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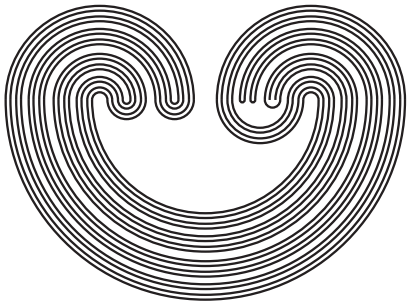

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BANACH SPACES OVER NONARCHIMEDEAN VALUED FIELDS

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Abstract

In this survey note we present the state of the art on the theory of Banach and Hilbert spaces over complete valued scalar fields that are not isomorphic to $\mathbb{R}$ or $\mathbb{C}$. For convenience we treat 'classical' theorems (such as Hahn-Banach Theorem, Closed Range Theorem, Riesz Representation Theorem, Eberlein-Šmulian's Theorem, Krein-Milman Theorem) and discuss whether or not they remain valid in this new context, thereby stating sometimes strong negations or strong improvements. Meanwhile several concepts are being introduced (such as norm orthogonality, spherical completeness, compactoidity, modules over valuation rings) leading to results that do not have -or at least have less important- counterparts in the classical theory.

1. The Scalar Field K

1.1. Non-archimedean Valued Fields and Their Topologies

In many branches of mathematics and its applications the valued fields of the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ play a

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fundamental role. For quite some time one has been discussing the consequences of replacing in those theories $\mathbb{R}$ or $\mathbb{C}$ -at first- by the more general object of a *valued field* $(K, |\quad|)$ i.e. a commutative field K, together with a *valuation* $|\quad| : K \to [0, \infty)$ satisfying $|\lambda| = 0$ iff $\lambda = 0$, $|\lambda{+}\mu| \leq |\lambda|{+}|\mu|$, $|\lambda\mu| = |\lambda|\,|\mu|$ for all $\lambda, \mu \in K$. Then, as in the real or complex case, $(\lambda, \mu) \mapsto |\lambda{-}\mu|$ is a metric on K making K into a *topological field* (i.e. addition, subtraction, multiplication and division are continuous operations). The following theorem essentially separates the absolute value on $\mathbb{R}$ or $\mathbb{C}$ from all other valuations enabling us to avoid carrying out mere generalizations.

Theorem 1. ([5],p.127) *Let* $(K, |\quad|)$ *be a valued field. Then either*
(i) K *is* (*isomorphic to*) *a subfield of* $\mathbb{C}$ *and the valuation induces the restriction topology on* K*, or*
(ii) *the valuation on* K *is non-archimedean* (*n.a.*) *i.e. it satisfies the strong triangle inequality*

$$|\lambda{+}\mu| \leq \max(|\lambda|, |\mu|) \qquad (\lambda, \mu \in K).$$

So, by excluding $\mathbb{C}$ and its subfields one obtains in return the strong triangle inequality (implying obviously the 'ordinary' triangle inequality), which will change the picture of Functional Analysis completely. As a first case in point notice that for each $\lambda \in K$ and $n \in \mathbb{N}$

$$|n\lambda| \leq \max(|\lambda|, |\lambda|, \ldots, |\lambda|) \leq |\lambda|$$

contrasting the Archimedean axiom '$\mathbb{N}$ is unbounded', explaining at the same time the expression '*non*-archimedean'.

The metric $(\lambda, \mu) \mapsto d(\lambda, \mu) := |\lambda{-}\mu|$ is a so-called *ultrametric* i.e. it satisfies the strong triangle inequality $d(\lambda, \nu) \leq \max\big(d(\lambda, \mu), d(\mu, \nu)\big)$. The following properties formulated for general ultrametric spaces illustrate the deviation from 'classical theory' (by this expression we hereafter indicate the mathematics in and over $\mathbb{R}$ and $\mathbb{C}$). They are quite easy to prove and well-known.

Proposition 2. *Let (X, d) be an ultrametric space.*
(i) *For each $a \in X$ and $r > 0$ the 'closed' ball $B(a, r) := \{x \in X : d(x, a) \leq r\}$ and the 'open' ball $B(a, r^-) := \{x \in X : d(x, a) < r\}$ are clopen (i.e. closed-and-open). The topology of X is zero-dimensional, hence totally disconnected.*
(ii) *Each point of any ball in X is a center.*
(iii) *Two balls in X are either disjoint or one is a subset of the other.*
(iv) (Isosceles Triangle Principle) *Let $x, y, z \in X$. Then the 'triangle' $\{x, y, z\}$ is isosceles. In fact, among $d(x, y)$, $d(x, z)$, $d(y, z)$ the largest and second largest distances are equal.*

1.2. Examples of n.a. Valuations

1. The *trivial valuation* can be put on any field and is defined by $|\lambda| = 1$ iff $\lambda \neq 0$. It leads to the trivial metric and the discrete topology. To avoid having to continually make exceptions we exclude it from now on.

2. The most important and famous n.a. valuation is without any doubt the so-called *p-adic valuation*, defined on $\mathbb{Q}$ for any prime p by $|0|_p := 0$ and

$$\left| p^k \frac{t}{n} \right|_p := p^{-k}$$

where $t, n, k \in \mathbb{Z}$, $t, n \neq 0$, $gcd(t, p) = gcd(n, p) = 1$. The completion of $(\mathbb{Q}, |\quad|_p)$ is in a natural way again a n.a. valued field $\mathbb{Q}_p$, the *p-adic number field*. According to Ostrowski's Theorem ([59],10.1) each valuation on $\mathbb{Q}$ is (equivalent to) either the absolute value function or (to) some p-adic one. Thus the various $\mathbb{Q}_p$, being completions of the rationals, can be viewed as possible alternatives to $\mathbb{R}$: all $\mathbb{Q}_p$ are nonisomorphic as fields ([59],33B), no $\mathbb{Q}_p$ is isomorphic to $\mathbb{R}$ ([59],16.7).

3. *The field $\mathbb{C}_p$ of p-adic complex numbers.* $\mathbb{Q}_p$ is not algebraically closed. In fact, its algebraic closure $\mathbb{Q}_p^a$ is infinite-dimensional over $\mathbb{Q}_p$([59],16.7). By a theorem of Krull $|\quad|_p$ can

be uniquely extended to a n.a. valuation on $\mathbb{Q}_p^a$ ([59],14.1,14.2). Although $\mathbb{Q}_p^a$ is not metrically complete ([59],16.6), fortunately its completion $\mathbb{C}_p$ is algebraically closed by Krasner's Theorem ([59],17.1). It is reasonable to call $\mathbb{C}_p$ the field of the *p-adic complex numbers*.

4. *Rational function fields.* Let F be any field. For $f = a_0 + a_1X + \cdots + a_nX^n \in F[X]$, set $|f| := 2^n$ if $a_n \neq 0$, $|0| := 0$. The function $|\ |$ extends uniquely to a n.a. valuation on the quotient field $F(X)$. It is a way to obtain n.a. valued fields of nonzero characteristic.

1.3. A Few More Non-archimedean Features

FROM NOW ON IN THIS NOTE $K = (K, |\ \ |)$ IS A COMPLETE N.A. VALUED FIELD. WE PUT $|K| := \{|\lambda| : \lambda \in K\}$.

The set $|K|\backslash\{0\}$ is a multiplicative group, called the *value group* of K. It need not be all of $(0, \infty)$. In fact, the value group of $\mathbb{Q}_p$ is $\{p^n : n \in \mathbb{Z}\}$. (See 1.2.2. Completion does not alter the value group!) Even the value group of $\mathbb{C}_p$ (1.2.3), $\{p^r : r \in \mathbb{Q}\}$, is not equal to $(0, \infty)$, according to a theorem ([59],16.2) stating that the value group of the algebraic closure of K is the divisible hull of the value group of K. We will call the valuation *discrete* or *dense* accordingly as the value group is discrete or dense in $(0, \infty)$.

By the strong triangle inequality the *closed unit disk* $B_K := \{\lambda \in K : |\lambda| \leq 1\}$ is an additive group and therefore a subring of K containing 1. The *open unit disk* $B_K^- := \{\lambda \in K : |\lambda| < 1\}$ (but recall that B_K and B_K^- are clopen see 1.1) is a maximal ideal in B_K since $B_K\backslash B_K^-$ consists of invertible elements. The quotient $k := B_K/B_K^-$ is called the *residue class field* of K. Even though it is an algebraic object it plays a key role in p-adic analysis (e.g. maximum principle for analytic functions, orthogonality theory). The residue class field of $\mathbb{Q}_p$ is the field $\mathbb{F}_p$ of p elements; the residue class field of $\mathbb{C}_p$ is the algebraic closure of $\mathbb{F}_p$ ([59],16.4). The following noteworthy result characterizes local compactness of K.

Theorem 3. ([59],12.2) *K is locally compact iff its residue class field is finite and its value group is discrete.*

$\mathbb{Q}_p$ is locally compact; $\mathbb{C}_p$ is not.

$(K, |\quad|)$ is called *maximally complete* if for any valued field extension $(L, |\quad|) \supset (K, |\quad|)$, for which K and L have the same value group and (naturally) isomorphic residue class fields, it follows that $L = K$. An ultrametric space is called *spherically complete* if each nested sequence $B_1 \supset B_2 \supset \ldots$ of balls has a nonempty intersection. Unlike ordinary completeness, there is no requirement that the diameters of the balls B_i approach 0. Spherical completeness of K is equivalent to maximal completeness ([57],4.47). Clearly spherical completeness implies ordinary completeness and locally compact, more generally, discretely valued complete fields are spherically complete. $\mathbb{C}_p$ is complete but not spherically complete ([59],20.6). Each $(K, |\quad|)$ admits a *spherical completion* $(\widetilde{K}, |\quad|)$ i.e. a, in some natural sense, 'smallest' spherically complete field extension of $(K, |\quad|)$, see ([57],4.49).

1.4. Outlook

The fields $\mathbb{R}$ and $\mathbb{C}$ are separable, locally compact and connected. $\mathbb{R}$ is ordered and $\mathbb{C}$ is algebraically closed. Our n.a. valued fields K in general have none of those properties. On the other hand we do have the strong triangle inequality which is responsible for exciting deviations from classical theory.
Non-archimedean theories have been developed in many areas such as Elementary Calculus, Number Theory, Algebraic Number Theory, Algebraic Geometry, the Theory of Analytic Functions in one or several variables and Functional Analysis. For an impression we refer to the Mathematics Subject Classification of the Mathematical Reviews. In this note we focus on Functional Analysis (MR 46S10, 47S10), in particular the theory of Banach spaces.

In this context we mention briefly a connection with theoretical physics. In 1987 I. Volovich [80] discussed the question as to whether at Planck distances ($\leq 10^{-34}$ cm) space must be disordered or disconnected. He suggested the use of p-adic numbers to build adequate mathematical models. It brought about a sequence of publications by several authors. See [79] (Hamiltonian equations on p-adic spaces), [29] (Cauchy problems, distribution theory, Fourier and Laplace transform), [30] (p-adic Quantum Mechanics and Hilbert spaces) for an impression of proposed applications of p-adic analysis to theoretical physics. See also the 'forerunning' papers [17] and [78].

2. Banach Spaces

2.1. Definition and Examples

Throughout §2 scalars are elements of a complete n.a. valued field $K = (K, |\quad|)$. A *norm* on a K-vector space E is a map $\|\quad\| : E \to [0, \infty)$ satisfying $\|x\| = 0$ iff $x = 0$, $\|\lambda x\| = |\lambda|\,\|x\|$, $\|x+y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in E$, $\lambda \in K$. (One may consider 'norms' that do not satisfy the strong triangle inequality, for example the obvious translations of ℓ^p-spaces ($1 \leq p < \infty$). However, such 'norms', called A-norms by some authors, see [37], [38], [39], are pathological from the viewpoint of duality theory and will therefore be ignored in this note.)
We often write E to indicate the *normed space* $(E, \|\ \cdot\ \|)$. A complete normed space is, as usual, called a *Banach space*.

The spaces c_0, c, ℓ^∞ consisting of all null, convergent, bounded sequences in K respectively, more generally the space $BC(X)$ of all bounded K-valued continuous functions on a topological space X and closed subspaces of $BC(X)$, equipped with the sup norm are obvious examples of Banach spaces. It is a curious fact that 'most' norms encountered in non-archimedean daily life turn out to be sup norms. Be that as it may, we provide two examples of norms of different nature.

1. Let L be a complete valued field containing K as a valued subfield. Then L is in a natural way a Banach space (even a normed division algebra) over K. Thus, $\mathbb{C}_p$ is an infinite-dimensional Banach space over $\mathbb{Q}_p$. Observe that $|\mathbb{C}_p|$ strictly contains $|\mathbb{Q}_p|$ showing that nonzero vectors cannot always be normalized. This leads to the question as to whether one can find on any Banach space an equivalent norm with values in $|K|$. This is still an open problem. For some discussion and partial solutions, see [55] II, [74].

2. A rather 'exotic' example of a Banach space over $\mathbb{Q}_p$ can be constructed as follows. Let L be an algebraically closed complete valued field with characteristic $p \neq 0$. One proves without pain that $G := \{x \in L : |1-x| < 1\}$ is a multiplicative group (!) that is divisible and torsion free. For each $x \in G$ the map $r \mapsto x^r$ $(r \in \mathbb{Q})$ is therefore well-defined; it extends by p-adic uniform continuity to a continuous function $\mathbb{Q}_p \to G$, written as $\lambda \mapsto x^\lambda$. Now let $(x, y) \mapsto xy$ $(x, y \in G)$ act as addition, let $(\lambda, x) \mapsto x^\lambda$ $(\lambda \in \mathbb{Q}_p, x \in G)$ be the scalar multiplication, and let the norm be defined by $\|x\| := -(\log|1-x|)^{-1}$ if $x \neq 1$, $\|1\| := 0$. Then this way G becomes a $\mathbb{Q}_p$-Banach space. (It is used in [61] to obtain results on character groups.)

2.2. Local Convexity

In the above spirit one defines *seminorms* and *locally convex spaces over* K i.e. vector spaces whose topology is induced by a collection of seminorms satisfying the strong triangle inequality in the usual way. For each seminorm p and $\varepsilon > 0$ the sets $\{x : p(x) \leq \varepsilon\}$ and $\{x : p(x) < \varepsilon\}$ are *absolutely convex* i.e. they are modules over the ring B_K (see 1.3). A subset C of a K-vector space is called *convex* if $x, y, z \in C$, $\lambda, \mu, \nu \in B_K$, $\lambda+\mu+\nu = 1$ implies $\lambda x+\mu y+\nu z \in C$. It is easily proved that C is convex iff C is either empty or an additive coset of an absolutely convex set.

Locally convex spaces will not be our main concern here; see [77], [60], [12] for some basic theory. But we will use the fact that absolutely convex sets, being B_K-modules, carry a richer algebraic structure than their archimedean counterparts (see 2.17). A natural generalization is the study of normed or locally convex B_K-modules, see [47]. The quotient A/B of two closed absolutely convex sets $B \subset A$ in a Banach space is a meaningful object (Banach B_K-module) in n.a. theory!

2.3. Basics on Banach Spaces

For Banach spaces E, F let $\mathcal{L}(E,F)$ be the space of all continuous linear maps $E \to F$. Each $T \in \mathcal{L}(E,F)$ is Lipschitz in the sense that there exists a Lipschitz constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in E$; let $\|T\|$ be the smallest Lipschitz constant. Then $T \mapsto \|T\|$ is a norm making $\mathcal{L}(E,F)$ into a Banach space. As is customary, we write $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$ and E' for $\mathcal{L}(E,K)$. The natural map $j_E : E \to E''$ is defined as in the classical case. E is *reflexive* if j_E is a bijective isometry. We will see in 2.7 that j_E need not be injective in general due to the restricted validity of the Hahn-Banach Theorem. In a finite-dimensional space all norms are equivalent (most classical proofs carry over). The Uniform Boundedness Principle, the Banach Steinhaus Theorem, the Closed Graph Theorem, the Open Mapping Theorem all rest on the Baire Category Theorem and some linearity considerations and therefore remain valid in our theory ([57],3.5,3.11,3.12, [18]).

A sequence $x_1, x_2, \ldots$ in a Banach space is summable iff $\lim_{n\to\infty} x_n = 0$ (this follows readily from the strong triangle inequality). As summability is automatically unconditional, this opens the way to defining, in a natural manner, sums over arbitrary indexing sets. Absolute summability plays no a role in n.a. analysis.

2.4. Orthogonality

The finest examples of classical Banach spaces are Hilbert spaces so one may look for a non-archimedean pendant. A complete discussion will take place in Section 3 from which we single out one particular, negative, result.

Theorem 4. *Let* $(\ ,\ )$ *be a bilinear* (*or Hermitean*) *form on a Banach space* E *such that* $|(x,x)| = \|x\|^2$ *for all* $x \in E$. *Suppose that each closed subspace admits an orthogonal projection. Then* E *is finite-dimensional.*

Even though Hilbert-like spaces may be absent in our theory (but see Section 3), we do have a powerful typically non-archimedean concept, valid in any Banach space. We say that a vector x is (*norm*)*orthogonal* to a vector y if the distance of x to the space Ky is just $\|x\|$. Of course this notion is also meaningful in classical theory but there it is rarely symmetric (actually only in Hilbert space and there it is equivalent to 'form' orthogonality in the usual sense). From our strong triangle inequality it follows easily that orthogonality is always symmetric. In fact, x and y are orthogonal iff for all $\lambda, \mu \in K$

$$\|\lambda x + \mu y\| = \max(\|\lambda x\|, \|\mu y\|).$$

This formula can naturally be extended so as to define orthogonality for an arbitrary collection of vectors.
All maximal orthogonal systems have the same cardinality (([57],5.2); the proof uses 'reduction' to vector spaces over the residue class field, see 1.3). Orthogonality is stable for small perturbations ([57],5.B), showing the contrast with form orthogonality in Hilbert space. An orthogonal system $(e_i)_{i\in I}$ of nonzero vectors, for some index set I, is called an *orthogonal base* of a Banach space E if each $x \in E$ can be expanded as $\sum_{i\in I} \lambda_i e_i$ (see 2.3) where $\lambda_i \in K$. This representation is unique and $\|x\| = \max_{i\in I} \|\lambda_i e_i\|$ ([57],pp. 170,171).

Examples. The unit vectors

$$e_1 = (1, 0, 0, \dots), \; e_2 = (0, 1, 0, \dots), \dots$$

form an orthonormal base of c_0, i.e. $e_1, e_2, \dots$ is an orthogonal base and $\|e_n\| = 1$ for each n. Mahler proved that the binomial polynomials $e_n : x \mapsto \binom{x}{n}$ $(n = 0, 1, 2, \dots)$ form an orthonormal base of $C(\mathbb{Z}_p \to \mathbb{C}_p)$, the space of all continuous (automatically bounded) functions with domain the (compact) unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$ and range $\mathbb{C}_p$. ([59],51.1). If K is algebraically closed the functions $x \mapsto x^n$ $(n = 0, 1, 2, \dots)$ on B_K form an orthonormal base of the space of analytic functions $B_K \to K$ ([59],42.1). (For the theory of n.a. analytic functions, see [16], [8], [9].) If K has a dense valuation, then ℓ^∞ does not have an orthogonal base ([57],5.19).

Spaces with an orthogonal base act like projective objects in the category of Banach spaces as the following theorem illustrates. A linear surjection T of a Banach space E to a Banach space F is called *strict* if for each $a \in E$, $r := \min\{\|x\| : x \in E, Tx = Ta\}$ exists and $\|Ta\| = \|T\|r$.

Theorem 5. ([57],5.7,5.8) (i) *Each Banach space is the quotient of some space with an orthogonal base by a closed subspace.*
(ii) *A closed subspace of a space with an orthogonal base has also an orthogonal base.*
(iii) *A Banach space E has an orthogonal base iff it is projective in the following sense. For each Banach space X and strict linear surjection $T : X \to E$ there is an $S \in \mathcal{L}(E, X)$ such that TS is the identity on E and $\|T\| \, \|S\| \leq 1$.*

2.5. Schauder Bases

The classical definition of separability is not suitable in n.a. theory, as K itself may not be separable. Therefore we 'linearize' this notion so as to obtain a convenient concept as follows. We say that a Banach space E is *of countable type* if there is a countable set in E whose linear hull is dense in E. It is easily seen that it coincides with separability in case K is separable. Like in the classical theory we say that a sequence $x_1, x_2, \dots$ is a *Schauder base* for E if each $x \in E$ has a unique expansion

$x = \sum_{n=1}^{\infty} \lambda_n x_n$, where $\lambda_n \in K$. By unconditional convergence (2.3), for every permutation σ of $\mathbb{N}$ the sequence $x_{\sigma(1)}, x_{\sigma(2)}, \ldots$ is a Schauder base.
As usual one applies the Open Mapping Theorem to show that the coordinate functions $x \mapsto \lambda_n$ are continuous. The following striking results show the simplicity of the non-archimedean theory.

Theorem 6. *Each Banach space of countable type has a Schauder base.*

The proof consists of carrying out a norm version of the Gram-Schmidt process and leads to a much stronger result ([57],5.5,3.16). In fact, if K is spherically complete one can choose the Schauder base to be orthogonal. In general one can find 'almost' orthogonal bases in the sense that for each $t \in (0,1)$ there is a Schauder base $e_1, e_2, \ldots$ such that $\| \sum_{n=1}^{m} \lambda_n e_n \| \geq t \max_n \|\lambda_n e_n\|$ for all $m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \in K$.

Theorem 7. ([57],3.7) *Each Schauder base is orthogonal with respect to an equivalent norm.*

Corollary 8. *All infinite-dimensional Banach spaces of countable type are linearly homeomorphic (to c_0).*

The theory of Schauder bases is becoming more varied in the context of locally convex spaces [11] or when the scalars are from an infinite rank valued field ([45],3.2-3.4).

2.6. $C(X)$

For a compact topological (Hausdorff) space X, let $C(X)$ be the Banach space of all continuous functions $X \to K$ with the norm $f \mapsto \|f\|_\infty := \max\{|f(x)| : x \in X\}$. We assume X to be zerodimensional in the sense that the clopen sets form a base for the topology; this demand is natural since it is equivalent to separation of points by continuous functions ([57],5.23). It is proved in

[57],5.22, that $C(X)$ has an orthonormal base consisting of K-valued characteristic functions of clopen subsets. It is not hard to show that $C(X)$ is of countable type iff X is ultra-metrizable. The *Stone-Weierstrass Theorem* (a unitary subalgebra of $C(X)$ separating the points of X is dense in $C(X)$) holds ([57],6.15); see [51] for wide generalizations as well as [4].
Let Y be a second zerodimensional Hausdorff compact space. As in the classical theory, a ring isomorphism between $C(X)$ and $C(Y)$ induces a homeomorphism between Y and X. But the *Banach-Stone Theorem* (if $C(X)$ and $C(Y)$ are isomorphic Banach spaces then X and Y are homeomorphic) does not hold in our theory, since $C(X)$ is linearly isometric to c_0 for all infinite metrizable X. This failure gave rise to the study of so-called *Banach-Stone maps* $C(X) \to C(Y)$ i.e. maps that induce homeomorphisms $Y \to X$, see [2] and [6] for this interesting theory.
A *measure* on X is a bounded additive map $\mu : \Omega \to K$, where Ω is the ring of clopen subsets of X. (Requiring σ-additivity on, say, the class of Borel sets leads to trivialities, see ([59],A.5).) A measure μ induces naturally an *integral* $\int_X f d\mu$, first for 'step functions' f (linear combinations of characteristic functions of clopen sets), next by continuity for continuous f. For integration theory, such as construction of L^1, measurable functions, also for more general X, see [57],Ch.7. Under obvious operations and the norm $\mu \mapsto \|\mu\| := \max\{|\mu(V)| : V \in \Omega\}$ the measures form a Banach space $M(X)$. The *non-archimedean Riesz Representation Theorem* (see [57],7.18, also for generalizations) asserts that the map assigning to each $\mu \in M(X)$ its integral $f \mapsto \int_X f d\mu$ is an isometrical bijection $M(X) \to C(X)'$. Its proof, however, is much simpler than in the archimedean case.
We consider an Ascoli type theorem in 2.13.

2.7. The Hahn-Banach Theorem

The proof of the Hahn-Banach Theorem for real scalars uses the fact that a collection of closed intervals with the finite intersection property has a non-empty intersection. In our case we need spherical completeness.

Theorem 9. ([59],A.8) *Let K be spherically complete, and let D be a subspace of a Banach space E. Then every $f \in D'$ can be extended to an $\overline{f} \in E'$ such that $\|\overline{f}\| = \|f\|$.*

If K is allowed to be non-spherically complete we have the following.

Theorem 10. ($(1+\varepsilon)$-Hahn-Banach [57],3.16vi.) *Let E be a Banach space of countable type, let D be a subspace of E, $f \in D'$, and $\varepsilon > 0$. Then f can be extended to an $\overline{f} \in E'$ such that $\|\overline{f}\| \leq (1+\varepsilon)\|f\|$.*

It is a corollary of the fact that closed subspaces of spaces of countable type have 'almost' orthogonal (closed) complements, see ([57],3.16v).

One cannot take $\varepsilon = 0$ in the above setting; there exist two-dimensional counterexamples ([57],p.68). If E is no longer of countable type the conclusion fails as well. For example the map $(a_1, a_2, \dots) \mapsto \sum a_n$ is in c_0' but cannot be extended to an element of $(\ell^\infty)'$ if K is not spherically complete ([57],4.15). Even worse, the dual of ℓ^∞/c_0 is $\{0\}$! ([57],4.3).
The converse of the above theorem –for nonspherically complete K, does the $(1+\varepsilon)$-Hahn-Banach property imply that E is of countable type?– is a long-standing open problem in n.a. analysis.

To deal properly with the limited validity of the Hahn-Banach Theorem we define a Banach space E to be *(norm)polar* if for each $x \in E$, $\|x\| = \sup\{|f(x)| : f \in \mathcal{F}\}$ for some collection $\mathcal{F} \subset E'$. Polarity is equivalent with the $(1+\varepsilon)$ Hahn-Banach property for finite-dimensional subspaces D and also with the isometrical property of $j_E : E \to E''$. The category of polar spaces is well-behaved with respect to duality theory; see [60]. Function spaces with the sup norm (such as $BC(X)$, ℓ^∞, E') are polar.
Sometimes a property, stronger than polarness, is needed, called property $(*)$:

'For each subspace D of countable type each $f \in D'$ can be extended to an $\overline{f} \in E''$'.

Banach spaces with an (orthogonal) base have $(*)$. The converse is an open problem, see ([55],Ch.V).

For geometrical versions of the Hahn-Banach Theorem (separation of convex sets by hyperplanes), see [56].

2.8. Spherically Complete Banach Spaces

Spherically complete Banach spaces play the role of injective objects; compare the dual properties of spaces with an orthogonal base in 2.4.

Theorem 11. (i) *Each Banach space has a 'spherical completion'* ([57],4.43).
(ii) *Quotients of spherically complete Banach spaces are spherically complete* ([57],4.2).
(iii) *A Banach space E is spherically complete iff it is injective in the following sense. For each Banach space X and linear isometry $T : E \to X$, there is an $S \in \mathcal{L}(X, E)$ such that ST is the identity on E* ([57],4.H).

Further Properties. K is spherically complete iff ℓ^∞ is spherically complete ([57],4.A). For every K the space ℓ^∞/c_0 is spherically complete ([57],4.1). If E is spherically complete and K is not, then $E' = \{0\}$ ([57],4.3). For any K, if ℓ^∞ is a subspace of a polar Banach space E and $\varepsilon > 0$, then there is a projection $E \to \ell^\infty$ of norm $\leq 1+\varepsilon$ ([67],1.2).

2.9. The Closed Range Theorem

For $T \in \mathcal{L}(E, F)$, where E, F are Banach spaces, we define as usual its *range* $R(T)$ by TE and its adjoint $T' \in \mathcal{L}(F', E')$ by $T'(f) := f \circ T$. The *weak topology* on E and the *weak* topology* on E' are defined as usual. The Hahn-Banach Theorem 9 can be applied to derive a direct translation of the classical Closed Range Theorem.

Theorem 12. ([70],6.7, [28]) *Let K be spherically complete, let E, F be Banach spaces. For any $T \in \mathcal{L}(E, F)$ the following are equivalent.*
(α) *$R(T)$ is closed.*
(β) *$R(T)$ is weakly closed.*
(γ) *$R(T')$ is closed.*
(δ) *$R(T')$ is weak*-closed.*

If K is not spherically complete, one has to impose severe restrictions; even $(\alpha) \Longrightarrow (\beta)$ is not always true as c_0 is not weakly closed in ℓ^∞, but weakly dense! ([57],4.15). The conclusion of Theorem 12 remains true if E, F are of countable type. The failure in the general case is demonstrated by an example of a Banach space E, with an orthogonal base, and a closed subspace D such that for the embedding $T : D \to E$ we have that $R(T)$ is weakly closed; $R(T')$ is not even norm closed while $R(T'')$ is w^*-closed ([70],6.8,6.12).

2.10. Reflexivity

One might expect a rich theory of reflexivity for spherically complete K as linear functions are well-behaved in this case (Hahn-Banach Theorem 9, Closed Range Theorem 12). Alas, we have the following disappointing fact.

Theorem 13. ([57],4.16) *Let K be spherically complete. Then the only reflexive Banach spaces are the finite-dimensional ones.*

Even more astonishing is the existence of infinite-dimensional reflexive spaces when K is not spherically complete. Classically the dual of c_0 is ℓ^1. The strong triangle inequality is responsible for the fact that in our case $c_0' = \ell^\infty$ (for any K).

Theorem 14. ([57],4.17,4.18) *Let K be not spherically complete. Then*
(i) *c_0 and ℓ^∞ are reflexive; they are in a natural way each other's dual,*
(ii) *each Banach space of countable type is reflexive.*

The n.a. theory of reflexivity is far from complete. One can extend (i) to natural generalizations $c_0(I)$ and $\ell^\infty(I)$ provided I is of nonmeasurable cardinal ([57],4.21). If E is reflexive, then so is E' ([57],4.L). If E'' is reflexive, then so is E' ([57],4.25). However, van Rooij constructed a non-reflexive closed subspace of the reflexive space ℓ^∞ over a non-spherically complete K ([57],4.J). The completed tensor product $\ell^\infty \widehat{\otimes} \ell^\infty$ with the usual π-norm is not reflexive ([50],2.3). For some time it was conjectured that every dual space with nonmeasurable cardinality over a nonspherically complete field is reflexive, but a counterexample was given in [15],7.33.
With the help of Goldstine's Theorem (see 2.17) one proves easily that a Banach space is reflexive iff its closed unit ball is weakly complete. For more, see 2.17.

In locally convex theory reflexivity is more varied. For example Fréchet spaces of countable type (K not spherically complete), Montel spaces (all K), and certain inductive limits are reflexive ([60], [12]).

2.11. Eberlein-Šmulian Theory [31]

For a subset X of a Banach space E, consider the following statements.
(α) X is weakly compact.
(β) X is weakly sequentially compact.
(γ) X is weakly countably compact.
The classical Eberlein-Šmulian Theorem states that (α), (β), and (γ) are equivalent, the interesting implications being $(\gamma) \Longrightarrow (\alpha)$ and $(\gamma) \Longrightarrow (\beta)$ (since $(\alpha) \Longrightarrow (\gamma)$ and $(\beta) \Longrightarrow (\gamma)$ are true in general). In the n.a. case we have:

Theorem 15. *Let X be a subset of a normpolar Banach space E. If either* (i) *K is spherically complete or* (ii) *E has* $(*)$ (see 2.7) *or* (iii) *E' is of countable type or* (iv) $\overline{[X]}$ (*the Banach space generated by X*) *is of countable type, then $(\alpha), (\beta), (\gamma)$ are equivalent. Moreover, each weakly convergent sequence in E is norm convergent and E is weakly sequentially complete.*

But the story does not stop here; there are strong improvements: If, in addition, K is *not* locally compact then X is weakly compact iff $f(X)$ is compact for every $f \in E'$. That this peculiar result does not hold for locally compact K (neither in the archimedean case) is easily seen by taking $E := K^2$, $X := \{x \in E : 0 < \|x\| \leq 1\}$. If (i) or (ii), then X is weakly compact iff X is norm compact! If neither (i), (ii), (iii), nor (iv) are satisfied, the conclusion of the Theorem fails. There are counterexamples to $(\alpha) \Longrightarrow (\beta)$ and to $(\beta) \Longrightarrow (\alpha)$.

2.12. The Banach-Dieudonné and Krein-Šmulian Theorem

Let E be a metrizable locally convex space. Let τ be the strongest topology on E' that coincides on equicontinuous sets with the w^*-topology. Let τ_{pc} be the topology on E' of uniform convergence on precompact subsets of E.

The classical Banach-Dieudonné theorem states that $\tau = \tau_{pc}$. Now let us consider the n.a. case. The conclusion $\tau = \tau_{pc}$ holds if K is locally compact ([49],1.2). For non-locally compact K there are Fréchet counterexamples, where τ is not even a linear topology ([49],2.7,2.8). A Banach counterexample is not known. But we do have:

Theorem 16. ([25],4.5, [49],1.5) *Let E be a polar metrizable locally convex space (i.e. the topology is generated by some collection of polar seminorms). Let τ_ℓ be the strongest linear topology on E' coinciding on equicontinuous sets with the w^*-topology. Then $\tau_\ell = \tau_{pc}$.*

As a corollary we obtain for polar Banach spaces E that an absolutely convex set A in E' is τ_{pc}-open iff $A \cap B$ is w^*-open in B for each ball B in E'. But also we have the n.a. version of the Krein-Šmulian Theorem.

Theorem 17. ([49],1.6) *Let E be a Banach space, let K be spherically complete. Let $A \subset E'$ be absolutely convex. If $A \cap B$ is w^*-closed in E' for each ball B in E' then A is w^*-closed.*

The conclusion fails if K is not spherically complete and E' is infinite-dimensional. But it does hold if E is of countable type and for edged absolutely convex A (for an absolute convex A we define $A^e := A$ if the valuation of K is discrete, $A^e := \bigcap \{\lambda A : \lambda \in K, |\lambda| > 1\}$ otherwise. A is called *edged* if $A = A^e$), and also in general polar Banach spaces for subspaces A of finite codimension ([49],1.10, [68],3.1).

2.13. Compactoids

Convex sets (see 2.2) consisting of at least two points contain line segments which are homeomorphic to the unit ball B_K of K. Thus, if K is not locally compact, convex compact sets in Banach spaces are trivial. To overcome this difficulty we 'convexify' the concept of (pre)compactness as follows. Recall that a set X in a (Hausdorff) locally convex space E is precompact if for each zero neighbourhood U there exists a finite set F in E such that $X \subset U + F$. We say that X is (*a*) *compactoid* in E if the above is true, where $X \subset U + F$ is replaced by $X \subset U + \text{co } F$. Here co F, *the absolutely convex hull of* F, equals $\{\sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_1, \ldots, x_n \in F, \lambda_1, \ldots, \lambda_n \in B_K\}$. It is easily seen that, if K is locally compact, compactoidity is the same as precompactness. Also, in the above we may choose F in the linear span of X [22] so that compactoidity does not depend on the embedding space E. The crucial property making this concept useful is that compactoidity of X implies compactoidity of co X. Intuitively (complete) compactoids should assume the role played by (compact) precompact sets in classical analysis.

The following properties are straightforward to prove. The closure of a compactoid is a compactoid. Its linear hull is of countable type. Bounded finite-dimensional sets are compactoid. If E has a compactoid zero neighbourhood then E is finite-dimensional. Compactoids are bounded.

The next results are somewhat more involved and show the analogy with the classical theory.

Theorem 18. *Let X be a subset of a Banach space E.*
(i) ([57],4.37) *X is a compactoid iff there is a sequence $e_1, e_2, \dots$ tending to 0 such that $X \subset \overline{\text{co}}\{e_1, e_2, \dots\}$ (the bar indicating topological closure).*
(ii) ([57],4.38) *If X is absolutely convex and compactoid and $\lambda \in K$, $|\lambda| > 1$, then there is a sequence $e_1, e_2, \dots$ in λX tending to 0 such that $X \subset \overline{\text{co}}\{e_1, e_2, \dots\}$. If K is spherically complete $e_1, e_2, \dots$ can be chosen to be orthogonal.*
(iii) ([69],2.5,2.6) *If X is a closed compactoid then any Hausdorff locally convex topology on E weaker than the norm topology coincides with that norm topology on X. If $T \in \mathcal{L}(E, F)$ for some Banach space F and X is in addition absolutely convex then $(TX)^e$* (see 2.12) *is closed. (In general TX is not closed.)*
(iv) *X is a compactoid iff each basic sequence in X tends to 0* [65]. (A sequence is called *basic sequence* if it is a Schauder base of its closed linear span.)

Theorem 19. ([72],3.1) (Non-archimedean Alaoglu Theorem.) *Let E be a polar Banach space. Then the unit ball of E' is, for the w^*-topology, a complete edged* (see 2.12) *compactoid.*

Theorem 20. ([35],Cor.3) (Non-archimedean Ascoli Theorem.) *Let X be a compact topological space. A subset of $C(X)$ is compactoid iff it is pointwise bounded and equicontinuous.*

In connection with compactoids, we should mention the study on Kolmogorov diameters of [15], [24], volume function [55], [56], and almost periodic functions [66], [14], [13]. A theory of compactoids in the context of locally convex B_K-modules can be found in [47].

2.14. C-Compactness

For spherically complete K the analogy between 'complete convex compactoid' and the classical 'compact convex' becomes particularly striking. We say [76] that a closed convex subset X of a locally convex space E is *c-compact* if every collection of closed

convex subsets of X with the finite intersection property has a nonempty intersection. This is, indeed, a 'convexification' of the definition of compactness by closed sets. However, since K is c-compact if and only if K is spherically complete, this notion is useful only for spherically complete K. The connection with 2.13 was first given by Gruson in [20]:

Theorem 21. ([56],6.15, [62],2.2) *Let X be a bounded absolutely convex subset of a locally convex space E over a spherically complete field K. Then X is c-compact iff it is a complete compactoid.*

C-compact sets behave slightly better than general complete compactoids. For example, the continuous linear image of a c-compact set is c-compact (compare (iii) of Theorem 18). If $x_1, x_2, \dots$ is a compactoid sequence in a Banach space over a spherically complete K, then there exist a_n in the convex hull of $\{x_n, x_{n+1}, \dots\}$ such that $a_1, a_2, \dots$ converges ([10],Prop.2). This is not true if K is not spherically complete: let $B_1 \supset B_2 \supset \dots$ be balls in K with empty intersection, choose $x_n \in B_n \backslash B_{n+1}$.

2.15. Compact Operators [74], [28], [64], [22]

Let E and F be Banach spaces. We call an operator $T \in \mathcal{L}(E,F)$ *compact* if the image of the unit ball of E is a compactoid (see 2.13). As in the classical theory the compact operators $E \to E$ form a closed two-sided ideal $C(E)$ in the algebra $\mathcal{L}(E)$, containing the finite rank operators. We even have that $T \in \mathcal{L}(E)$ is compact iff it is the (norm) limit of a sequence of finite rank operators.

The Riesz theory on compact operators can pretty well be extended. The *spectrum* $\sigma(T) := \{\lambda \in K : T - \lambda I \text{ is not invertible}\}$ of a compact operator is not only compact, it is at most countable with 0 as its only possible accumulation point; moreover, each nonzero element of the spectrum is an eigenvalue. One also has the Fredholm alternative. For polar Banach spaces E and compact $T \in \mathcal{L}(E)$ we have the spectral formula $\max\{|\lambda| : \lambda \in \sigma(T)\} = \inf_n \|A^n\|^{\frac{1}{n}}$ in case K is algebraically closed.

For further studies, see e.g. [40], [41] (connection with semi-Fredholm operators), [48] (Calkin algebra $\mathcal{L}(E)/C(E)$).

Note. There is almost nothing known on spectral theory for non-compact $T \in \mathcal{L}(E)$. The spectral formula does not hold, some operators have empty spectrum, some do not even have non-trivial closed invariant subspaces [58].

2.16. The Krein-Milman Theorem

For a compact convex set A in a real locally convex space, the set ext A of extreme points (points a in A such that $A\backslash\{a\}$ is convex) has the following remarkable properties. First, the closed convex hull of ext A is again A (Krein-Milman Theorem) and second, it is minimal in the sense that if A is the closed convex hull of some closed set X, then $X \supset$ ext A.

Attempts to formulate a n.a. Krein-Milman Theorem have met with little success([36], [7]) . To explain the difficulties, consider the closed unit ball B_K of K. Not only is it the closed convex hull of the points 0 and 1, but also of any two points $\alpha, \beta \in B_K$ provided $|\alpha - \beta| = 1$, since for each $\gamma \in B_K$ we have $\gamma = \frac{\gamma-\beta}{\alpha-\beta}\alpha + \frac{\alpha-\gamma}{\alpha-\beta}\beta$. So there seems to be no way to select two unique points with special geometrical or algebraic properties that act like extreme points as in the classical case. However, by using the module structure for absolutely convex sets, we have the following 'Krein-Milman-type' results.

Let us define an absolutely convex set A in a locally convex space E to be a *Krein-Milman* (KM-) *compactoid* if A is complete and $A = \overline{\text{co}}\, X$, where X is compact. If the valuation of K is discrete, every closed absolutely convex compactoid is KM; otherwise, $A := B_K^-$ is complete but not KM.

Theorem 22. [63] *Let A be an absolutely convex set in a locally convex space E. Then A is KM iff A is isomorphic as a topological module to some power of B_K. If E is a Banach space then A is KM iff there exists a basic sequence $e_1, e_2, \ldots$ tending to 0 such that $A = \overline{\text{co}}\{e_1, e_2, \ldots\}$. Then $\{0, e_1, e_2, \ldots\}$ is a minimal element among the compact sets Y with $A = \overline{\text{co}}\, Y$.*

Theorem 23. (Proof to be published elsewere.) *Let $A = \overline{\text{co}}\, X$ be a KM-compactoid in a polar locally convex space E (i.e. the topology is generated by a family of polar seminorms), where X is compact. Let μ be a measure on X* (see 2.6)*, where $\|\mu\| = |\mu(X)| = 1$. Then there exists a unique $z_\mu \in E$ such that $f(z_\mu) = \int_X f\, d\mu$ for all $f \in E'$. This z_μ lies in A (and may be called the μ-barycenter of X).*

For related work on Šilov boundaries, see [3].

2.17. The Anti-equivalence [71]

For a polar Banach space E, let $B_{E'}$ be the closed unit ball of E'. Equipped with the w^*-topology $B_{E'}$ is a topological B_K-module that is complete, edged and compactoid by the Alaoglu Theorem 19. The category $\mathcal{C}$ of those modules is, surprisingly, equal to the category of *all* complete edged compactoid subsets of locally convex spaces. There is also a more abstract description of $\mathcal{C}$, see [70]. For polar Banach spaces E, F and $T \in \mathcal{L}(E, F)$, $\|T\| \leq 1$, its adjoint $T' : F' \to E'$ induces, by restriction a (w^*-) continuous B_K-module map $T^d : B_{F'} \to B_{E'}$. But conversely, each continuous B_K-module map $B_{F'} \to B_{E'}$ is of the form T^d for some unique contraction $T \in \mathcal{L}(E, F)$. For the proof Goldstine's Theorem: *$j_E : E \to E''$ is a homeomorphism $(E, w) \to (E'', w^*)$ with dense image, $j_E(B_E)$ is w^*-dense in $B_{E''}$*, and some properties of the bounded w^*-topology are used. Thus the category $\mathcal{B}$ of polar Banach spaces is anti-equivalent to $\mathcal{C}$. For any statement on Banach spaces, there exists therefore a 'dual', equivalent, statement on compactoids and conversely.

A few examples. E is of countable type iff $B_{E'}$ is metrizable. E has an orthogonal base iff $B_{E'}$ is a KM-compactoid. E is reflexive iff each homomorphism $B_{E'} \to A$ is automatically continuous for each $A \in \mathcal{C}$.

The theory is by no means finished. Of course the results generate new interest in the general theory of topological B_K-modules, see [47].

3. Exotic Hilbert Spaces

3.1. Valuations of Arbitrary Rank

When looking at the requirements in 1.1 for a non-archimedean valuation, one notices that, unlike for the archimedean triangle inequality, addition of real numbers does not play a role; one only needs multiplication and ordering. This leads to the following generalization. Let G be a commutative multiplicative totally ordered group i.e. one has the compatibility requirement that $g_1 \leq g_2$ implies $hg_1 \leq hg_2$ for all $h, g_1, g_2 \in G$. Adjoin an element 0 and put $0 \cdot g = g \cdot 0 = 0 \cdot 0 = 0$, $0 < g$ for all $g \in G$. A (*Krull*) *valuation* on a field K is a surjective map $|\ |\colon K \to G \cup \{0\}$ satisfying $|\lambda| = 0$ iff $\lambda = 0$, $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$, $|\lambda\mu| = |\lambda|\,|\mu|$ for all $\lambda, \mu \in K$. (Note. In most text books G is additively written.) To indicate the impact of this generalization, let us define a subset X of G to be *convex* (this has nothing to do with the convexity of 2.2) if $g_1, g_2 \in X$, $h \in G$, $g_1 \leq h \leq g_2$ implies $h \in X$. The set of convex subgroups $\neq \{1\}$ of G is linearly ordered by inclusion. If it is finite, its cardinality is called the *rank* of G (or the valuation), otherwise the rank is called *infinite*. A basic result is that G has rank 1 iff G is isomorphic, as an ordered group, to a subgroup of $(0, \infty)$ ([5],3.4). Then the corresponding valuation is real-valued as in Sections 1 and 2. An example of an infinite rank group is given by $G = \bigoplus_{n\in\mathbb{N}} G_n$ where each G_n is a (multiplicative) copy of $\mathbb{Z}$ with the antilexicographic ordering. The proper convex subgroups $\neq \{1\}$ form a chain $H_1 \subset H_2 \subset \dots$ where $H_m = \bigoplus_{n=1}^{m} G_n$ for each m. To show how G may act as a value group, let $F := \mathbb{R}(X_1, X_2, \dots)$ be the field of rational functions with real coefficients in countably many variables $X_1, X_2, \dots$. The requirements $|r| = 1$ if $r \in \mathbb{R}\backslash\{0\}$ and $|X_n| = (1, 1, \dots, g_n, 1, 1, \dots) \in G$, where, for each n, g_n is the generator of G_n , $g_n > 1$, determine a Krull valuation on F with value group G. The completion of F has the same value group. TO SIMPLIFY MATTERS FROM NOW ON $K = (K, |\ |)$ DENOTES THIS COMPLETE VALUED FIELD.

3.2. Exotic Hilbert Spaces, an Algebraic Introduction

Let F be a field, let $\lambda \mapsto \lambda^*$ be an involution in F (i.e. an automorphism of order 2 or the identity). An *inner product* on an F-vector space E assigns to every ordered pair x, y in E a number (x, y) in F such that $(x, x) = 0$ iff $x = 0$, for each $a \in E$ the map $x \mapsto (x, a)$ is linear, $(x, y) = (y, x)^*$ for all $x, y \in E$. For each $X \subset E$ we denote as usual $\{y \in E : (y, x) = 0 \text{ for all } x \in X\}$ by $X^\perp$. E is called an *orthomodular space* if for any subspace D of E

$$D = D^{\perp\perp} \iff E = D \oplus D^\perp.$$

Notice the purely algebraic character of this definition. Of course classical Hilbert spaces are orthomodular. The existence of infinite-dimensional non-classical orthomodular spaces has been open for quite some time but A. Keller found one in 1980 [26]; see 3.3.

Such spaces must be kind of weird according to the following theorem of M.P. Solèr [75]: *Let E be an orthomodular space, and suppose it contains an orthonormal sequence $e_1, e_2, \ldots$ (in the sense of the inner product). Then the base field is $\mathbb{R}$ or $\mathbb{C}$ and E is a classical Hilbert space.*

3.3. Exotic Hilbert Spaces

In Keller's example (see 3.2 above) the base field is the valued field K described in 3.1 and $D^{\perp\perp} = \overline{D}$ for every subspace D. Also, the map $x \mapsto |(x, x)|$ satisfies the strong triangle inequality. So, one may define a n.a. norm by $x \mapsto \sqrt{|(x, x)|}$. Its values are in $\sqrt{G} \cup \{0\}$ where $\sqrt{G} := \bigoplus_{n\in\mathbb{N}} \sqrt{G_n}$, and, for each n, $\sqrt{G_n}$ is the free group generated by $g_n^{1/2}$, with, like in 3.1, g_n the generator of G_n that is > 1. $\sqrt{G}$ is in a natural way an ordered group containing G. Concretely, Keller's example consists of all sequences $(\eta_i)_{i\geq 0}$ in K for which $\sum_{i=0}^{\infty} \eta_i^2 X_i$ converges (here $X_0 := 1$ and $X_1, X_2, \ldots$ are as in 3.1) with componentwise operations. The inner product is defined as follows. If $x = (\alpha_i)_{i\geq 0}$, $y = (\beta_i)_{i\geq 0}$ then $(x, y) = \Sigma \alpha_i \beta_i X_i$, the norm is $x \mapsto \sqrt{|(x, x)|} = \max_{i\geq 0} |\alpha_i| \sqrt{|X_i|} \in \sqrt{G} \cup \{0\}$. Here K has the trivial involution.

So let us define an orthomodular space E over K to be an *exotic Hilbert space* (also called *G-K-K space* in [46] or *form Hilbert space* in [45]), if $x \mapsto \sqrt{|(x,x)|}$ is a norm and $D^{\perp\perp} = \overline{D}$ for every subspace D. We have the following version of Solèr's Theorem.

Theorem 24. ([45],4.3.7) *Every bounded orthogonal sequence in an exotic Hilbert space tends to* 0.

It is unknown whether this conclusion holds for orthomodular spaces in general. The reason for this 'strange' behaviour of orthogonal sequences lies in the following property of $\sqrt{G}$. Let $y_1 = (g_1^{1/2}, 1, 1, \dots) \in \sqrt{G}$, $y_2 := (1, g_2^{1/2}, 1, \dots) \in \sqrt{G}, \dots$ and let $v_1, v_2, \dots \in G$ be such that $v_n y_n \leq 1$ for all n. Then $\inf_n v_n y_n = 0$.

The above theorem holds -mutatis mutandis- for any n.a. valued scalar field. If it has rank 1, we can multiply elements of an orthogonal sequence of nonzero vectors by suitable scalars so as to obtain a sequence bounded away from 0. As a corollary we therefore obtain Theorem 4 on the nonexistence of infinite-dimensional 'Hilbert-like' spaces in case the valuation is real-valued.

Now we list some properties (most can be found in [45]) showing that such spaces deserve the name 'Hilbert'. We have seen that an exotic Hilbert space E has an inner product and that $x \mapsto \|x\| := \sqrt{|(x,x)|}$ is a norm satisfying the strong triangle inequality. We have the Cauchy-Schwarz inequality $|(x,y)| \leq \|x\|\,\|y\| \quad (x, y \in E)$ [19].

For each closed subspace D we have $D = D^{\perp\perp}$, so D has an orthogonal complement; therefore the Theorem 9 version of the Hahn-Banach Theorem holds. Form orthogonality implies norm orthogonality in the sense of 2.4 but not conversely. E is complete and has a countable form orthogonal base. Each maximal form orthogonal system of nonzero vectors is an orthogonal base. For each $f \in E'$ there exists a unique $a \in E$ with $f(x) = (x, a)$ for all $x \in E$, and $\|f\| = \|a\|$. E is reflexive. For each $A \in \mathcal{L}(E)$ there exists a unique $A^* \in \mathcal{L}(E)$ and that $(Ax, y) = (x, A^*y)$ for

all $x, y \in E$. This leads to the definition of hermitean, normal, unitary,... operators. Spectral theory is, as yet, rather unknown territory.

On the other hand there are enough features to justify the term 'exotic' for such spaces E. We have already seen that bounded orthogonal sequences tend to 0. In addition, each $A \in \mathcal{L}(E)$ can be approximated uniformly on bounded sets by finite rank operators, making every $A \in \mathcal{L}(E)$ compact in the sense of [73]. This ties in with the fact that the unit ball of E is a compactoid ([45],4.3.7). If $A \in \mathcal{L}(E)$ is injective and AE is closed then A is surjective! Each surjective $A \in \mathcal{L}(E)$ is bijective! Thus, E is never linearly homeomorphic to any proper closed subspace [73]. All these properties make exotic Hilbert spaces E resemble finite-dimensional spaces, behaving more 'rigidly' than classical Hilbert spaces. Another peculiarity is that continuous linear operators need not be bounded (Lipschitz). The theory of exotic Hilbert spaces is far from complete; more research is needed. A recent result [27] is the absence of operators A and B satisfying the commutation relation $AB - BA = I$, which is of interest for n.a. Quantum Mechanics. (As a contrast, on the space c_0 over a rank 1 valued field there *do* exist operators A, B satisfying $AB - BA = I$! [33])

3.4. Banach Spaces Over Fields with an Infinite Rank Valuation

Traditionally, norms have values in $[0, \infty)$. This has also been our choice in §2 for spaces over a field with a rank 1 valuation. However, for our infinite rank valued scalar field K of 3.1 it is not at once clear what a 'natural home' for norm values should be. The range of the valuation is $\{0\} \cup G$, with $G = \bigoplus_n G_n$, and the natural norm on exotic Hilbert spaces has values in $\{0\} \cup \sqrt{G}$. For several constructions such as norms in quotient spaces, norms of linear operators, we would like to be able to take infima and suprema of bounded sets in our range. Now G and $\sqrt{G}$ are not Dedekind complete, so one is tempted to consider the Dedekind completions $G^{\#}$ and $\sqrt{G}^{\#}$, constructed in the usual way by cuts.

Unfortunately, although there are at least two natural ways to extend the multiplication of, say, G to $G^{\#}$, it will never be a group, only a semigroup, as it is easily seen that for a proper convex subgroup H, we will get $(\sup H)^2 = \sup H$. On the other hand, one may not need a multiplication in $\sqrt{G}^{\#}$ or $G^{\#}$ in many situations. So, what is really needed? A norm $\| \ \|$ on a space over K should satisfy $\|x\| = 0 \iff x = 0$, $\|\lambda x\| = |\lambda| \, \|x\|$, $\|x + y\| \leq \max(\|x\|, \|y\|)$, so the range set X should be linearly ordered and allow a multiplication of elements of G with elements of X. This leads to the following.

Definition. ([45],1.5) Let G be the value group of K. A *G-module* is a totally ordered set X together with an action $G \times X \to X$ such that $g \geq h$, $x \geq y$ implies $gx \geq hy$ for all $g, h \in G$, $x, y \in X$, and such that for each $\varepsilon > 0$ there is a $g \in G$ for which $gx < \varepsilon$ (to make scalar multiplication continuous).
Adjoin an element 0_X to X with the properties $g \cdot 0_X = 0 \cdot 0_X = 0_X$ and $0_X < x$ for all $g \in G$, $x \in X$, and let us write from now on 0 instead of 0_X. An *X-norm* on a K-vector space E is a map $\| \ \| : E \to X \cup \{0\}$ satisfying the rules mentioned above.

One proves directly that the *Dedekind completion* $X^{\#}$ of X is again a G-module in a natural way. Examples of G-modules are: any union of cosets of the value group in $(0, \infty)$ in the case of a rank 1 (real valued) valuation, but also the sets G, $\sqrt{G}$, $G^{\#}$ and $\sqrt{G}^{\#}$ mentioned above for the field K of 3.1.
This definition opens the way to building a theory of Banach spaces over K - and over more general infinite rank valued fields L - in the spirit of Section 2. This theory is not yet fully developed; see [45] for recent results. Several theorems remain valid in this general setting such as the Hahn-Banach Theorem for spherically complete L, the theory of norm-orthogonality, the Open Mapping Theorem, but there are also differences, such as for example the existence of a Banach space of countable type without Schauder base ([71]; compare Theorem 6). There is a

quite satisfactory theory for so-called *norm Hilbert spaces* (NHS) i.e. spaces E such that for each closed subspace D there is a linear projection P of E onto D with $\|Px\| \leq \|x\|$ for all x. This class properly contains the category of the exotic Hilbert spaces, but it shares the 'exotic' properties of (3.3).

A final remark. The notion of a general X-normed space is by no means restricted to the infinite rank case. For example, let L be a n.a. valued complete field with a dense valuation $L \to [0,\infty)$, let $E := \ell^\infty$. For $x = (x_1, x_2, \dots) \in \ell^\infty$ put $r := \sup_n \|x_n\|$. Define

$$\|x\| := \begin{cases} r^+ & \text{if } \max\limits_n |x_n| \text{ exists} \\ r^- & \text{otherwise} \end{cases}$$

where r^- is an 'immediate predecessor' of r^+. Formally $\|\ \|$ takes its nonzero values in $X := (0,\infty) \times \{0,1\}$, a $(0,\infty)$-module with the lexicographic ordening, and obvious operations. $r^+ := (r,1)$, $r^- := (r,0)$.
This way the norm $\|x\|$ can store more information on the vector x than in the traditional ones. This area is still unexplored.

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Note to the references. No attempts have been made to be complete; references are just meant to be a help to the reader to find details and further investigations. Accessibility has been the main criterion of listing.
For general introduction on n.a. valued fields and elementary analyse see [1], [5], [32], [34], [54], [59], for infinite rank valuations see [5], [53], [81]. Books on n.a. Functional Analysis are [44], [43], [52], [57], and survey papers [42], [21].

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