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LCA HYPERGROUPS

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Abstract

This is an expository introduction to LCA hypergroups with an indication of the basic theory. The point of view of hypergroups clarifies the study of various generalizations of LCA groups. Applications to Markov chains on the hypercube are also discussed. The latter refers to joint work with the author's colleague, Daming Xu.

Part I: Introduction to Hypergroups

Hypergroups, as I understand them, have been around since the early 1970's when Charles Dunkl, Robert Jewett and René Spector independently created locally compact hypergroups with the purpose of doing standard harmonic analysis. As one would expect, there were technical differences in their definitions. The standard, in the non-Soviet world, became Jewett's 101-page paper [J] because he worked out a good deal of the basic theory that people would want. Bloom and Heyer's book [BH] is a report on a good deal of the mathematics that has been done on the basis of Jewett's axioms.

In August of 1993 the first conference on hypergroups was held in Seattle. This first conference was an international conference and was well attended by people from all over the world.

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* dedicated to Edwin Hewitt, 1920-1999

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See [CGS]. As the preface to the proceedings states, "This led to fireworks. Hypergroups occur so often and in so many different and important contexts, that mathematicians all over the world have been discovering the same mathematical structure hidden in very different applications, and publishing theorems about these structures, in many cases without even knowing that they were talking about hypergroups."

In particular, structures very like the Dunkl-Jewett-Spector creations of the 1970's had been studied in the early 1950's by Berezansky and colleagues. The axioms, terminology and language were all different, and the connection was not realized by most workers until the Seattle conference in 1993. These connections are explained in the nice article [BK1], where it is noted that the ideas of hypergroups appear in works of Delsarte and Levitan published in 1938 and 1940. See also the very recent article [BK2]. A fundamental part of their axioms is their "structure measures" $c(A, B, r)$ which for locally compact groups with Haar measure m reduce to $m((A - r) \cap B)$. I think the axioms involving $c(A, B, r)$, which have no counterpart in the axioms of a locally compact group, are unnatural compared to the axioms of Jewett and others. In any case, in the proceedings [CGS] of the conference there was an effort to standardize notation. In many of the papers, the hypergroups studied by Jewett (and used in the book of Bloom and Heyer) are referred to as DJS-hypergroups. These are the hypergroups that I will be talking about.

Incidentally, there was a follow-up conference at Oberwolfach in 1994 that was organized by Herbert Heyer. The proceedings [H] contains many interesting articles, including several on hypergroups.

Here are the axioms for a DJS-hypergroup. As I learned from [K], they have been neatly rephrased by Lasser [L], so I will give Lasser's version. We begin with a locally compact Hausdorff space K, and denote $\boldsymbol{M}(K)$ for the space of all finite complex regular measures on K. $\boldsymbol{M}^1(K)$ will denote the probability measures in $\boldsymbol{M}(K)$. Point masses will be denoted by δ_x. A hypergroup is determined by K and the following data:

(H*) A continuous mapping $(x, y) \to \delta_x * \delta_y$ from $K \times K$ into $\boldsymbol{M}^1(K)$, where $\boldsymbol{M}^1(K)$ has the weak topology with respect to the space $C_c(K)$ of continuous complex-valued functions with compact support. [convolution]

(Hinv) An involutive homeomorphism $x \to \check{x}$ from K to K. [an involution]

(Hid) A fixed element e in K. [an identity element]

After identification of x with δ_x, the mapping in (H*) extends uniquely to a continuous bilinear mapping $(\mu, \nu) \to \mu * \nu$ from $\boldsymbol{M}(K) \times \boldsymbol{M}(K)$ into $\boldsymbol{M}(K)$. And the involution on K gives an involution $\mu \to \mu^*$ on $\boldsymbol{M}(K)$, where $\mu^*(E) = \overline{\mu(\check{E})}$ for each Borel set E in K.

Now a DJS-hypergroup is the quadruple $(K, *, \check{}, e)$ satisfying

(H1) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$ for all $x, y, z \in K$.

(H2) $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ for all $x, y \in K$.

(H3) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$.

(H4) e is in the support $\operatorname{supp}(\delta_x * \delta_{\check{y}})$ if and only if $x = y$.

(H5) $\operatorname{supp}(\delta_x * \delta_y)$ is compact for all $x, y \in K$.

(H6) The mapping $(x, y) \to \operatorname{supp}(\delta_x * \delta_y)$ of $K \times K$ into the space of nonvoid compact subsets of K is continuous, where the latter space is given the "Michael" topology in [J], § 2.5.

Every locally compact group G is a hypergroup satisfying these axioms. In this case, $\delta_x * \delta_y = \delta_{xy}$ for all $x, y \in G$ and $\check{x}$ is the inverse of x. Axioms (H1) – (H4) are clear. Note that axiom (H4) says that $xy^{-1} = e$ if and only if $x = y$. Axiom (H5) is very clear since each $\operatorname{supp}(\delta_x * \delta_y)$ consists of the single element xy. The technical axiom (H6) can also be verified in this setting.

I will only give one rather special class of examples in this talk, but I will briefly list various families of examples. First, as mentioned above, all locally compact groups are hypergroups. Given a compact group, the space of all conjugacy classes forms a compact commutative (!) hypergroup. The space of all finite-dimensional irreducible representations is a discrete commutative hypergroup. Moreover, it turns out that these two commutative hypergroups can be viewed as duals to each other — in a sense that will be discussed in this talk.

If G is a locally compact group and B is a compact group of automorphisms of G, then the space G_B of B-orbits forms a hypergroup in a similar way. In fact, this is just a generalization of the conjugacy-class example mentioned above. Another family of examples consists of double-coset spaces. Finally, some very interesting hypergroups arise from the study of orthogonal polynomials.

A key concept in locally compact groups and hypergroups is that of an invariant measure. For the group case, Alfred Haar proved in 1933 that a (second countable) locally compact group has a left-invariant Borel measure m, which we now call a left Haar measure. Later authors proved that the left Haar measure is unique up to a positive constant. By left-invariant we mean that $m(xE) = m(E)$ for all x in G and all Borel sets E in G. This is equivalent to the requirement that $\delta_x * m = m$ for all x in G, where the convolution here is extended in a natural way so as to apply even if m is an infinite measure. Similarly, a right Haar measure m' is one that satisfies $m' * \delta_x = m'$ for all x in G. Locally compact groups also have right-invariant Haar measures.

As in the group case, a measure m on a DJS-hypergroup K is called a left Haar measure if $\delta_x * m = m$ for all x in K, with a similar definition for right Haar measure. Does every K have a Haar measure? Each of the pioneers Dunkl, Jewett and Spector proved that every compact hypergroup K has a left Haar measure. I believe that each of them also proved the same result for discrete hypergroups. In any case, the latter result is easy. The formula is $m(x) = ([\delta_{\check{x}} * \delta_x](\{e\}))^{-1}$ for all $x \in K$.

The case for commutative hypergroups was substantially harder, but Spector [S] proved that every commutative hypergroup has a Haar measure. Incidentally, Spector uses weaker axioms than the DJS-axioms and pays a price. He doesn't assume axiom (H5). His proofs simplify somewhat if you use the DJS-axioms. Remarkably, *the general question of whether every DJS-hypergroup has a Haar measure is still open.* This is certainly the biggest open question in the subject. By the way, just as in the group case, when the Haar measures exist they are unique up to a constant. This was shown by Jewett [J].

As in the group case, we say that a hypergroup is unimodular if the left and right Haar measures coincide. Groups that are either commutative, compact, or discrete are unimodular. Obviously commutative hypergroups are unimodular, and it is not hard to show that compact hypergroups are also unimodular. Surprisingly, on page 39 of Bloom and Heyer's book it is noted that "in contrast to the group case, it is unknown whether all discrete hypergroups are necessarily unimodular." However, the paper [KW] contains a construction for a class of non-unimodular discrete hypergroups! They arise as double-coset hypergroups induced by the transitive action of a non-unimodular group of permutations on an infinite set.

The most work has been done on commutative hypergroups for the simple reason that they are easier to deal with. As in the case of groups, Fourier and Fourier-Stieltjes transforms play a big role. These are functions that, in the group case, are defined on the character group. In the case of a hypergroup K, they are defined on the space $\hat{K}$ of all hypergroup characters, which might or might not be a hypergroup in its own right. There are even three-element hypergroups K for which $\hat{K}$ is not a hypergroup.

A *hypergroup character* χ on a hypergroup K is a bounded complex-valued continuous function that is not identically zero and satisfies

(i) $\chi(\check{x}) = \overline{\chi(x)}$ for all $x \in K$,

(ii) $\chi(x * y) = \chi(x)\chi(y)$ for all $x, y \in K$.

Each of these requirements deserves comment. With the other requirements, (i) holds automatically for locally compact groups because $xx^{-1} = e$ for all x. This remark is also true on compact hypergroups, but in general there are functions χ that satisfy (ii) and not (i). Property (ii) needs clarification because $x * y$ isn't defined. This very suggestive notation is due to Jewett. In general,

$$f(x * y) = \int_K f d\,\delta_x * \delta_y.$$

In the group case, this is $f(xy)$ just like it should be, but in general we will only use the notation $x * y$ in connection with a function.

Many familiar results carry over to general and to commutative hypergroups. I will only mention the Levitan-Parseval identity, which I will state in the easy case that K is finite and commutative. Then $\hat{K}$ has a Plancherel measure ν, and

Parseval's identity. $\sum_{x \in K} |f(x)|^2 m(x) = \sum_{\chi \in \hat{K}} |\hat{f}(\chi)|^2 \nu(\chi).$

If m is normalized so that $m(K) = 1$, then ν is normalized so that $\nu(1) = 1$.

Let me end part I with a family of very simple hypergroups that will play a role in part II of this lecture. Here are **all** of the two-element hypergroups. For $0 < \theta \leq 1$, the set K will be $\{0, 1\}$, 0 will serve as the identity and the involution will be the identity map. Since 0 is the identity, the three products $\delta_0 * \delta_0 = \delta_0$, $\delta_0 * \delta_1 = \delta_1 * \delta_0 = \delta_1$ are automatic. The only interesting product is

$$\delta_1 * \delta_1 = \theta\delta_0 + (1 - \theta)\delta_1.$$

Since the coefficients have to be nonnegative and add to 1, we must have $0 \leq \theta \leq 1$. Since 0 has to be in the support of $\delta_1 * \delta_1$, we have to have $\theta > 0$. It is easy to check that we get a hypergroup for each θ, which we denote by $Z_\theta(2)$. Note that for $\theta = 1$, we get the familiar two-element group $Z(2)$.

$Z_\theta(2)$ has two characters, the function 1 and the function χ where $\chi(0) = 1$ and $\chi(1) = -\theta$. The normalized Haar measure m_θ is given by $m_\theta(0) = \frac{\theta}{1+\theta}$ and $m_\theta(1) = \frac{1}{1+\theta}$. Again, note that if $\theta = 1$, we get the correct characters and Haar measure on $Z(2)$. If we think of θ as varying continuously starting at 1, then we can view the family $\{Z_\theta(2) : 0 < \theta \leq 1\}$ of hypergroups as a deformation of the group $Z(2)$.

In part II, I will really be interested in the group $Z(2)^d$ of all d-tuples of 0's and 1's. This can be deformed into a big product of hypergroups, namely $Z_\theta(2)^d$. The hypergroup characters of $Z_\theta(2)^d$ are easy to determine in terms of the characters of each factor $Z_\theta(2)$. And the Haar measure m_θ on $Z_\theta(2)^d$ is just the product of the Haar measures on each factor. Thus

$$m_\theta(\boldsymbol{x}) = \frac{\theta^{d-H(\boldsymbol{x})}}{(1+\theta)^d} \quad \text{for all} \quad \boldsymbol{x} \text{ in } Z_\theta(2)^d.$$

Remember, $\boldsymbol{x}$ is a string of 0's and 1's; here $H(\boldsymbol{x})$ is the number of these terms that are equal to 1.

Part II: Markov Chains and Deformations of Hypergroups

I am going to begin by discussing "random walks." The walk needs to be governed by some rules: initially our walks will be in finite groups. Random, as the word suggests, means that at each step the walk will be governed by a probability Q. I will assume that a particle starts at the identity of the group. Q will describe the various first steps and their probabilities. $Q^{(n)}$ will describe the position of the particle after n steps. It turns out that $Q^{(n)}$ is the convolution of Q n-times.

Let U be the uniform probability measure on a finite group G. This is, of course, Haar measure normalized to give the group total mass 1. It has been known since at least the 1940's that $Q^{(n)}$ converges to U unless there are obvious impediments, like the support of Q lying in a proper subgroup of G. In other words, if there were many particles, the particles would be evenly mixed after a while. In the past twenty years, under the guidance of the guru Persi Diaconis, there has been renewed interest in studying how fast $Q^{(n)}$ converges to U. First, one needs a measurement of closeness. Here's a commonly accepted measurement, called the *total variation distance*:

$$\begin{aligned} ||Q^{(n)} - U|| &= \max\{|Q^{(n)}(A) - U(A)| : A \subseteq G\} \\ &= \frac{1}{2} \sum_{y \in G} |Q^{(n)}(y) - U(y)|. \end{aligned}$$

Here are some examples.

(a) Repeated shuffling of a deck of cards can be viewed as a random walk on the large non-abelian group of all permutations of the deck. If one shuffles with the same randomness, then there is a unique probability Q that describes the shuffles. Diaconis and his colleagues have studied the difficult question of determining the rate at which $||Q^{(n)} - U||$ converges to 0. This depends, of course, on the choice of Q but there are interesting choices of Q and choices that reflect real shuffles by real people. The papers [BD] and [AD] provide nice introductions to this subject. The book [D] is the basic "textbook" of the subject.

(b) Interesting random walks occur in much simpler abelian groups, like the cyclic group $Z(q)$ on q elements. One natural Q is the "nearest neighbor random walk" where $Q(1) = Q(q-1) = 0.5$. If q is even, then this lives on a proper subgroup. To avoid this sort of parity problem, we sometimes study the "nearest neighbor or stay at home random walk." Here $Q(1)$, $Q(q-1)$, and $Q(0)$ may all be set equal to one-third.

Before going on, let me mention a powerful, but simple tool, that Diaconis and his co-workers have used. They call it their Upper Bound Lemma. They have a version for non-abelian groups, but I will only discuss finite abelian groups today. This is an easy consequence of Parseval's identity after bounding the norm by an ℓ^2-norm.

Upper Bound Lemma. $\quad 4||Q^{(n)} - U||^2 \le \sum_{\chi \neq 1} |\hat{Q}(\chi)|^{2n}.$

The easiest non-trivial application is to $Z(3)$ with the nearest neighbor random walk $Q(1) = Q(2) = 0.5$. We worked this out in [RX3] and found that $||Q^{(n)} - U||^2 \le 2^{-2n-1}$. For $n = 10$, this yields $||Q^{(10)} - U|| \le 0.000691$. Direct calculation shows that $||Q^{(10)} - U||$ is approximately 0.000650.

(c) Now consider the cube $Z(2)^d$ of all d-tuples of 0's and 1's. For each $\boldsymbol{x}$ in $Z(2)^d$, let $H(\boldsymbol{x})$ be the number of coordinates of $\boldsymbol{x}$ equal to 1. This is the Hamming distance from $\boldsymbol{x}$ to the origin. There's a natural graph of this group where one connects two elements if they differ in exactly one coordinate, i.e., if $H(\boldsymbol{x} - \boldsymbol{y}) = 1$. A natural random walk is the "nearest neighbor random walk" where each of the d elements with $H(\boldsymbol{x}) = 1$ has probability $\frac{1}{d}$. The only difficulty is that there can be parity problems. To avoid these we again use the "nearest neighbor or stay at home" random walk. I.e., $Q(\mathbf{0}) = \frac{1}{d+1}$ and $Q(\boldsymbol{x}) = \frac{1}{d+1}$ for all $\boldsymbol{x}$ with $H(\boldsymbol{x}) = 1$. The upper bound lemma can be used to estimate $||Q^{(n)} - U||$. Some combinatorial estimates are needed. This quantity goes to 0 exponentially with n. For example, if $d = 2$, we find that $4||Q^{(n)} - U||^2 \le (\frac{1}{3})^{2n-1}$.

We can view every random walk on a finite group G as a Markov chain. In fact, the transition matrix M corresponding to Q is given by $M(x, y) = \delta_x * Q(y)$ for all $x, y \in G$. Here δ_x denotes the unit point mass at x and $*$ denotes convolution. Note that $M(e, y) = Q(y)$, where e is the identity of the group. It follows easily by induction that the matrix power $M^n(x, y) = [\delta_x * Q^{(n)}](y)$ for $x, y \in G$. If we want to study Markov chains, there are lots of techniques available. However, if it turns out that the Markov chain is actually a random walk, then we also have the tools of elementary harmonic analysis at our disposal, as illustrated by the use of the Upper Bound Lemma.

Diaconis and Hanlon [DH] studied some special Markov chains called Metropolis Markov chains. As I will explain, the ones that they studied are based on familiar random walks on finite groups, but are not themselves random walks. Metropolis Markov chains are so-named, because of an old paper [M] with five authors, among them Nicholas Metropolis and Edward Teller.

Metropolis Markov chains make sense on any finite set X. Let π be a probability on X with $\pi(x) > 0$ for all $x \in X$. The Metropolis algorithm is a classical Markov chain simulation method for sampling from π, which is effective when the ratios $\frac{\pi(x)}{\pi(y)}$ are available. We begin with a "base chain" $B(x, y)$ which is partly symmetric, i.e., $B(x, y) = 0$ if and only if $B(y, x) = 0$. For us, B will be generated by a random walk. We will use the ratios

$$r_{y,x} = \frac{\pi(y)B(y, x)}{\pi(x)B(x, y)},$$

where we decree that this is 0 if $B(x, y) = B(y, x) = 0$. Here is the corresponding Metropolis Markov chain:

$$\begin{aligned} M(x, y) &= B(x, y) && \text{if } y \neq x \text{ and } r_{y,x} \geq 1; \\ &= B(x, y)r_{y,x} && \text{if } r_{y,x} < 1; \\ &= B(x, x) + E(x) && \text{if } y = x, \end{aligned}$$

where $E(x) = \sum B(x, z)(1 - r_{z,x})$, summed over all $z \neq x$ with $r_{z,x} < 1$. I.e., $E(x)$ is the value needed so that $\sum_{y \in X} M(x, y)$ is equal to 1.

The stationary distribution for this Markov chain will be the original probability π. Diaconis and Hanlon were interested in the convergence rate of these Markov chains, but note that now this may well depend on the starting point. So the variation distance is defined by

$$||M^n(x, \_) - \pi|| = \frac{1}{2} \sum_{y \in X} |M^n(x, y) - \pi(y)|.$$

Consider again the nearest neighbor walk on $Z(2)^d$: $Q(\boldsymbol{x}) = \frac{1}{d}$ if and only if $H(\boldsymbol{x}) = 1$. Note that the corresponding Markov chain B is given by $B(\boldsymbol{x}, \boldsymbol{y}) = \delta_{\boldsymbol{x}} * Q(\boldsymbol{y}) = Q(\boldsymbol{y} - \boldsymbol{x}) = \frac{1}{d}$ if and only if $H(\boldsymbol{x} - \boldsymbol{y}) = 1$. All we need now, to specify a Metropolis Markov chain, is the stationary distribution. Diaconis and Hanlon consider the measures π_θ defined on $Z(2)^d$ by

$$\pi_\theta(\boldsymbol{x}) = \frac{\theta^{H(\boldsymbol{x})}}{(1+\theta)^d}, \quad \text{where} \quad 0 < \theta \leq 1.$$

When $\theta = 1$, this is just the uniform probability measure on the group $Z(2)^d$. Diaconis and Hanlon find the eigenvectors and eigenvalues of the Metropolis Markov chain M in terms of Krawtchouk polynomials, and they establish the rate of convergence of $||M^n(\mathbf{0}, \_) - \pi_\theta||$. I won't give you the complicated result, but this will be small if n is roughly $d \log(d) + c$, where c does not depend on d.

In [RX1], Daming Xu and I analyzed the Diaconis-Hanlon result from a little different point of view. First, recall that the hypergroups $Z_\theta(2)^d$ have Haar measure given by

$$m_\theta(\boldsymbol{x}) = \frac{\theta^{d-H(\boldsymbol{x})}}{(1+\theta)^d}.$$

This is very like the π_θ studied by Diaconis-Hanlon. Since $H(\boldsymbol{x})$ counts the number of 1's in $\boldsymbol{x}$, and $d - H(\boldsymbol{x})$ counts the number of 0's in $\boldsymbol{x}$, a simple change of variable, $\boldsymbol{x} \to \mathbf{1} - \boldsymbol{x}$, in the Diaconis-Hanlon Markov chain changes their stationery distribution to the Haar measure. With this trivial change, we see that they estimated the rate of convergence of $||M^n(\mathbf{1}, \_) - m_\theta||$.

Since random walks on a finite group converge to the Haar measure of the group, unless there are obvious impediments, this suggests that perhaps M is, in fact, a random walk but **on the hypergroup** $Z_\theta(2)^d$. This is the case and the probability measure that generates this random walk is exactly the nearest neighbor random walk Q. The Metropolis Markov chains studied by Diaconis and Hanlon, which can be viewed as deformations of the nearest neighbor random walk, are precisely the nearest neighbor random walk on the hypergroup deformations of the group $Z(2)^d$.

The Upper Bound Lemma mentioned before carries over to hypergroups with no difficulty. It now reads

Upper Bound Lemma. $$4||Q^{(n)} - m||^2 \leq \sum_{\chi \neq 1} |\hat{Q}(\chi)|^{2n} \nu(\chi).$$

Here ν is the Plancherel measure on $\hat{K}$. Since ν is counting measure in the group case, this result is an exact generalization of the original Upper Bound Lemma. For the nearest neighbor random walk Q on the hypergroup $Z_\theta(2)^d$, the summands in the Upper Bound Lemma are easily calculated. Some careful algebraic estimations then lead to exactly the same bounds that Diaconis and Hanlon obtained. We get one benefit, though, because it is now clear the bounds work no matter what point our random walk starts at. That is, the bounds that Diaconis and Hanlon obtained for $||M^n(\mathbf{0}, \_) - \pi_\theta||$ hold for $||M^n(\boldsymbol{x}, \_) - \pi_\theta|| = ||M^n(\mathbf{1} - \boldsymbol{x}, \_) - m_\theta||$.

Using the same methods, Daming Xu and I also obtained similar estimates for the Metropolis Markov chain associated with the nearest neighbor random walk on $Z(3)^d$. In [RX2], we worked hard to obtain similar results involving the groups S_n of all permutations of an n-element set. Again the basic random walk is the nearest neighbor random walk starting at the identity permutation. The nearest neighbors are transpositions. Diaconis and Hanlon studied the Metropolis Markov chain in this setting. They "lumped" the chain to the space K_n of conjugacy classes. The Markov chains so obtained are still not random walks, but they are if we then deform K_n into an object that isn't even a hypergroup. The objects are called *signed hypergroups*.

[RX3] contains a nice expository account of all of this, and much more. In particular, we work out the details for some Metropolis Markov chains on S_3^d and on S_4^d.

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