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INFIMA OF RING TOPOLOGIES

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Abstract

We consider when the infima of two field topologies, considered as members of four different lattices (the lattices of all topologies, of all group topologies, of all ring topologies, and of all field topologies) are the same and when they are distinct.

1. Introduction

The range of a sequence $\{x_n\}$ will be denoted by range$\{x_n\}$. The nonzero elements of an additive group G (e.g., the additive group of a ring) will be denoted by G^*. We denote the set of integers, the set of positive integers and the set of rational numbers by $\mathbf{Z}$, $\mathbf{Z}_{>0}$ and $\mathbf{Q}$, respectively. We denote the usual topology on any subfield of the complex numbers by $\mathcal{T}_\infty$.

By a *group* topology on an additive group we mean a topology with respect to which addition is jointly continuous and negation is continuous. By a *ring* topology we mean a group topology on the additive group of a ring with respect to which multiplication is jointly continuous; and by a *field* topology we mean a ring topology on a field with respect to which inversion is a continuous function on the set of nonzero elements of the

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field. We let $\mathcal{L}_t(K)$ (respectively, $\mathcal{L}_g(K)$, $\mathcal{L}_r(K)$, $\mathcal{L}_f(K)$) denote the set of all topologies (all group, ring, field topologies) on a set (group, ring, field) K. Each of these four sets of topologies is considered to be partially ordered by containment, and it is well-known that each of these sets is a complete lattice with this order (see, e.g., [10, pp.30–31]).

The least upper bound (greatest lower bound) of a subset E of a totally ordered group or of the lattice all *ring* topologies on a ring will be denoted by $\vee E$ ($\wedge E$); we let $a \vee b = \vee\{a, b\}$ and $a \wedge b = \wedge\{a, b\}$.

We denote the trivial topology and the discrete topology on any set by $\mathbf{0}$ and $\mathbf{1}$, respectively.

Recall (see, e.g., [10, Definition 3.4.1]) that a family of topologies on a set is called *independent* if $\cap U_i = \emptyset$ for $U_i \in \mathcal{T}_i$, where $\mathcal{T}_i$ are distinct members of a finite subfamily of the family, implies one of the sets U_i must be empty.

The neighborhood filter at the identity of a group topology $\mathcal{T}$ will be denoted by $\mathcal{B}(\mathcal{T})$. A group topology on a group G is called *minimal* if it is a minimal element of the set of all Hausdorff group topologies on G. Compact Hausdorff group topologies are obvious examples of minimal group topologies.

A ring topology on a field is either Hausdorff or trivial (see, e.g., [10, Theorem 1.3.1]). For certain subsets (near orders) A of a field K, the sets $\{xA : x \in K^*\}$ form a neighborhood base at zero for a Hausdorff ring topology on K; this topology is denoted by $\mathcal{T}_A$ (see, e.g., [10, Chapter 4]). In particular, the topology having $\{x\mathbf{Z} : x \in \mathbf{Q}^*\}$ as a neighborhood base at zero is denoted by $\mathcal{T}_{\mathbf{Z}}$.

By a *valuation* on a field we mean either a nonarchimedean valuation with values in a totally ordered abelian group (whose operation we write as multiplication) or the usual absolute value, denoted by $|\ |_\infty$, on a subfield of the complex numbers. For a valuation v on a field K, the sets $U_g(a) = \{x \in K : v(x-a) < g\}$ are called *spheres*. The spheres about zero form a neighborhood base at zero for a topology which we call the topology induced by the valuation. A nondiscrete topology induced by a valuation on K is a field topology which is minimal among the set of all Hausdorff ring topologies on K (see, e.g., [10, Theorem 3.3.1]). Any family of nondiscrete topologies induced by valuations is independent (see [16, Corollary 2.3]). The topology induced by the valuation v is denoted by $\mathcal{T}_v$. Thus $\mathcal{T}_{|\,|_\infty} = \mathcal{T}_\infty$.

A class of topologies called *direct topologies* was defined in [18] and developed in [11], [12], [13], and [14]. These topologies generalized important examples of ring topologies on $\mathbf{Q}$ defined in [8] and [9]. Further important examples of direct topologies appeared in [5] and [6].

Definition: A sequence of subsets $\{M_{i\,n}\}_{i,n\in\mathbf{Z}_{>0}}$ of a ring S will be called a *direct system* in S if

(D1) $0 \in M_{i\,n}$ for all i and n and $M_{i\,n} = \{0\}$ if $i < n$;
(D2) $-M_{i\,n} = M_{i\,n}$ for all i and n;
(D3) $M_{i\,n+1} + M_{i\,n+1} \subset M_{i\,n}$ for all i and n;
(D4) $[\sum_{i=1}^{k-1}(M_{i\,1} + M_{i\,1} + M_{i\,1})] \bigcap (M_{k\,1} + M_{k\,1} + M_{k\,1}) = \{0\}$ for all $k > 1$;
(D5) for all i and n, $\sum_{j\vee k=i} x_j y_k \in M_{i\,n}$ whenever $x_j \in M_{j\,n+1}$ and $y_k \in M_{k\,n+1}$; and
(D6) for all $a \in S$ and $n > 0$, there exists $k \geq 0$ such that, for all i, $aM_{i\,n+k} \subset M_{i\,n}$ and $M_{i\,n+k}a \subset M_{i\,n}$.

For each positive integer n, $U_n = \cup_{k=1}^{\infty} \sum_{i=1}^{k} M_{i\,n}$. The sequence $\{U_n\}$ will be called the neighborhood base associated with the direct system. A ring topology with a neighborhood base at zero which is the neighborhood base associated with a direct system will be called a *direct topology.*

The following construction of direct topologies is used in some of the examples and in the final theorem.

Let $(K, |\,|)$ be a field with a nontrivial absolute value, and let R be a ring with quotient K such that R is discrete in the topology induced by the absolute value. Let γ be a positive real number less than or equal to 1; let $\{\alpha_{i\,n}\}$, $i, n \in \mathbf{Z}_{>0}$, $i \geq n$, be real numbers greater than or equal to $\wedge|R^*|$; and let P_i, $i \in \mathbf{Z}_{>0}$, be positive real numbers satisfying $P_1 \geq 1/\gamma$ and

$$\alpha_{i\,n} \geq \alpha_{i\,n+1}(2\sum_{j=n+1}^{i-1} \alpha_{j\,n+1}P_j + \alpha_{i\,n+1}P_i);$$
$$\tfrac{\gamma P_{i+1}}{2\cdot 2^{3^i}} \geq \sum_{j=1}^{i}\alpha_{j\,1}P_j.$$

Let $\{p_i\}$ be a sequence in R such that $\gamma P_i \leq |p_i| \leq P_i$, for all i. Then the sets U_n, $n \geq 1$, defined below form a neighborhood base at zero for a ring topology on K (see [12, Theorem 12]):

$$U_n = \{\sum_{i=n}^{k} \frac{a_i}{b_i} p_i : a_i, b_i \in R, k \geq n, \left|\frac{a_i}{b_i}\right| \leq \alpha_{i\,n}, 0 < |b_i| \leq 2^{3^{i-n}}\}.$$

For a given field K, with a fixed choice of $|\ |$ and R, the set of all topologies of this form on K will be denoted by **mut1**(K). For $K = \mathbf{Q}$, we choose the usual absolute value and let $R = \mathbf{Z}$. The reader is referred to [12, Theorem 13] for the definition of a second similar class of topologies referred to in Example 6 as **mut2**(K). If J is an infinite subset of positive integers and $J(i)$ denotes the ith element of J listed as an increasing sequence, then a topology finer than the topology with base $\{U_n\}$ may be defined by taking the sets U'_n consisting of sums in U_n such that $a_i = 0$ for $i \notin J$. This is referred to as the *condensation* of the topology to J ([12, pp.199-200]); this definition of condensation generalizes readily to all direct topologies). A second topology finer than the one defined by $\{U_n\}$ is obtained by replacing the parameters defining U_n by

$$\alpha'_{i\,n} = \alpha_{J(i)\,n}, \qquad\qquad p'_i = p_{J(i)}.$$

We refer to this as the *Mutylin condensation* of the original topology to J (see [12, p.211]).

2. Infima

In [1] (or see [10, Theorem 3.2.1]) the authors show that for each Hausdorff ring topology $\mathcal{T}$ on a field K, the quotient topology for the division function

$$\begin{aligned}(K,\mathcal{T})\times(K^*,\mathcal{T}|_{K^*}) &\longrightarrow K\\ (x,y) &\longmapsto x/y\end{aligned}$$

is a Hausdorff field topology coarser than or equal to $\mathcal{T}$. Mutylin [9] showed that if $\mathcal{B}$ is a neighborhood base at zero for $\mathcal{T}$, then

$$\{\frac{U}{1+U\backslash\{-1\}} : U\in\mathcal{B}\},$$

is a neighborhood base at zero for a Hausdorff field topology weaker than $\mathcal{T}$. (Weber [15] showed Mutylin's result generalized readily to topological rings.) Actually these two constructions yield the same field topology: both may be verified to be the finest field topology coarser than or equal to $\mathcal{T}$ (see, e.g., [10, Theorem 3.2.1]). We will call this topology $D\mathcal{T}$.

For a field K and $\mathcal{S},\mathcal{T}\in\mathcal{L}_f(K)$, the supremum of $\mathcal{S}$ and $\mathcal{T}$ is the same whether calculated in the lattice $\mathcal{L}_f(K)$, $\mathcal{L}_r(K)$, $\mathcal{L}_g(K)$ or $\mathcal{L}_t(K)$. Analogous statements for ring topologies on a ring and group topologies on a group are true. A neighborhood base at a point x with respect to the supremum of a family $\mathbf{T}$ of topologies on any set can be described explicitly: all sets of the form $\cap_{\mathcal{T}\in\mathbf{T_0}}U_{\mathcal{T}}$, where $U_{\mathcal{T}}$ belongs to a $\mathcal{T}$-neighborhood base at x and $\mathbf{T_0}$ ranges over all finite subsets of $\mathbf{T}$.

The situation is quite different for infima: For an arbitrary set X, $\mathcal{S}\cap\mathcal{T}$ is the infimum of two topologies in $\mathcal{L}_t(X)$. For an additive group G, the sets $U+V$, where U varies over an $\mathcal{S}$-neighborhood base at zero and V varies over a $\mathcal{T}$-neighborhood base at zero, are easily seen to form a neighborhood base at zero for the infimum of $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{L}_g(G)$; we denote this topology by $\mathcal{S}+\mathcal{T}$. The notation given above for the finest ring topology weaker than both of the ring topologies $\mathcal{S}$ and $\mathcal{T}$ is $\mathcal{S}\wedge\mathcal{T}$.

The infimum in $\mathcal{L}_f(K)$ of two topologies $\mathcal{S}$ and $\mathcal{T}$ is easily verified to be $D(\mathcal{S} \wedge \mathcal{T})$. Clearly

$$D(\mathcal{S} \wedge \mathcal{T}) \leq \mathcal{S} \wedge \mathcal{T} \leq \mathcal{S} + \mathcal{T} \leq \mathcal{S} \cap \mathcal{T}.$$

The purpose of this paper is to consider when these inequalities are strict.

Theorem 1. *If $\mathcal{S}$ and $\mathcal{T}$ are first countable Hausdorff group topologies on an additively written abelian group G, then $\mathcal{S} \cap \mathcal{T} = \mathcal{S} + \mathcal{T}$ if and only if $\mathcal{S}$ and $\mathcal{T}$ are comparable (i.e., $\mathcal{S} \leq \mathcal{T}$ or $\mathcal{T} \leq \mathcal{S}$).*

Proof. Suppose $\mathcal{S} \not\leq \mathcal{T}$ and $\mathcal{T} \not\leq \mathcal{S}$. Then there exist symmetric sets $U_0 \in \mathcal{B}(\mathcal{S})$ and $V_0 \in \mathcal{B}(\mathcal{T})$ such that $V \not\subset U_0$ and $U \not\subset V_0$ for any $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$. Thus, there exists a sequence $\{x_n\}$ of elements in U_0 such that $x_n \xrightarrow{\mathcal{S}} 0$, and $x_n \notin V_0$ for any n. If the range of x_n has any $\mathcal{T}$-cluster points, we replace the sequence $\{x_n\}$ by a subsequence which is $\mathcal{T}$-convergent to, say, x; $\lim_{\mathcal{T}} x_n$ will denote the set ($\emptyset$ or $\{x\}$) of all $\mathcal{T}$-limits of $\{x_n\}$. Analogously, choose a sequence $\{y_n\}$ of elements in V_0 such that $y_n \xrightarrow{\mathcal{T}} 0$, $y_n \notin U_0$ for any n, and $\lim_{\mathcal{S}} y_n = \{y\}$ or $\{y_n\}$ has no $\mathcal{S}$-cluster points.

We show that

$$E = \operatorname{range}\{x_n + y_n\} \cup \lim_{\mathcal{T}} x_n \cup \lim_{\mathcal{S}} y_n$$

is not $(\mathcal{S} + \mathcal{T})$-closed: Since $x_n + y_n \xrightarrow{\mathcal{S}+\mathcal{T}} 0$, 0 is in the $(\mathcal{S} + \mathcal{T})$-closure of E. But, when they exist, $x, y \neq 0$; and $x_n + y_n = 0$ would imply $y_n = -x_n \in U_0$, a contradiction. So $0 \notin E$.

However, $\operatorname{range}\{x_n + y_n\} \cup \lim_{\mathcal{T}} x_n$ is $\mathcal{T}$-closed, so E is $\mathcal{T}$-closed. Similarly E is $\mathcal{S}$-closed. I.e.,

$$G \backslash E \in (\mathcal{S} \cap \mathcal{T}) \backslash (\mathcal{S} + \mathcal{T}). \qquad \square$$

The hypothesis of the theorem is not a necessary condition: if $\mathcal{S}$ is a minimal Hausdorff group topology and $\mathcal{T} \not\geq \mathcal{S}$ is Hausdorff, then $\mathcal{S} + \mathcal{T}$ is not T_0, but all cofinite sets are in $\mathcal{S} \cap \mathcal{T}$, so $\mathcal{S} + \mathcal{T} < \mathcal{S} \cap \mathcal{T}$.

Theorem 2. *If A is a ring and $\mathcal{S}, \mathcal{T} \in \mathcal{L}_r(A)$, then these conditions are equivalent:*

(1) $\mathcal{S} \wedge \mathcal{T} = \mathcal{S} + \mathcal{T}$;
(2) $\mathcal{S} + \mathcal{T}$ is a ring topology;
(3) given $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$, there exists $U_1 \in \mathcal{B}(\mathcal{S})$ and $V_1 \in \mathcal{B}(\mathcal{T})$ such that $U_1V_1 \subset U+V$ and $V_1U_1 \subset U+V$.

The routine verification is omitted.

Lemma 3.1 *Group topologies $\mathcal{S}$ and $\mathcal{T}$ on a group $(G, \cdot)$ are independent if and only if $UV = G$ for all $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$. In particular, if G is abelian and its group operation is addition, then $\mathcal{S}$ and $\mathcal{T}$ are independent if and only if $\mathcal{S}+\mathcal{T} = \mathbf{0}$.*

Proof. The second statement follows from the first, and the easy proof in [16, Theorem 1.6] of the first statement does not use the assumed commutativity: If $UV = G$ for all $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$ and if A and B are nonempty $\mathcal{S}$-open and $\mathcal{T}$-open sets, respectively, with $a \in A$ and $b \in B$, then $a^{-1}b \in G = (a^{-1}A)(B^{-1}b)$, so that there exists $a' \in A$ and $b' \in B$ such that $a^{-1}b = a^{-1}a'(b')^{-1}b$. Hence, $b' = a' \in A \cap B$. Conversely, if $\mathcal{S}$ and $\mathcal{T}$ are independent, $g \in G$, $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$, then there exists $x \in U^{-1}g \cap V$. That is $x = u^{-1}g = v$ for some $u \in U$ and $v \in V$. Then $g = uv \in UV$. $\square$

Theorem 3. *If $\mathcal{S}$ and $\mathcal{T}$ are ring topologies on a field K such that*

(1) $\mathcal{T}$ is minimal among Hausdorff ring topologies on K,
(2) $\mathcal{S} \not\geq \mathcal{T}$, and
(3) $\mathcal{S}$ and $\mathcal{T}$ are not independent;

then $\mathbf{0} = \mathcal{S} \wedge \mathcal{T} < \mathcal{S} + \mathcal{T}$.

Proof. (1) and (2) imply the stated equality, and Lemma 3.1 states that (3) is equivalent to the statement $\mathcal{S} + \mathcal{T} \neq \mathbf{0}$. $\square$

Example 1: In the field $\mathbf{Q}$,

$$\mathcal{T}_{\mathbf{Z}} \wedge \mathcal{T}_{\infty} = \mathbf{0} < \mathcal{T}_{\mathbf{Z}} + \mathcal{T}_{\infty} < \mathcal{T}_{\mathbf{Z}} \cap \mathcal{T}_{\infty}.$$

Example 2: For $\mathcal{T} \in \mathbf{mut1}(K)$,

$$\mathcal{T} \wedge \mathcal{T}_{|\,|} = \mathbf{0} < \mathcal{T} + \mathcal{T}_{|\,|} < \mathcal{T} \cap \mathcal{T}_{|\,|}.$$

Example 3: Let $\mathcal{T}$ be a nondiscrete direct field topology, and let $\mathcal{T}^o$ and $\mathcal{T}^e$ denote the condensations of $\mathcal{T}$ to the set of odd, respectively, even, positive integers (with both condensations with respect to the same direct system). Then, by [12, Theorem 11], $\mathcal{T}^o$ and $\mathcal{T}^e$ are field topologies, and, from the definition of the neighborhoods defining a direct topology,

$$\mathcal{T} = D(\mathcal{T}^o \wedge \mathcal{T}^e) = \mathcal{T}^o \wedge \mathcal{T}^e = \mathcal{T}^o + \mathcal{T}^e < \mathcal{T}^o \cap \mathcal{T}^e.$$

For $\mathcal{T} \in \mathbf{mut1}(\mathbf{Q})$, let $\mathcal{S}^o$ and $\mathcal{S}^e$ be the Mutylin condensations of $\mathcal{T}$, to the sets of odd and even positive integers, respectively, with defining neighborhood bases at zero $\{U_n^o\}$ and $\{U_n^e\}$. Then

$$U_n^o + U_n^e = \{\sum \frac{a_i}{b_i} p_i : \left|\frac{a_i}{b_i}\right| \leq \alpha_{i\,n}, 0 < |b_i| \leq 2^{3^{\alpha}}\},$$

where α is the integer $\frac{i}{2} - n$ or $\frac{i+1}{2} - n$. One readily verifies (by the same methods as in [7, pp.161–162]—or see [10, p.27]) that

$$U_{n+1}^o U_{n+1}^e \subset U_n^o + U_n^e.$$

Also,

$$P_{4n}/2^{3^{3n}} \in U_n \backslash (U_2^o + U_2^e).$$

Therefore

$$\mathcal{T} < \mathcal{S}^o \wedge \mathcal{S}^e = \mathcal{S}^o + \mathcal{S}^e.$$

Example 4: Let D be a Dedekind domain with quotient field K; let V and W be D-submodules of K; and let $\mathbf{lin}(K, D)$ be the collection of all D-linear ring topologies on K (see [3, p.476] for definitions and terminology). Then

$$\mathcal{T}_V + \mathcal{T}_W = \mathcal{T}_V \wedge \mathcal{T}_W,$$

and $\mathcal{T}_V + \mathcal{T}_W$ obviously is D-linear, so the infimum of $\mathcal{T}_V$ and $\mathcal{T}_W$ computed in the lattice $\mathbf{lin}(K, D)$, viz., $\mathcal{T}_{V+W}$, is also $\mathcal{T}_V + \mathcal{T}_W$.

Hence, the only natural misinterpretation of the symbol "inf" in [3] that would be erroneous is $\mathcal{T}_V \cap \mathcal{T}_W$.

Theorem 4. (cf. [17, 2.2–2.4]) *If $\{\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2\}$ is an independent set of group topologies, then*

$$(\mathcal{T}_1 \vee \mathcal{T}) + (\mathcal{T}_2 \vee \mathcal{T}) = \mathcal{T}.$$

If $\{\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2\}$ is an independent set of ring topologies, then also

$$(\mathcal{T}_1 \vee \mathcal{T}) \wedge (\mathcal{T}_2 \vee \mathcal{T}) = \mathcal{T}.$$

If $\{\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2\}$ is an independent set of field topologies, then also

$$D[(\mathcal{T}_1 \vee \mathcal{T}) \wedge (\mathcal{T}_2 \vee \mathcal{T})] = \mathcal{T}.$$

Proof. Choose a set of the form

$$W = (U_1 \cap U) + (U_2 \cap U); \qquad U_i \in \mathcal{B}(\mathcal{T}_i),\ i = 1, 2; \quad U \in \mathcal{B}(\mathcal{T}).$$

The collection of all such sets is a base for $(\mathcal{T}_1 \vee \mathcal{T}) + (\mathcal{T}_2 \vee \mathcal{T})$. Pick $V \in \mathcal{B}(\mathcal{T})$ such that $V - V \subset U$. Choose $z \in V$ and

$$x \in U_1 \cap (z - V) \cap (z - U_2).$$

Then

$$z = x + (z - x) \in W;$$

i.e., $V \subset W$, so

$$(\mathcal{T}_1 \vee \mathcal{T}) + (\mathcal{T}_2 \vee \mathcal{T}) \leq \mathcal{T}.$$

On the other hand, obviously

$$\begin{aligned} \mathcal{T}\ [\ &\leq\ D[(\mathcal{T}_1 \vee \mathcal{T}) \wedge (\mathcal{T}_2 \vee \mathcal{T})] \leq (\mathcal{T}_1 \vee \mathcal{T}) \wedge (\mathcal{T}_2 \vee \mathcal{T})] \\ &\leq\ (\mathcal{T}_1 \vee \mathcal{T}) + (\mathcal{T}_2 \vee \mathcal{T}). \end{aligned}$$ $\square$

Example 5: ([17, 2.3–2.4]) Let E be a set of inequivalent valuations on a field K, and, for $A \subset E$, let $\mathcal{T}_A$ be the supremum of the valuation topologies $\mathcal{T}_v, v \in A$. If A, B and C are disjoint subsets of E, then $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$ are independent. Since $\mathcal{T}_A \vee \mathcal{T}_C = \mathcal{T}_{A\cup C}$, $\mathcal{T}_B \vee \mathcal{T}_C = \mathcal{T}_{B\cup C}$ and $\mathcal{T}_C$ is a field topology,

$$D(\mathcal{T}_{A\cup C} \wedge \mathcal{T}_{B\cup C}) = \mathcal{T}_{A\cup C} \wedge \mathcal{T}_{B\cup C} = \mathcal{T}_{A\cup C} + \mathcal{T}_{B\cup C} = \mathcal{T}_C.$$

Theorem 5. *Suppose $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ are ring topologies on a field K satisyfing the following conditions:*

(1) $\mathcal{U}$ is induced by a valuation.
(2) All zero neighborhoods of $\mathcal{S} \vee \mathcal{T}$ are $\mathcal{U}$-unbounded.
[In the presence of condition (1), condition (2) above and (2′) below are equivalent:
(2′) $\mathcal{S} \vee \mathcal{T} \not\geq \mathcal{U}$.]
(3) $\mathcal{T}$ and $\mathcal{U}$ are not independent.
(4) $\mathcal{S}$ and $\mathcal{U}$ are independent.

Then $(\mathcal{S} \vee \mathcal{T}) \wedge (\mathcal{S} \vee \mathcal{U}) < (\mathcal{S} \vee \mathcal{T}) + (\mathcal{S} \vee \mathcal{U})$.

Proof. Let S, T and U, with or without subscripts or primes, denote open neighborhoods of zero in $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$, respectively. Let $U_g(0)$ be a sphere of radius g with respect to a valuation v inducing $\mathcal{U}$.

Hypothesis (3) implies there exists $c \in K$, and U and T such that

$$(c + U) \cap T = \emptyset.$$

Choose symmetric U_1 such that $U_1 + U_1 \subset U$. Then, for any $a \in c + U_1$,

$$a \notin (S \cap T) + (S \cap U_1).$$

Suppose the product $(S' \cap T') \cdot (S' \cap U')$ is given, and $U_r(0) \subset U' \cap U_1$. Using (2), choose $x \in S' \cap T'$ such that $v(\frac{c}{x}) < r'$, where $r' = r$ if v is nonarchimedian and $r' = r/2$ otherwise. Let $t = (r/v(x)) \wedge r'$ and choose $y \in K$ such that

$$y \in S' \cap \left(\frac{c}{x} + U_t(0)\right)$$

(using (4)). Then

$$y \in \frac{c}{x} + U_t(0) \subset U_r(0) \subset U'.$$

Therefore, $y \in S' \cap U'$ and $xy \in (S' \cap T') \cdot (S' \cap U')$. However,

$$v(xy - c) = v(x)v(y - \frac{c}{x}) < r,$$

so

$$xy \in c + U_r(0) \subset c + U_1.$$

The desired conclusion now follows from Theorem 2. □

Example 6: Let q be a prime integer, and let $\mathcal{L}_q{}^*(\mathbf{Q})$ denote the set of all Hausdorff ring topologies on $\mathbf{Q}$ which are coarser than or equal to $\mathcal{T}_A$, where A is the near order $\{m/q^n : m, n \in \mathbf{Z}\}$.

For $\mathcal{U} = \mathcal{T}_\infty$ on $\mathbf{Q}$, the hypotheses of Theorem 5 are satisfied whenever $\mathcal{S} \in \mathcal{L}_q{}^* \cup \mathbf{mut2}(\mathbf{Q})$ and $\mathcal{T} \in \{\mathcal{T}_\mathbf{Z}\} \cup \mathbf{mut1}(\mathbf{Q})$, provided the same parameters $\{p_i\}$ are used to define $\mathcal{S}$ and $\mathcal{T}$ when $\mathcal{S} \in \mathbf{mut2}(\mathbf{Q})$ and $\mathcal{T} \in \mathbf{mut1}(\mathbf{Q})$.

Although our results indicate that $\mathcal{S} \cap \mathcal{T}$ and $\mathcal{S} \wedge \mathcal{T}$ are generally distinct, there are examples in the literature ([4, p.165], [18, pp.40–43 and pp.66–69]), and [11, Theorem 8] of *families* $\{\mathcal{S}_i\}$ of group and ring topologies such that

$$\cap_i \mathcal{S}_i = \bigwedge_i \mathcal{S}_i.$$

We gave explicit descriptions for the neighborhoods of zero for the topologies $\mathcal{S} \vee \mathcal{T}$, $\mathcal{S} \cap \mathcal{T}$ and $\mathcal{S} + \mathcal{T}$; and the neighborhoods of zero in $D(\mathcal{S} \wedge \mathcal{T})$ are described explicitly in terms of the neighborhoods of zero in $\mathcal{S} \wedge \mathcal{T}$. When $\mathcal{S} \wedge \mathcal{T}$ and $\mathcal{S} + \mathcal{T}$ coincide, we have a description of the neighborhoods of zero of the former. It would be useful to have a description for the $(\mathcal{S} \wedge \mathcal{T})$-neighborhoods of zero in the general case.

Theorem 6. *Let $(K, |\,|)$ be a field with a nontrivial absolute value. Suppose K contains a discrete subring whose quotient is K. Then $\mathcal{L}_r(K)$ is not a distributive lattice.*

Proof. Choose $\mathcal{T} \in \mathbf{mut1}(K)$ and a condensation $\mathcal{T}_A > \mathcal{T}$. The result follows from [2, pp.69–70] and this diagram:

$$\begin{tikzcd}[column sep=small, row sep=small]
& \mathbf{1} \arrow[dl, dash] \arrow[ddr, dash] & \\
\mathcal{T}_A \arrow[dd, dash] & & \\
& & \mathcal{T}_{||} \arrow[ddl, dash] \\
\mathcal{T} \arrow[dr, dash] & & \\
& \mathbf{0} &
\end{tikzcd}$$

□

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