\{U_n\}_{n\in\omega}$ with $U_n \in \mathcal{U}_n$ such that β almost belongs to U_n for every $n \in \omega$.*

Proof. Assume the contrary. Then for each $\beta \in E$ there exists an $n_\beta \in \omega$ for which β does not almost belong to any $U \in \mathcal{U}_{n_\beta}$. Then there is a stationary $E' \subset E$ and $n \in \omega$ such that $n_\beta = n$ for every $\beta \in E'$. Let

$$P' = \{\alpha \in \omega_1 : \text{ there is a sequence } \{\alpha_n\} \subset E' \text{ with } \alpha_{n+1} > \alpha_n \text{ and } \alpha_n \longrightarrow \alpha\}.$$

As in the proof of Lemma 3.2 for P, one can show that P' is a club. The axiom $\diamondsuit^+$ implies the existence of a club C such that $E' \cap \alpha \in \mathcal{A}_\alpha$ for every $\alpha \in C$. Hence, $P' \cap C$ is club, and so $P' \cap C \cap G$ is stationary.

Let $\alpha \in P' \cap C \cap G$. Then $E' \cap \alpha = A^k_\alpha \in \mathcal{A}_\alpha$ for some $k \in \omega$. As $\alpha \in P'$, for every $\delta < \alpha$ there is a $k \in \omega$ such that $\alpha_k \in E'$ and $\delta \leq \alpha_k < \alpha$. As a consequence, A^k_α is cofinal in α. Therefore there is a sequence $s^\alpha_k \subset A^k_\alpha \cap E \cap (\beta_k, \alpha]$ converging to α. Since $A^k_\alpha \cap E = E' \cap \alpha$ it follows that $s^\alpha_k \subset E'$. The set $U = ((\beta^\alpha_n, \alpha] \cap A_\alpha) \cup \{\alpha\}$ belongs to $\mathcal{U}_n$. Since $s^\alpha_k \longrightarrow \alpha$ we have $s^\alpha_k \cap (\beta^\alpha_n, \alpha) \neq \emptyset$. It is easy to see that every $\gamma \in s^\alpha_k \cap (\beta^\alpha_n, \alpha] \subset E'$ almost belongs to U, which is a contradiction.

Let $\beta \in E$ and let $\{U_n\}_{n\in\omega}$ be as in the claim. Consider the centered family of closed sets $\{F_n\}_{n\in\omega}$, where $F_i = U_i \cap \{\gamma_n(\beta) : n \geq i\}$. Then, $F_i \neq \emptyset$, $F_i \subset U_i \in \mathcal{U}_i$ for each $i \in \omega$ and $\bigcap_{i\in\omega} F_i = \emptyset$, which contradicts the completeness assumed for $\{\mathcal{U}_n\}_{n\in\omega}$. Thus we proved that X is not Čech-complete.

In [BaGrTk, Example 3.5], it was shown that for each $n \in \omega$, the space X^n is the union of two metrizable subspaces A_0 and A_1, where $A_0 = \bigoplus_{\delta\in G} X^\delta$, $A_1 = \bigoplus_{\delta\notin G} X^\delta$, and

$$X^\delta = \{(\alpha_i)_{i<n} \in X^n : \max\{\alpha_i : i < n\} = \delta\}.$$

Each point of X^δ has a countable neighborhood with a unique non-isolated point. This neighborhood is Čech-complete, so every X^δ is locally Čech-complete. Now, every metrizable locally Čech-complete space is Čech-complete and the discrete sum of Čech-complete spaces is Čech-complete. Consequently, A_0 and A_1 are Čech-complete. □

4. Čech-completeness in Topological Groups

We are going to prove that Čech-completeness is finitely additive in topological groups. The result that follows is surely known in folklore but we include its proof for completeness.

Lemma 4.1. *Let X be a topological group. If X is locally Čech-complete then X is Čech-complete.*

Proof. Let $e \in X$ be the identity element and let U be a Čech-complete neighborhood of e. Any Čech-complete space is of pointwise countable type so there is a compact subspace $K \subset U$ of countable character with $e \in K$. Let $\{U_i\}_{i\in\omega}$ be a base at K.

Consider a family $\{W_i\}_{i\in\omega}$ such that $W_i \in \mathcal{T}(e,X)$, $W_i \subset U_i$ and $W_{i+1}^2 \subset W_i = W_i^{-1}$ for all $i \in \omega$. Since $W_{i+1}^2 \subset W_i$, we have $\overline{W_{i+1}} \subset W_i$ for all $i \in \omega$. This implies $\bigcap_{i\in\omega} W_i = \bigcap_{i\in\omega} \overline{W_i}$. The compact set $H = \bigcap_{i\in\omega} W_i \subset K$ is a subgroup. Take the canonical function $f : X \to X/H$ given by $f(x) = Hx$. Since H is compact, it follows that f is closed and perfect too, because $f^{-1}(Hx) = Hx$ is a compact subspace for every $x \in X$. As H has countable character in K and K has countable character in X, the quotient space X/H has countable character. Then X/H is metrizable [Po], and hence it is a paracompact space. Any perfect inverse image of a paracompact space is paracompact. Since $X = f^{-1}(X/H)$, the space X is paracompact. By hypothesis, X is locally Čech-complete, and by [En, problema 5.5 (c)], the space X is Čech-complete. □

Theorem 4.2. *Let X be a topological group. If $X = A_1 \cup A_2 \cup \ldots, \cup A_n$ where A_i is locally Čech-complete for $1 \leq i \leq n$, then X is Čech-complete.*

Proof. By Theorem 2.3, the space X has an open non-empty Čech-complete subspace. Being a homogeneous space, X is locally Čech-complete. Lemma 4.1 implies that X is Čech-complete. □

A topological group X is called *complete in the sense of Raikov* if for every topological group H such that X is a subgroup of H, the set X is a closed subspace of H.

Proposition 4.3. *If a topological group X has a dense Čech-complete subspace then X is complete in the sense of Raikov.*

Proof. Let H a topological group such that $X < H$. As $\overline{X} < H$ we may suppose that $\overline{X} = H$. We will show that $X = H$. If $H - X \neq \emptyset$, take a $g \in H - X$. Then $gX \cap X = \emptyset$. If G is Čech-complete and dense in X, then $G \cap gG \subset X \cap gX = \emptyset$. The subspaces G and gG are dense G_δ-subsets of H and so is their intersection. This means that a countable intersection of dense open sets in H is empty, contradicting Baire property of H. □

Proposition 4.3 and [Ch, Theorem 1] imply

Corollary 4.4. *If a topological group X has a dense Čech-complete subspace (which is not necessary a subgroup) then X is Čech-complete.*

Lemma 4.5. *Let X, Y be topological groups. If $f : X \to Y$ is an open map and X is Čech-complete then Y is Čech-complete.*

Proof. As X is almost Čech complete and f is open, the space Y is almost Čech-complete [AaLu, Theorem 4.2.1]. As a consequence, there is a Čech-complete subspace $D \subset Y$ such that $\overline{D} = Y$. Corollary 4.4 implies that Y is Čech-complete. □

5. Unsolved Problems

We are going to list here some problems that we could not solve while working on this paper.

Problem 5.1. *Let $X = \bigcup_{n\in\omega} U_n$, where $U_n \in \mathcal{T}(X)$ and U_n is Čech-complete for all $n \in \omega$. Is X Čech-complete?*

Problem 5.2. *Let $X^2 = X_0 \cup X_1$ where X_i is locally Čech-complete for $i = 0, 1$. Is X locally Čech-complete?*

Problem 5.3. *Is there a ZFC example of a space X which is not Čech-complete, while $X^2 = X_0 \cup X_1$, where X_i is Čech-complete for $i = 0, 1$?*

Problem 5.4. *Suppose that $X^\omega = \bigcup_{i\in\omega} X_i$, where X_i is Čech-complete for all $i \in \omega$. Is X^ω Čech-complete?*

Problem 5.5. *Suppose that $X^\omega = \bigcup_{i\in\omega} X_i$ where X_i is locally Čech-complete for all $i \in \omega$. Is then X (locally) Čech-complete?*

References

[Tk] V. V. Tkachuk, *Finite and countable additivity topological properties in nice spaces*, Trans. Amer. Math. Soc. (1994), vol. **341**, N2, 585-601.

[OhYa] Haruto Otha and Kohzo Yamada, *Simple examples showing that various topological properties are not finitely additive in the sense of V. V. Tkachuk*, Math. Japonica (1998), Vol. **47**, N1, 93-95.

[En] R. Engelking, *General Topology*, Heldermann Verlag Berlin, 1989.

[Po] L. Pontriagin, *Topological Groups*, Mir. 1977.

[AaLu] J. M. Aarts and D. J. Lutzer, *Completeness properties designed for recognizing Baire spaces*, Dissertationes Mathematicae CXVI, Warszawa 1974.

[ChČoNa] J. Chaber, Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fundamenta Mathematicae (1974), vol. **84**, 107-119.

[Ch] M. M. Choban, *On completion of topological groups* (in Russian), Vestnik Mosk. Univ. Matematika, (1970), N1, 33-38.

[Ku] Kenneth Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in logic and the foundations of mathematics, Volume **102**, North Holland Publishing Company, 1980.

[BaGrTk] Zoltan Balogh, Gary Gruenhage and Vladimir Tkachuk, *Additivity of metrizability and related properties*, Topology and its Applications (1998), vol. **84**, 91-103.

[Fro] Z. Frolík, *Locally G_δ-spaces*, Czech. Math. J.**12**, (1962), 346-355.

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