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INVERSE LIMITS ON CIRCLES USING WEAKLY CONFLUENT BONDING MAPS

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ABSTRACT. In this paper we consider the circle-like continua which are inverse limits on circles using weakly confluent bonding maps. These include all proper circle-like continua and some indecomposable chainable continua. In an attempt to understand the nature of the chainable continua in this class of continua, we include an investigation of a specific parameterized family of mappings of $[0, 1]$ which arise from certain weakly confluent, inessential mappings of the circle.

0. INTRODUCTION

In 1954, Capel [2] showed that the inverse limit of an inverse sequence $\{X_i, f_i\}$ where, for each positive integer i, $X_i = S^1$ (the unit circle in the plane) and f_i is a monotone surjection is homeomorphic to S^1. In 1985, J. J. Charatonik showed that inverse limits on circles with open bonding maps produce the solenoids as do inverse limits on circles using confluent bonding maps [3]. The latter result also follows from theorems of Krupski [8], W. Charatonik [4] and Capel [2]. These observations prompted W. J. Charatonik to ask in seminar at the University of Missouri – Rolla in the fall of 1999 which continua arise as an inverse limits on circles using weakly confluent bonding maps. In Section 2 of this paper we provide a solution to this problem. Attempting to refine this classification

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led the author to the investigation of the family of inverse limits studied in Section 3 of this paper.

By a *continuum* we mean a compact, connected subset of a metric space. By a *mapping* we mean a continuous function. A point x is said to be an *endpoint* of a continuum M provided whenever H and K are subcontinua of M containing x then H is a subset of K or K is a subset of H. A continuum is said to be *decomposable* if it is the union of two of its proper subcontinua and is called *indecomposable* if it is not decomposable. If $X_1, X_2, X_3, \cdots$ is a sequence of topological spaces and $f_1, f_2, f_3, \cdots$ is a sequence of mappings such that, for each positive integer i, $f_i : X_{i+1} \to X_i$, then by the *inverse limit* of the inverse sequence $\{X_i, f_i\}$ we mean the subset of $\prod_{i>0} X_i$ to which the point x belongs if and only if $f_i(x_{i+1}) = x_i$ for $i = 1, 2, 3, \cdots$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is denoted $\varprojlim \{X_i, f_i\}$. It is well known that if each factor space, X_i is a continuum, the inverse limit is a continuum. In case we have a single factor space, I, and a single bonding map, f, we denote the inverse limit by $\varprojlim \{I, f\}$. We denote the projection of the inverse limit into the nth factor space by π_n. If K is a subcontinuum of the inverse limit, we denote $\pi_n[K]$ by K_n. If $f : X \to Y$ is a mapping and $f[X] = Y$, we write $f : X \twoheadrightarrow Y$. If $f : X \twoheadrightarrow Y$ is a mapping and K is a subcontinuum of Y, we say the f is *weakly confluent with respect to* K provided some component of $f^{-1}(K)$ is thrown by f onto K. A mapping $f : X \twoheadrightarrow Y$ is *weakly confluent* provided it is weakly confluent with respect to each subcontinuum of Y.

1. Weakly confluent maps of the circle to itself

Suppose $f : S^1 \to S^1$ is a mapping. Denote by *exp* the covering map of the reals, R, onto S^1 given by $exp(t) = e^{it}$. Following Cook [5], there is a map $f^* : R \to R$ such that $exp \circ f^* = f \circ exp$. Indeed, there are infinitely many choices for f^* but we merely choose one. Note that f is inessential if and only if $f^*(0) = f^*(2\pi)$. Consequently, f is essential if and only if f^* is a surjection. Denote by $M(f)$ the number $\frac{f^*(b)-f^*(a)}{2\pi}$ where $f^*(b)$ is the maximum value of $f^*|[0, 2\pi]$ and $f^*(a)$ is the minimum value of $f^*|[0, 2\pi]$. The number $M(f)$ is independent of the choice of f^* and is denoted by

$R(f)$ in [6] and by $AS(f)$ in [9]. Intuitively, $M(f)$ is the largest "number" of times any arc lying on S^1 is wrapped around S^1. The following lemma is the heart of our arguments for weak confluence of maps of the circle.

Lemma. *Suppose $f : S^1 \twoheadrightarrow S^1$ and $f^* : R \to R$ are mappings such that $exp \circ f^* = f \circ exp$. If K is a subcontinuum of S^1 then f is weakly confluent with respect to K if and only if $exp|f^*[R]$ is weakly confluent with respect to K.*

Proof: Suppose K is an arc lying in S^1 and $exp|f^*[R]$ is weakly confluent with respect to K. Then, there is an arc α lying in $f^*[R]$ such that $exp[\alpha] = K$. Since f^* is weakly confluent, there is an arc β lying in R such that $f^*[\beta] = \alpha$. If H is $exp[\beta]$, then H is an arc lying in S^1 such that $f[H] = K$.

On the other hand, if K is a subcontinuum of S^1 and H is a subcontinuum of S^1 such that $f[H] = K$, there is an arc β lying in R such that $exp[\beta] = H$. But, $exp \circ f^*$ throws β onto K so $f^*[\beta]$ is a subarc of $f^*[R]$ such that $exp[f^*[\beta]] = K$. □

Theorem 1. (Feuerbacher) *If $f : S^1 \to S^1$ is essential, then f is weakly confluent.*

Proof: Since f is essential, f^* is surjective. Thus, if K is a subcontinuum of S^1, there is an arc α such that $exp[\alpha] = K$. By the lemma, f is weakly confluent with respect to K. It follows that f is weakly confluent. □

In his dissertation at the University of Houston in 1974 which he published in Fundamenta Mathematicae in 1980, Gary Feuerbacher proved Theorem 1 [6 , Lemma 6, p. 6]. The proof we presented above is different from that of Feuerbacher. As a consequence of Theorem 1, the problem of characterizing the weakly confluent maps of the circle to itself reduces to determining which inessential maps of the circle are weakly confluent. In the following theorem we see that an inessential map of S^1 onto itself is weakly confluent if and only if some arc lying in S^1 is wrapped around S^1 twice. The results in Theorem 2 and its corollary have been known at least ten years [9, 13.69, p. 309], however, the author has not seen that proof and since this paper is for the most part about weakly confluent maps of the circle which are inessential, we include its proof for the sake of completeness.

Theorem 2. *Suppose $f : S^1 \twoheadrightarrow S^1$ is inessential. Then, f is weakly confluent if and only if $M(f) \geq 2$.*

Proof: Suppose $f : S^1 \twoheadrightarrow S^1$ and $M(f) \geq 2$. Since $M(f) \geq 2$, and f is inessential, $f^*[R]$ is an arc whose length is at least 4π. So, if K is a subarc of S^1 there is an arc α lying in $f^*[R]$ such that $f^*[\alpha] = K$. So $exp|f^*[R]$ is weakly confluent with respect to K. By the Lemma, f is weakly confluent with respect to K and, so, f is weakly confluent.

On the other hand, suppose $M(f) < 2$ and a and b are numbers in $[0, 2\pi]$ such that $\frac{f^*(b)-f^*(a)}{2\pi} = M(f)$. Choose numbers c and d such that $f^*(b) < c < d < f^*(a)+4\pi$ and let K be the arc $exp[d-2\pi, c]$. There are two intervals which intersect $f^*[R]$ and are mapped by exp onto K. These are $[d-2\pi, c]$ and $[d-4\pi, c-2\pi]$. However, since $f^*(b) < c$ and $d - 4\pi < f^*(a)$, no subinterval of $f^*[R]$ is thrown by exp onto K, so by the Lemma, f is not weakly confluent. □

Corollary. (Brooks) *Suppose $f : S^1 \twoheadrightarrow S^1$ is a mapping. Then, f is weakly confluent if and only if f is essential or $M(f) \geq 2$.*

Two intervals are said to be non-overlapping if they are mutually exclusive or they share only a common endpoint. If f is a mapping of an interval I onto an interval J and n is a positive integer, we say that f satisfies the *n-pass condition* provided there are n mutually non-overlapping subintervals of I each of which is thrown by f onto J. An inverse limit on intervals produces an indecomposable inverse limit if each bonding map satisfies the two-pass condition [7, Theorem 6.3].

Theorem 3. *If $f : S^1 \twoheadrightarrow S^1$ is an inessential, weakly confluent map, $f^* : R \to R$ is a map such that $exp \circ f^* = f \circ exp$ and $[u, v]$ is an interval of length at least 4π, then $f^*|[u, v]$ satisfies the three-pass condition.*

Proof: Suppose $f : S^1 \twoheadrightarrow S^1$ is an inessential, weakly confluent map and $f^* : R \to R$ is a map of the reals such that $exp \circ f^* = f \circ exp$. Since f is inessential, $f^*[R]$ is an interval. Since f^* is weakly confluent and the length of $[u, v]$ is at least 4π, there is an interval $[c, d]$ lying in $[u, u + 2\pi]$ such that $f[c, d] = f^*[R]$ with $\{f^*(c), f^*(d)\}$ being the endpoints of $f^*[R]$. It is easy to see that $[c, d]$, $[d, c+2\pi]$ and $[c+2\pi, d+2\pi]$ all lie in $[u, v]$ and each is thrown by f^* onto $f^*[R]$. □

2. Inverse limits

We now consider inverse limits on circles using weakly confluent bonding maps. Since each essential map of the circle is weakly confluent, this class of continua includes all the circle-like continua which are not chainable (the *proper* circle-like continua). Thus, we are led to consider inverse limits on circles using inessential, weakly confluent bonding maps.

Theorem 4. *If $\{X_i, f_i\}$ is an inverse limit system where, for each positive integer i, $X_i = S^1$ and f_i is an inessential, weakly confluent map, then $\varprojlim \{X_i, f_i\}$ is homeomorphic to $\varprojlim \{[0,1], g_i\}$ where, for each i, g_i satisfies the three-pass condition.*

Proof: Let $M = \varprojlim \{S^1, f_i\}$ where, for each i, f_i is inessential and weakly confluent. Since each f_i is inessential, for each i, there is a map $\phi_i : S^1 \to R$ such that $f_i = exp \circ \phi_i$. For each positive integer i, let $f_i^* = exp \circ \phi_i$. Since f_i is inessential and $exp \circ f_i^* = f_i \circ exp$, we have $f_i^*[R]$ is an interval of length at least 4π and $f_i^*|f_{i+1}^*[R]$ throws $f_{i+1}^*[R]$ onto $f_i^*[R]$. By the Subsequence Theorem [7, Corollary 1.7.1], M is homeomorphic to $\varprojlim \{f_i^*[R], f_i^*|f_{i+1}^*[R]\}$. Rescaling this inverse limit on intervals yields the desired inverse limit on $[0, 1]$. □

Theorem 5 provides an answer to Charatonik's question.

Theorem 5. *If the continuum M is homeomorphic to an inverse limit on circles using weakly confluent bonding maps then M is a proper circle-like continuum or M is homeomorphic to an inverse limit on intervals with each bonding map satisfying the three-pass condition.*

Burgess showed [1, Theorem 7, page 657] that if M is a chainable continuum then in order that M be circle-like it is necessary and sufficient that M be indecomposable or 2-indecomposable. It follows from Theorem 5 that any chainable continuum which arises as an inverse limit on circles using weakly confluent, inessential bonding maps must be indecomposable. It would be interesting to know if each indecomposable chainable continuum is homeomorphic to an inverse limit on circles using inessential, weakly confluent bonding maps.

3. A parameterized family

In an attempt to obtain a better understanding of the rich variety of these chainable continua, the author decided to investigate a specific parameterized family of maps which arise from weakly confluent, inessential maps of the circle. When one employs the author's scheme for representing maps by which one "fattens up" the range and then draws the domain inside according to the pattern of the map, one obtains pictures like those of Figure 1. Specifically, in Figure 1, two maps of an entire family of maps of the circle are represented. The map changes with the choice of where one places the point $(1,0)$ on the domain circle and two possible choices are shown in Figure 1. We shall see that one of these two maps produces the classic B-J-K continuum [7] while the other produces a chainable continuum with no endpoints. Moreover, we shall also see that if n is a positive integer, there are members of this family with only n end points. If f is a member of the family of maps of the circle indicated in Figure 1, $f^*[R]$ is an interval of length 4π and $f^*|f^*[R]$ throws $f^*[R]$ onto itself. When we consider the maps of $[0,1]$ produced by rescaling the maps $f^*|f^*[R]$, we obtain maps such as those shown in Figure 2 with the map shown below being more "typical." The two maps shown in Figure 2 are in fact the maps produced by those from Figure 1. In general, the parameterized family of maps of $[0,1]$ that results from the family of circle maps is given by

$$f_t(x) = \begin{cases} 4x + t & 0 \le x \le \frac{1-t}{4} \\ -4x + (2-t) & \frac{1-t}{4} \le x \le \frac{2-t}{4} \\ 4x - (2-t) & \frac{2-t}{4} \le x \le \frac{3-t}{4} \\ -4x + (4-t) & \frac{3-t}{4} \le x \le \frac{4-t}{4} \\ 4x - (4-t) & \frac{4-t}{4} \le x \le 1 \end{cases}$$

where the parameter t lies in $[0,1]$.

We now focus our attention on inverse limits of inverse systems using a single bonding map chosen from this family of maps. If $0 \le t \le 1$, we denote by M_t the inverse limit of the inverse sequence $\{[0,1], f_t\}$. We shall see that for each non-negative integer n, there is a parameter value t such that M_t has only n endpoints.

Theorem 6. *If* 0 *or* 1 *is periodic of period* p *for* f_t *then* M_t *is an indecomposable continuum with only* p *endpoints.*

Proof: Suppose 0 is periodic of period p. For convenience of notation we denote f_t by f and M_t by M. Let E denote the set to which the point x of M belongs if and only if $x_i = 0$ for infinitely many integers i. It is clear that the cardinality of E is p. Moreover, each point of E is an endpoint of M. To see this suppose x is a member of E and H and K are subcontinua of M containing x. Then, for infinitely many integers i, H_i is a subset of K_i or, for infinitely many integers i, K_i is a subset of H_i since $x_i = 0$ for infinitely many i. However, if $i > j$ and H_i is a subset of K_i then H_j is a subset of K_j since $f^{i-j}[H_i] = H_j$ and $f^{i-j}[K_i] = K_j$. Thus, for every i, H_i is a subset of K_i or, for every i, K_i is a subset of H_i. Consequently, H is a subset of K or K is a subset of H, so x is an endpoint of M.

If x is not in E, we shall now show that x is not an endpoint of M by showing that there exist two arcs α and β lying in M such that $\alpha \cap \beta = \{x\}$. Let $\mathcal{O} = \{1\} \cup \{0, f(0), \ldots, f^{p-1}(0)\}$. There exists a positive integer N such that if $i \geq N$ then x_i is not in $\mathcal{O}$. There exist two arcs α_N and β_N lying in $[0,1]$ such that $\alpha_N \cap \beta_N = \{x_N\}$ and $\alpha_N \cup \beta_N$ does not intersect $\mathcal{O}$. There exist arcs α_{N+1} and β_{N+1} lying in $[0,1]$ such that $\alpha_{N+1} \cap \beta_{N+1} = \{x_{N+1}\}$ and such that f throws α_{N+1} and β_{N+1} onto α_N and β_N, respectively. Continuing this process we build inverse systems $\{\alpha_i, f|\alpha_i\}$ and $\{\beta_i, f|\beta_i\}$ (for $i \leq N$, take α_i and β_i to be the f^{N-i} images of α_N and β_N, respectively) whose inverse limits are the arcs α and β. $\square$

Theorem 7. *For* $t = \frac{1}{2}$, M_t *has no endpoints.*

Proof: As in the proof of Theorem 5 we denote f_t by f and M_t by M. Let x be a point of M. If $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots)$ then, as in the second half of the proof of Theorem 5, we can produce arcs α and β such that $\alpha \cap \beta = \{x\}$, so x is not an endpoint of M. If x is not $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots)$, since $f(0) = \frac{1}{2}$ and $\frac{1}{2}$ is fixed by f, there is a positive integer N such that if $i \geq N$ then x_i is not in $\{0, \frac{1}{2}, 1\}$. Now, again as in the second half of the proof of Theorem 5, we can produce arcs α and β such that $\alpha \cap \beta = \{x\}$, so x is not an endpoint of M. $\square$

We end this paper with some computations to show that, for each positive integer p, there is a member of the family $\{f_t\}$ which has 0 in a periodic orbit of period p. Since $f_t(\frac{2-t}{4}) = 0$ and $f_t(\frac{4-t}{4}) = 0$ and $f(0) = t$, we get 0 to be in a periodic orbit of period 2 when $t = \frac{2-t}{4}$ and when $t = \frac{4-t}{4}$. Thus, when $t = \frac{2}{5}$ and when $t = \frac{4}{5}$ we have 0 in a periodic orbit of period 2.

To obtain periodic orbits for 0 of higher orders, we introduce some notation. Let us denote by $f_{t,2}$ the inverse of $f_t|[\frac{1-t}{4}, \frac{2-t}{4}]$ and by $f_{t,3}$ the inverse of $f_t|\frac{2-t}{4}, \frac{3-t}{4}]$. Solving the equation $t = f_{t,2}(\frac{2-t}{4})$ yields $t = \frac{6}{19}$. Thus, for $t = \frac{6}{19}$, $f_t^2(t) = 0$, so 0 is periodic of period 3. We could also get 0 in a periodic orbit of period 3 by choosing t to be the solution to the equation $t = f_{t,3}(\frac{2-t}{4})$. This yields $t = \frac{10}{21}$. In order to obtain a periodic orbit of period p for 0, one could solve $t = (f_{t,3})^{p-2}(\frac{2-t}{4})$ for t. For each p, there is a solution to this equation since the sequence $(f_{t,3})^n(\frac{2-t}{4})$ increases to the fixed point for f_t in $[\frac{2-t}{4}, \frac{3-t}{4}]$. Of course, the map $f_{t,2}$ could have been employed in place of $f_{t,3}$ to produce a different parameter value for producing a periodic orbit of 0 of period p. Yet another choice in producing a periodic point of period p would be to mix up the choice between $f_{t,2}$ and $f_{t,3}$ (e.g., to get 0 in an orbit of period 4, solve $t = f_{t,2} \circ f_{t,3}(\frac{2-t}{4})$ which yields $t = \frac{22}{75}$). It might be interesting to know whether the set of parameter values which produce periodic orbits of 0 is dense in some interval.

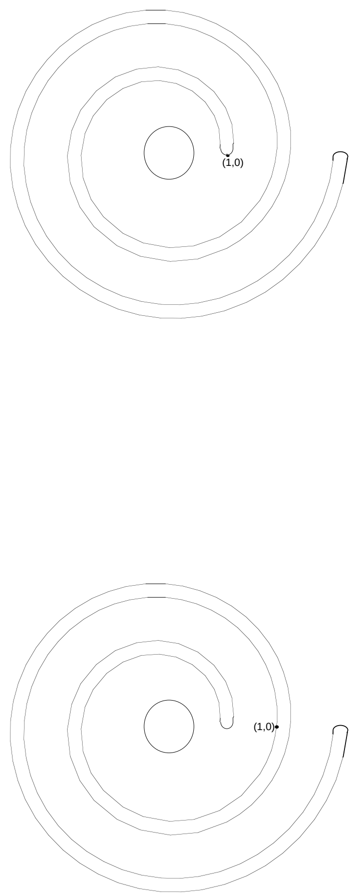

Figure 1

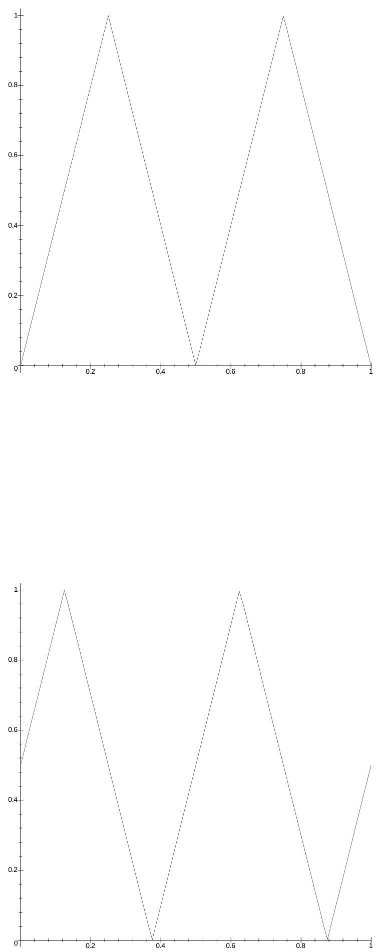

Figure 2

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