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TOPOLOGICAL CHARACTERIZATION OF EQUIVALENT UNIFORMITIES IN TOPOLOGICAL GROUPS

SALVADOR HERNÁNDEZ

Abstract. We characterize the topological groups with equivalent left and right uniform structures (SIN groups) by means of their uniformly discrete subsets. As a consequence, it follows that a ω-bounded non-Archimedean group G is SIN if and only if every left uniformly continuous real-valued function on G is right uniformly continuous. And, in general, a topological group is SIN if and only if every set of left (right) equiuniformly real-valued functions on G is right (left) equiuniformly continuous.

A topological group G is said to be a *SIN-group* if the left and right uniform structures on G coincide. Equivalently, when for every neighbourhood V of the neutral element in G, there is a neighbourhood U of the neutral element such that $Ux \subset xV$ for all $x \in G$. It is known that every compact (even precompact) topological group is SIN but there are many examples of topological groups that are not SIN (see [5, 10].) All topological groups here are assumed to be Hausdorff.

Following [8] we say that G is a *functionally SIN-group* (*FSIN*, for short) if every bounded left uniformly continuous real-valued function on G is right uniformly continuous. (Equivalently, every

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bounded right uniformly continuous function on G is left uniformly continuous.)

While the former property obviously implies the latter, it is still an open question, due to Itzkowitz, (cf. [6]) whether or not the converse is also satisfied. Nevertheless, there is much information about this problem and it is known that the answer is positive for most relevant classes of topological groups. For instance, a locally compact or metrizable group that is FSIN must be SIN. (See [7, 3, 13] and the references there to find the most recent contributions along this line.)

In this paper, we present a new class of FSIN groups that are SIN, namely, the class of $\aleph_0$-bounded groups G whose topology is generated by the G_δ subsets of G. This result is obtained using a previous charaterization of SIN groups which are $\aleph_0$-bounded and non-Archimedean. Firstly, we shall recall the following definitions that have been taken from [8], [12] and [10] respectively.

Definition 1. *A subset A of a topological group G is said to be* left neutral in G *if for every neighbourhood V of the identity in G there is a neighbourhood U of the identity such that $UA \subset AV$. In a similar way we define* right neutral *subsets. A subset that is both left and right neutral is said to be* neutral.

Definition 2. *A subset A of a topological group G is said to be* thin *in G if for any neighbourhood V of the identity there exists a neighbourhood U of the identity such that $x^{-1}Ux \subset V$ for all $x \in A$.*

Definition 3. *A subset A of a topological group is said to be* left uniformly discrete *in G if it is uniformly discrete with respect to the left uniform structure, that is, for a suitable neighbourhood V of the identity the left translates aV and bV are disjoint whenever $a, b \in A$ and $a \neq b$. In like manner are defined* right uniformly discrete subsets.

Neutral subgroups were introduced in [10]. Later, this notion was extended to arbitrary subsets of a topological group to investigate Dugundji compacta in a coset space, (cf. [4]). The concept appears again in [8] in connection to the FSIN-property. In the same paper the authors also introduce the definitions of left (right) uniformly discrete subset. Finally, the idea of a thin subset is due to Tkačenko who studied a slight variant of the definition considered here mainly in relation to the study of free topological groups, (cf. [12, 11]). Notice that every SIN group is a thin subset in itself. Itzkowitz and Protasov (see [7, 9]) use the terms *balanced* group and *refined* (resp. *F-refined*) subset for the same concepts that we have denoted SIN group and thin (resp. left neutral) subset. Finally, a topological group G is called $\aleph_0$-*bounded* when for every neighbourhood U of the neutral element of G there is a countable subset $S \in G$ such that $G = S \cdot U$.

We shall make use of the following two results stated in [8].

Lemma 1. (Megrelishvili, Nickolas and Pestov, [8]) *Let V be a neighbourhood of the identity in a topological group G. Then there exists a set $A \subset G$ such that the left translates aV and bV are disjoint whenever $a, b \in A$ and $a \neq b$, and such that $AVV^{-1} = G$.*

Theorem 1. (Megrelishvili, Nickolas and Pestov, [8]) *Every left uniformly discrete subset of a FSIN group is left neutral.*

The equivalence of (1) and (2) below had been obtained by Itzkowitz in [7], we include a proof of it for the reader's sake.

Lemma 2. *For any topological group G the following assertions are equivalent:*

(1) *The group G is SIN.*
(2) *For every left (right) uniformly discrete subset A in G it holds that A is thin in G.*
(3) *For every left (right) uniformly discrete subset A in G it holds that $\Delta A = \{(a, a) : a \in A\}$ is a left (right) neutral in $G \times G$.*

Proof: *1. implies 2.* and *1. implies 3.* are obvious.

2. implies 1.: Let V be an arbitrary neighbourhood of the identity in G. Clearly, there is a symmetric neighbourhood of the identity W such that $W^9 \subset V$. Applying Lemma 1, there is a subset A of G such that the left translates aW^2 and bW^2 are disjoint whenever $a, b \in A$ and $a \neq b$, and such that $AW^4 = G$. Hence, A is left uniformly discrete and, by hypothesis, thin in G. Then there exists a neighbourhood of the identity U such that $Ua \subset aW$ for all $a \in A$.

Now, let g be arbitrarily picked in G. We have that $g = ah$ with $a \in A$ and $h \in W^4$. If we take any $u \in U$ then $ug = uah$ and, since

$Ua \subset aW$, there exists an element $w \in W$ such that $ug = uah = awh$. Thus, $ug = ah(h^{-1}wh) = g(h^{-1}wh) \in gW^4WW^4 \subset gW^9 \subset gV$. That is, we have verified that $Ug \subset gV$ for all $g \in G$ and we are done.

3. implies 2.: Let A be any left uniformly discrete subset in G; that is, there is a neighbourhood of the identity V such that the left translates aV and bV are disjoint whenever $a, b \in A$ and $a \neq b$. It is readily seen that $\Delta A = \{(a, a) : a \in A\}$ is left uniformly discrete in $G \times G$ and, therefore, left neutral as well. Hence, for any neighbourhood of the identity W in G there is a neighbourhood of the identity U such that $(U \times U) \cdot \Delta A \subset \Delta A \cdot (V \cap W \times V \cap W)$. Now, if we pick any singleton $a \in A$ it holds that, for any $u \in U$, there is $b \in B$ such that $(ua, a) \in (b, b) \cdot (V \cap W, V \cap W)$. Then $a \in bV$ and, as a consequence, $a = b$. Thus $ua \in aW$ for all $a \in A$ or, equivalently, $UA \subset AW$. This completes the proof. □

Following Enflo (cf. [1]) we shall say that a subset M in a topological group G is *uniformly open* if there is a neighbourhood of the identity V such if $x \in M$ then $Vx \subset M$. A topological group G is said to be *locally uniformly open* (or *non-Archimedean*) if there is a base for the neighbourhood system of the identity which consists of uniformly open neighbourhoods. It is easily verified that G is locally uniformly open if and only if it contains a base for the neighbourhood system consisting of open subgroups (cf. [1, th. 1.3.1]). Using the theorem above, we obtain the following characterization the SIN property for non-Archimedean groups.

Theorem 2. *Let G be an $\aleph_0$-bounded locally uniformly open topological group. Then G is a SIN group if and only if every left uniformly continuous real-valued function on G is right uniformly continuous.*

Proof: Let A be a left uniformly discrete subset in G and let V be an arbitrary neighbourhood of the identity in G. Let W be a neighbourhood of the identity such that aW and bW are disjoint whenever $a, b \in A$ and $a \neq b$. Then, since the group is $\aleph_0$-bounded, we deduce that A is a countable subset of G. Set $A = \{a_n\}_{n<\omega}$ and let N be an open subgroup in G with $N \subset V \cap W$. Then the family $\{gN : g \in G\}$ defines a clopen partition of G in countably many subsets at most. Moreover, as $N \subset W$, we have that a_nN and a_mN are disjoint whenever $n, m < \omega$ and $n \neq m$. Define a real-valued function f on G by $f(g) = n$ if $g \in a_nN$, $n < \omega$ and $f(g) = 0$ if $g \notin \cup\{a_nN : n < \omega\}$. It is easily verified that f is left uniformly continuous on G. (Indeed, if $g^{-1}h \in N$ and $g \in xN$ with $x \in G$ then $h \in xN$ too, and $f(g) = f(h)$.) Hence, by hypothesis, f will be right uniformly continuous as well. So that there is a neighbourhood of the identity M such that $M \subset N$ and if $xy^{-1} \in M$ then $|f(x) - f(y)| < 1/3$. Now, applying Lemma 2, there is a neighbourhood of the identity U such that $UA \subset A(V \cap M \cap N)$. Define $L = M \cap N \cap U$.

If $g \in La$ with $a \in A$, then $g = ua$ with $u \in L \subset U$. Hence there is $b \in A$ and $v \in V \cap M \cap N$ with $g = bv$. We claim that $a = b$. Indeed, $b^{-1}g = v \in V \cap M \cap N \subset N$ and, as a consequence, $f(g) = f(b)$. Analogously, $ga^{-1} = u \in L \subset M$, so that $|f(g) - f(a)| < 1/3$. Thus, $|f(a) - f(b)| < 1/3$ or, equivalently, $f(a) = f(b)$. But, according to fashion in which f was defined, it follows that $a = b$. Hence, $ua = av$ with $v \in V$. This proves that $a^{-1}La \subset V$ for all $a \in A$. Therefore, A is a thin subset in G. This completes the proof. □

Corollary 1. *Let G be an $\aleph_0$-bounded topological group whose topology is generated by the G_δ subsets of G. Then G is SIN if and only if is FSIN.*

Proof: Suposse that G is FSIN. By Theorem 2, we just need to verify that each left uniformly continuous real-valued function f on G is right uniformly continuous. Now, for any $n < \omega$, set $f_n = \sup\{\inf\{f, n\}, -n\}$. Clearly, each f_n is bounded and left uniformly continuous. Therefore, by hypothesis, each f_n is right uniformly continuous as well. Thus, given any $\epsilon > 0$, there is a neighborhood of the neutral element, say U_n, such that $gh^{-1} \in U_n$ yields $|f_n(g) - f_n(h)| < \epsilon$. Set $U = \cap_{n<\omega} U_n$ and assume that $gh^{-1} \in U$. Obviously, there is some $m < \omega$ such that $f(g) = f_m(g)$ and $f(h) = f_m(h)$. Therefore, $|f(g) - f(h)| = |f_m(g) - f_m(h)| < \epsilon$. Since U is a neighborhood of the neutral element, the proof is done. □

Corollary 2. (Protasov, [9]) *Let G be a topological group such that $G \times G$ is FSIN then G is SIN.*

Proof: It follows directly from Theorem 1 and Lemma 2. □

The class of topological groups that are known to satisfy that FSIN implies SIN is large. For example, each topological group which is a quasi-k-space (a topological space X is called a quasi-k-space if it has the property that a subset A of X is closed whenever $A \cap K$ is closed in K for every countably compact subspace K of X) or each locally connected topological group belongs to this class. Nevertheless, there are examples of topological groups where Theorem 2 applies and which are not included in either of the preceding classes: it is enough to take G as any Lindelöf topological group with the property that each G_δ subset of G is open (for instance, the free topological group $F(X)$ generated by a Lindelöf topological P-space X). Indeed, it is easily verified that G is a locally uniformly open group which may not be a quasi-k-space because each countably compact subset of G is finite.

We finish this note giving a characterization of the SIN property by means of the sets of equiuniformly continuous functions. In order to do it we need the following definition.

Definition 4. (Isbell, [2, pag. 43]) *A set of real-valued functions* $\{f_i\}_{i\in I}$ *defined on a topological group* G *is called* left (right) equiuniformly continuous *when for every* $\varepsilon > 0$ *there is a neighbourhood of the identity* U *such that for all* $x, y \in G$ *with* $x^{-1}y \in U$ *(*$xy^{-1} \in U$*) it holds that* $|f_i(x) - f_i(y)| < \varepsilon$ *for each* $i \in I$*.*

Theorem 3. *Let* G *be a topological group. Then* G *is SIN if and only if every set of real-valued left (right) equiuniformly continuous functions is right (left) equiuniformly continuous.*

Proof: Necessity is obvious. For the sufficiency, consider a uniformly discrete subset A of G. Using the left uniformly discreteness of A and passing to a smaller neighbourhood if necessary, we can assume without loss of generality that V is a symmetric neighbourhood of the identity such that the left translates aV^2 and bV^2 are disjoint whenever $a, b \in A$ and $a \neq b$. Now, repeating the same arguments given in [8, lemma 1], we can obtain a left uniformly continuous function $f : G \longrightarrow \mathbb{R}$ with the properties $f(e_G) = 1$, $f|_{G\setminus V} \equiv 0$, and $0 \leq (x) \leq 1$ for all $x \in G$. Then, again as in [8, lemma 1], for every $a \in A$, we define the function $f_a : G \longrightarrow \mathbb{R}$ by $f_a(x) = f(a^{-1}x)$. It is easily checked that the set of real-valued functions $\{f_a : a \in A\}$ is left equiuniformly continuous and, therefore, right equiuniformly continuous. This means that there is a neighbourhood $U$of the neutral element such that for all $x, y \in G$ with $xy^{-1} \in U$ it holds that $|f_a(x) - f_a(y)| < 1/2$ for each $a \in A$. Now take any element $x \in Ua$ with $a \in A$, arbitrarily chosen. We have that $xa^{-1} \in U$ and as a consequence $f_a(x) > f_a(a) - 1/2 = 1/2$. This means that $f(a^{-1}x) \neq 0$ and, from the way in which f was taken, it follows that $a^{-1}x \in V$ or, equivalently, that $a^{-1}Ua \subset V$ for all $a \in A$. The proof is completed by applying Lemma 2. □

In [11] Sipacheva and Tkačenko characterize the thin subsets of a free topological group $F(X)$ (in the sense of Markov) for any topological space X which is not a P-space. As a part of their main result they obtain that, whenever A is a thin subset in $F(X)$ and X is not a P-space, it holds that supp(A) is functionally bounded in X; that is, the restriction to A of every real-valued continuous function defined on X is bounded in A. Here, supp(A) has the usual meaning as the union of all the "letters" that appear in the words belonging to A. As a consequence of this characterization and Theorem 2 above, it follows for example that if X is a topological space which is neither a P-space and nor pseudocompact then $F(X)$ is not SIN.

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Universidad Jaume I, 12071-Castallón, Spain

E-mail address: hernande@mat.uji.es