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UPWARDS PRESERVATION BY ELEMENTARY SUBMODELS

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Abstract. Given a topological space X and an elementary submodel M we can define a new topological space X_M. We investigate for which topological properties P it is true that if X_M has P, then X has P. We first look at this question in general and then we impose conditions on M. In particular, we show some preservation results assuming M to be ω-covering and we also show that, under CH, the properties of being ω-covering and countably closed are equivalent for any elementary submodel M. After, we investigate how much we can weaken the hypothesis of M being ω-covering.

1. Introduction

Let $\langle X, \mathcal{T} \rangle$ be a topological space and let M be an elementary submodel of some "large enough" $H(\theta)$ (see e.g. [11], [3] or [20]). In [10] we considered the new topological space $X_M = X \cap M$ with the topology generated by $\mathcal{T} \restriction M = \{U \cap M : U \in \mathcal{T} \cap M\}$ and we investigated, in particular, which topological properties of X were preserved when we take X_M. This was a first step in the systematic study of topological spaces induced by elementary submodels. The motivation of this study was the fact that elementary submodels have been used in set-theoretic topology with increasing frequency over the past 20 years; see for example [3], [4], [2], [20], [1].

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In this paper we take a slightly different approach than the one considered on [10]. There the approach was "downwards", i.e. going from X to X_M, here it is "upwards". More precisely, we investigate for which topological properties P the statement "X_M has P implies X has P" is true. Through this article, when this statement is true we will say that P is "preserved upwards".

This approach was considered also in [19], [14], [16], where some upwards preservation results were proved and some questions about recovering X from X_M were answered. We refer the reader to any of the papers mentioned above for a more detailed introduction on the topic.

In this paper we investigate the upwards preservation of some basic topological properties as separability, compactness, second countability, etc. In [19], F. D. Tall proved that T_2 and T_3 are preserved upwards. It is easy to see that T_0 and T_1 are also preserved upwards (this is similar to the T_2 case). On the other hand, it is easy to see that $T_{3\frac{1}{2}}$ and T_4 are not preserved in general. For instance, let X be any regular space that is not $T_{3\frac{1}{2}}$ (or not T_4) and take M a countable elementary submodel. Then X_M will be a regular second countable space (since M is countable) and therefore it is metrizable. This example also shows that we cannot expect many properties to be preserved in general. For a less trivial example of non-preservation of normality (one where X_M is uncountable) we have the following:

Example 1.1. *There is a non-normal space X and an elementary submodel M such that X_M is uncountable and normal.*

Proof: Let A and B be two disjoint stationary subsets of some cardinal κ such that for each $\alpha < \kappa$ we have that $A \cap \alpha$ and $B \cap \alpha$ are non-stationary. Let M be any elementary submodel such that $M \cap \kappa$ is an ordinal $\gamma < \kappa$.

We take $X = A \times B$. By a result of [12], we have that X is not normal (it is a product of two stationary disjoint subsets). However, the same result implies that $X_M = (A \cap \gamma) \times (B \cap \gamma)$ is normal.

Remark 1.2. A consistent first countable example is to take X to be the Niemytzky plane. Assume $MA + \neg CH$ and take M of size $\aleph_1$. Then X_M is normal because $M \cap \mathbb{R}$ is a Q-set (see e.g. [18]).

However, there are some general positive preservation results which are shown in the second section of this paper. In the third section we show some preservation results assuming M to be ω-covering and in the fourth section we investigate how much we can weaken this hypothesis on M. In the last section we connect preservation of non-metrizability to Hamburger's question.

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2. Some general preservation results

In this section we show that there are some properties that are upwards preserved for any elementary submodel M.

Theorem 2.1. *Let X be a topological space and let M be an elementary submodel. If X_M is compact, then X is compact.*

Proof: Suppose X is not compact. This means that

$$\exists \mathcal{C} \in \mathcal{P}(\mathcal{T}) \text{ such that} \bigcup \mathcal{C} = X \text{ and } \forall \mathcal{C}' \subseteq \mathcal{C} \text{ finite } \exists x \in X \setminus \bigcup \mathcal{C}'.$$

By elementarity, we can get $\mathcal{C} \in M$. Fix such $\mathcal{C}$. We then have that

$$H(\theta) \models \bigcup \mathcal{C} = X \text{ and } \forall \mathcal{C}' \subseteq \mathcal{C} \text{ finite } \exists x \in X \setminus \bigcup \mathcal{C}'.$$

Since $\mathcal{C} \in M$, by elementarity, we have that $\bigcup(\mathcal{C} \cap M) \supseteq X \cap M$. But then $\mathcal{C} \cap M$ is an open cover of $X \cap M$ without finite subcover (by elementarity, $\mathcal{C} \cap M$ has no finite subcover in M, but all finite subsets of $\mathcal{C} \cap M$ are in M). This contradicts the compactness of X_M and we are done.

Note that a similar argument will show:

Theorem 2.2. *Let X be a topological space and M be an elementary submodel. If X_M is countably compact, then X is countably compact.*

Proof: Just have to repeat the previous proof but assuming $\mathcal{C}$ countable.

The main theorem of section 3 in [10] implies, in particular, that if X is compact Hausdorff and M is an elementary submodel with $X \in M$, then there is $Y \subseteq X$ such that X_M is a perfect preimage of Y. To show this result, it was defined, for every $x \in X_M$, the set

$$K_x = \bigcap \{V \in \mathcal{T} \cap M : x \in V\}.$$

Using this notation and the proof of this theorem we can show the following result (M will always be an elementary submodel satisfying $X \in M$):

Lemma 2.3. *If X_M is compact, then for every $x \in X$ there is $x' \in X \cap M$ such that $x \in K_{x'}$.*

Proof: Suppose not. Then there is $x \in X$ such that for every $x' \in X \cap M$ we have $x \notin K_{x'}$. Fix such x. By the definition of $K_{x'}$, this implies that

$$\text{for all } x' \in X \cap M \text{ there is } V_{x'} \in \mathcal{T} \cap M \text{ such that}$$
$$x' \in V_{x'} \text{ and } x \notin V_{x'}.$$

But then $\{V_{x'} : x' \in X \cap M\}$ is an open cover of X_M. Therefore, by compactness of X_M, it has a finite subcover $\mathcal{V}$. Since $\mathcal{V} \subseteq M$ and $\mathcal{V}$ is finite, we have that $\mathcal{V} \in M$. Also, $M \models \mathcal{V}$ covers X. Thus, by elementarity, $\mathcal{V}$ covers X, a contradiction because $x \notin \bigcup \mathcal{V}$.

We can now show:

Theorem 2.4. *If X_M is compact and T_2, then there is a perfect map from X to X_M.*

Proof: Take Y as in the proof of Theorem 3.2 in [10], namely $Y = \bigcup \{K_x : x \in X_M\}$. By the previous lemma we have that $Y = X$. Also, by theorem 2.1, X is compact and therefore regular (it is T_2 because X_M is T_2 and being T_2 is upwards preserved [19]). We can now apply Theorem 3.2 in [10] to get the perfect map from X to X_M.

We then have the following:

Corollary 2.5. *X_M is compact T_2 if and only if X is compact T_2 and X_M is a perfect image of X.*

F. D. Tall showed in [19] that the property of being locally compact is also preserved upwards for any elementary submodel M. It is not hard to show that connectedness is also always preserved (this was independently shown by R.G.A. Prado and F. D. Tall [17]):

Theorem 2.6. *For any elementary submodel M, if X_M is connected then X is connected.*

Proof: Suppose X is not connected. Then there are two disjoint clopen subsets of X, A and B, such that $X = A \cup B$. By elementarity, there are such A and B in M. But then $X_M = (A \cup B) \cap M$, and $A \cap M$ and $B \cap M$ are clopen in X_M, a contradiction.

3. M ω-covering

Some basic topological properties involving countable sets, like separability, second countable, etc, are not preserved in general (for instance we can take X regular and not having one of these properties and M countable, as done in the introduction). But they will be preserved if we make extra assumptions on M. The more natural assumption would be to assume M countably closed, which seems a bit strong. We will show that it is enough to assume M ω-covering, which is in general weaker (we can have M ω-covering of size $\aleph_1$ [3]). However, we will also show that, under CH, the properties of being ω-covering and countably closed are equivalent.

Definition 3.1. An elementary submodel M is *countably closed* if it is closed for countable subsets, i.e. ${}^{\omega}M \subseteq M$.

Definition 3.2. An elementary submodel M is *ω-covering* if for every countable subset A of M there is a countable set $B \in M$ such that $A \subseteq B$.

In [10] we proved some results in which we assumed M to be countably closed and raised the question if this hypothesis could be weakened to M ω-covering. We had examples that showed that in some cases it cannot, if one assumes $\neg CH$, but there was still the case of what happens under CH. We show here that, under CH, these two properties are equivalent, which solves the problem.

We separate the following lemma from the proof because it will be used later:

Lemma 3.3. *If M is an ω-covering elementary submodel, then $\omega_1 \subseteq M$.*

Proof: Suppose that there is $\alpha < \omega_1$ such that $M \cap \omega_1 = \alpha$ (note that $M \cap \omega_1$ is always an ordinal). But then, by ω-covering, there is a countable $A \in M$ such that $\alpha \subseteq A$. Since A is countable, $A \in M$ implies that $A \subseteq M$. Also, we can suppose that $A \subseteq \omega_1$ (just take $A' = A \cap \omega_1$). Now, $\alpha \notin M$ and $A \in M$ imply that $\alpha \subsetneq A$. Thus, there is $\beta \in (M \cap \omega_1) \setminus \alpha$, contradicting the choice of α.

Theorem 3.4. *CH is equivalent to every ω-covering elementary submodel of H_θ is countably closed.*

Proof: It is clear that if $\neg CH$, then there is an elementary submodel M which is ω-covering but not countably closed (just take any ω-covering elementary submodel of size $\aleph_1$). Assume now that we have CH and that M is an ω-covering elementary submodel.

By CH, $H_\theta \models$ there is a bijection $f : \omega_1 \longrightarrow \mathcal{P}(\omega)$. Then, by elementarity, there is $f \in M$, such that $f : \omega_1 \longrightarrow \mathcal{P}(\omega)$ is a bijection. Since $f \in M$ and $\omega_1 \subseteq M$, we then have that $\mathcal{P}(\omega) \subseteq M$.

Fix $A \subseteq M$ countable. We have to show that $A \in M$. By ω-covering, there is $B \in M$ countable such that $A \subseteq B$. Since B is countable, there is a bijection $g : B \longrightarrow \omega$. By elementarity we can take $g \in M$ (since $B \in M$).

Let $C = \{n \in \omega : \text{there is } a \in A \text{ such that } g(a) = n\} = f(A)$. Then $C \subseteq \omega$ and therefore $C \in M$ (since $\mathcal{P}(\omega) \subseteq M$). But $g \in M$, so $g^{-1}(C) = A \in M$ and we have what we wanted.

We now give some preservation results. The result for second countable spaces is essentially proved in [3] and the result for countable tightness is due to R.G.A. Prado and is in [16].

Theorem 3.5. *The following topological properties are preserved upwards if we assume M to be ω-covering:*

(a) second countability;

(b) first countability;

(c) Lindelöfness;
(d) separability;
(c) countable tightness.

Proof: (a) Since X_M is second countable, there is $\mathcal{B} \subseteq \mathcal{T} \cap M$ countable such that it is a base of X_M. But then, by ω-covering, there is $\mathcal{B}' \in M$ countable such that $\mathcal{B} \subseteq \mathcal{B}'$. We can take $\mathcal{B}' \subseteq \mathcal{T}$ (since $\mathcal{T} \in M$).

Since $\mathcal{B} \subseteq \mathcal{B}'$, we have that $\mathcal{B}'$ is also a base of X_M. Thus, using $\mathcal{B}' \in M$ we have

$$M \models \mathcal{B}' \text{ is a base of } X,$$

which, by elementarity, implies that $\mathcal{B}'$ is a base of X. Thus, X is second countable.

(b) This is similar to (a) but we work with a base $\mathcal{B}_x$ for $x \in X \cap M$ instead of $\mathcal{B}$.

(c) We carry out the proof as in the compact case (theorem 2.1), but with a countable subcover instead of a finite subcover. The only difference will be that we do not necessarily have that all countable subsets of $\mathcal{C} \cap M$ are in M. But we can use ω-covering to get that for every $\mathcal{C}' \subseteq \mathcal{C}$ countable, there is $\mathcal{C}'' \in M$ countable satisfying $\mathcal{C}' \subseteq \mathcal{C}''$. Since $\mathcal{C} \in M$ we can take $\mathcal{C}''$ also satisfying $\mathcal{C}'' \subseteq \mathcal{C}$. So, if $\mathcal{C} \in M$ is the open cover witnessing that X is not Lindelöf, then $\mathcal{C} \cap M$ has no countable subcover in M. But X_M Lindelöf implies that there is a countable subcover $\mathcal{C}'$ of $\mathcal{C} \cap M$, so there is a countable subcover $\mathcal{C}'' \in M$ as above, a contradiction.

(d) The idea is the same as in the previous proofs. If X_M is separable, then there is a countable set $D \subseteq X \cap M$ dense in X_M. But M ω-covering gives us a countable set $E \in M$ such that $D \subseteq E$ and we can take $E \subseteq M \cap X$. Since $D \subseteq E$, E is dense in X_M, and therefore $M \models X$ has a countable dense subset. Thus, by elementarity, X is separable.

(e) Suppose not, then there is $A \subset X$ and there is $x \in cl(A)$ such that for every $B \subset A, |B| \leq \aleph_0$, we have that $x \notin cl(B)$. By elementarity, we can take $A \in M$ and $x \in M$. Note that $x \in cl(A)$ implies $x \in cl_{\tau_M}(A \cap M)$ (also by elementarity).

As $t(X_M) = \aleph_0$, there is $B \subset A \cap M$ such that $|B| = \aleph_0$ and $x \in cl_{\tau_M}(B)$. Using that M is $\omega-$covering, there is $C \in M, |C| =$

$\aleph_0$ and $B \subset C$. Call $D = C \cap A$. Notice that $D \in M, D \subset A, |D| = \aleph_0$ and $x \in cl_{\tau_M}(D \cap M)$.

To finish the proof note that $x \in cl(D)$. For otherwise

$$\exists V \in \mathcal{T} \text{ such that } x \in V \text{ and } V \cap D = \emptyset.$$

This implies,

$$M \models \exists V \in \mathcal{T} \text{ such that } x \in V \text{ and } V \cap D = \emptyset.$$

Pick this V in $\mathcal{T} \cap M$. The previous sentence tells us that $V \cap D \cap M = \emptyset$. This contradicts the fact that $x \in cl_{\tau_M}(D \cap M)$. So $t(X) = \aleph_0$.

Corollary 3.6. *If X_M is separable metrizable and M is an ω-covering elementary submodel with $X \in M$, then X is separable metrizable.*

Proof: If X_M is separable metrizable, then X_M is second countable and regular and therefore X is second countable regular. Thus X is separable metrizable.

It is natural to ask if we can get Lemma 2.2 for X Lindelöf and M ω-covering. The same proof shows that the result is true if we assume M to be countably closed:

Theorem 3.7. *For a countably closed elementary submodel M and a space X of pointwise countable type (with $X \in M$), if X_M is Lindelöf, then X_M is a perfect image of X.*

We next show that, for regular spaces, pointwise countable type is also preserved upwards if M is ω-covering:

Theorem 3.8. *Suppose X is regular, M is an ω-covering elementary submodel with $X \in M$ and that X_M has pointwise countable type. Then X has pointwise countable type.*

Proof: Fix $x \in X_M$. Then there is a compact K (in X_M) such that $x \in K$ and $\chi(K, X_M) = \aleph_0$. Let $\mathcal{B} = \{U_n : n \in \omega\}$ be a countable outer base of K. Note that each U_n is an open set in X_M but we may have $U_n \notin M$.

We claim that without loss of generality we can suppose $\mathcal{B} \subseteq \mathcal{T} \cap M$. To see that, fix $n \in \omega$. For each $y \in K$, there is $V_y \in \mathcal{T} \cap M$ such that $y \in V_y \subseteq U_n$. Then $\mathcal{V} = \{V_y : y \in K\}$ is an open cover of

K and therefore it has a finite subcover $\mathcal{V}'$. Since $\mathcal{V}' \subseteq M$ and it is finite, $\mathcal{V}' \in M$, so $\bigcup \mathcal{V}' \in M$. Also $\bigcup \mathcal{V}' \supseteq K$ and $\bigcup \mathcal{V}' \subseteq U_n$. Take then $V_n = \bigcup \mathcal{V}'$. Clearly $\{V_n : n \in \omega\}$ is still an outer base of K.

Now, M ω-covering, $\mathcal{B} \subseteq M$ and $\mathcal{B}$ countable imply that there is $\mathcal{B}' \in M$ countable such that $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{T} \cap M$. Define $\mathcal{B}_x = \{B \in \mathcal{B}' : x \in B\}$. Note that $\mathcal{B}_x \in M$, since x and $\mathcal{B}'$ are in M, and that $\mathcal{B}_x$ is countable. We can then enumerate (in M) $\mathcal{B}_x = \{B_n : n \in \omega\}$. Also, by taking finite intersections we can make $\mathcal{B}_x$ decreasing and we still have a family in M. Therefore, without loss of generality, we can suppose $\mathcal{B}_x$ is decreasing.

Still working in M, using the regularity of the space, we can inductively pick $V_n \in \mathcal{T} \cap M$ such that $x \in V_n \subseteq \overline{V_n} \subseteq B_n \cap V_{n-1}$, for each $n \in \omega$. Let $\mathcal{B}'_x = \{V_n : n \in \omega\}$. Thus $\mathcal{B}'_x \in M$ and for every $n \in \omega$, we have $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n$.

We then have that $K' = \bigcap \mathcal{B}'_x$ is closed and it is in M, since $\mathcal{B}'_x \in M$. Also $K' \subseteq K$ (since $\overline{V_n} \subseteq B_n$ and $\mathcal{B}_x \supseteq \mathcal{B}$). Thus, K' is a closed subset of K, so it is compact (in X_M), and it has countable character.

We have shown that for every $x \in X \cap M$ there is a compact (in X_M) $K' \in M$ such that $x \in K'$ and $\chi(X', X_M) = \aleph_0$. By elementarity, this is true in H_θ which implies that X has pointwise countable type.

The next example shows that some assumption on M is needed in the previous theorem:

Example 3.9. *Let $X = \mathbb{N}^{\omega_1}$ and suppose M is such that $M \cap \omega_1$ is countable. Then X does not have pointwise countable type but X_M does.*

Proof: X does not have pointwise countable type because it is an uncountable power of a non-compact space (see e.g. [6]). Since $M \cap \omega_1$ is countable, we have that X_M is homeomorphic to a subspace of $\mathbb{N}^\omega$, which is first countable and therefore it has pointwise countable type.

Remark 3.10. We can have $M \cap \omega_1$ countable without M being countable. This is the case if we have Chang's Conjecture. But if one assumes that $0^\#$ does not exist, M uncountable implies $\omega_1 \subseteq M$ [14].

We finish this section with the following preservation results (for M countably closed) and examples from R.G.A. Prado's thesis [16]. Since these results do not appear elsewhere we include the proofs here (with R.G.A. Prado's kind permission):

Theorem 3.11. *Suppose that M is a countably closed elementary submodel and that $X \in M$ is a topological space. If X_M is sequential then X is sequential.*

Proof: Suppose not, i.e. suppose there are A sequentially closed and $x \in cl(A) \setminus A$. By elementarity, we can pick such A and x in M.

Note that $A \cap M$ is sequentially closed in X_M. To see that, let $f : \omega \longrightarrow A \cap M$ be a convergent sequence to $y \in X \cap M$ (we mean convergence in the $\mathcal{T}_M$ sense) and call $B = \{f(n) : n \in \omega\}$. We then have that $f \subset \omega \times B \subset M$ and f is countable. So, using that M is countably closed we can conclude that $f \in M$ and $B \in M$. Then,

$$M \models f : \omega \longrightarrow B \text{ converges to } y \text{ (in the } \mathcal{T} \text{ sense) },$$

so by elementarity, $f : \omega \longrightarrow B \subset A$ converges to y (in the $\mathcal{T}$ sense). As A is sequentially closed, $y \in A$, so $y \in A \cap M$.

As X_M is sequential, we have that $A \cap M$ is closed in X_M. But, by our choice of A and x and by elementarity, we have $M \models x \in cl(A) \setminus A$. This implies that $x \notin A \cap M$ and, by the definition of $\mathcal{T}_M$, also implies that $x \in cl_{\tau_M}(A \cap M) = A \cap M$, a contradiction.

Theorem 3.12. *Suppose that M is a countably closed elementary submodel and that $X \in M$ is a topological space. If X_M is Fréchet then X is Fréchet.*

Proof: Suppose that X is not a Fréchet space. Then $M \models X$ is not a Fréchet space. We get then, $M \models \exists A \subset X,\ \exists x \in X$ such that $x \in cl(A)$ but no sequence contained in A converges to x. Again, pick x and A in M.

First note that $x \in cl_{\tau_M}(A \cap M)$. Otherwise $\exists V \in \mathcal{T} \cap M$ such that $x \in V$ and $V \cap A \cap M = \emptyset$. This is equivalent to $M \models x \in V$ and $V \cap A = \emptyset$, contradicting our assumption.

As X_M is Fréchet we can pick $f : \omega \longrightarrow A \cap M$ a sequence converging to x. Now, M is countably closed, $f \subset M$ and f is countable so $f \in M$. As in the previous proof $M \models f$ converges to x (in the $\mathcal{T}$ sense), a contradiction.

It is important to notice that some restriction on the elementary submodel is necessary for the above results to hold. For a trivial example, we can again take any countable elementary submodel M and any $X \in M$ with uncountable tightness.

The condition that M is not only $\omega-$covering but countably closed required for Theorem 3.11 and Theorem 3.12 is also (consistently) necessary (by theorem 3.4, under CH these two conditions are equivalent). In [16], there is an example, due to A. Dow, assuming $\mathbf{p} > \omega_1$, of a topological space X and an $\omega-$covering elementary submodel M of size $\aleph_1$ containing X such that X is not sequential but X_M is Fréchet.

4. Weakening the hypothesis of M being ω-covering

As we mentioned before none of the properties of theorem 3.5 are preserved in general, since we can take M countable. One could ask what happens if we assume M uncountable. Since the main point there was not the size of M but the size of $M \cap \mathcal{T}$, we can have an example similar to 3.9:

Example 4.1. *Suppose there is an uncountable elementary submodel M such that $M \cap \omega_1$ is countable. Then there is X not first countable such that X_M is a separable metrizable space.*

Proof: Let $X = 2^{\omega_1}$. Since $M \cap \omega_1$ is countable, X_M is homeomorphic to a subspace of 2^α, where α is a countable ordinal [10], and therefore it is a separable metrizable space.

A natural question then is to ask if we can weaken "M ω-covering" to "$\omega_1 \subseteq M$" in theorem 3.5. Recall that in lemma 3.3 we showed that M ω-covering implies $\omega_1 \subseteq M$. Also, $0^\#$ does not exist and $|M| \geq \aleph_1$ imply $\omega_1 \subseteq M$ [14]. We will show that we can, if we make extra assumptions on the space. We will start by giving some examples, which shows not only that these assumptions are necessary, but also that the main point actually seems to be the cofinality of $M \cap \kappa$, for a certain cardinal κ.

Note that, if κ is a cardinal, we can take $M = \bigcup_{n\in\omega} M_n$, where each M_n is an elementary submodel including ω_1 such that

$M_n \subset M_{n+1}$ and $M_n \cap \kappa = \alpha_n < \kappa$. Then $\gamma = M \cap \kappa$ has countable cofinality.

Example 4.2. *There is a regular space X and an elementary submodel M such that $\omega_1 \subseteq M$ and X is not first countable, but X_M is.*

Proof: Let $X = \kappa + 1$, where κ is a regular cardinal, $\kappa \geq \omega_2$, with the following topology: κ is discrete and the basic open sets at κ are of the form $(\alpha, \kappa]$, for $\alpha < \kappa$. Clearly X is not first countable. However, if M is an elementary submodel such that $M \cap \kappa$ has countable cofinality, then X_M is first countable.

Remark 4.3. Note that in the previous example, X does not have countable tightness, but X_M does.

Example 4.4. *There is a non-Lindelöf space X and an elementary submodel M with $\omega_1 \subseteq M$ such that X_M is Lindelöf.*

Proof: Take $X = \omega_2$ with the ordinal topology and M an elementary submodel such that $\omega_1 \subseteq M$ and $M \cap \omega_2 = \alpha$, α with countable cofinality. Then X is not Lindelöf, but $X_M = \alpha$ is Lindelöf, since it is σ-compact (if $\{\alpha_n : n \in \omega\}$ is a sequence cofinal in α, then we can write $\alpha = \bigcup_{n \in \omega} \alpha_n + 1$).

Remark 4.5. The above example is also an example of a space X and an elementary submodel M such that X is not σ-compact but X_M is an uncountable σ-compact space.

Example 4.6. *There is a space X and an elementary submodel M such that X_M is separable uncountable but X is not separable.*

Proof: Let $X \subseteq 2^{\omega_2}$ be such that $f \in X$ if and only if there is $\alpha < \omega_2$ such that $f(\beta) = 0$ for every $\beta > \alpha$. Note that X is not separable: if $D = \{f_n : n \in \omega\}$ is a countable subset of X, then for every $n \in \omega$ there is $\alpha_n < \omega_2$ such that $f_n(\beta) = 0$, for every $\beta > \alpha_n$; but now for $\alpha > sup\{\alpha_n : n \in \omega\}$, $V = \{f \in X : f(\alpha) = 1\}$ is open and $V \cap D = \emptyset$, which implies that D is not dense.

Fix an elementary submodel M such that $M \cap \omega_2 = \delta$, where $cf(\delta) = \omega$. We will show that X_M is separable.

Let $\{\delta_n : n \in \omega\} \subseteq M$ be an increasing sequence of ordinals cofinal in δ. Note that by the definition of X and by elementarity, $f \in X \cap M$ if and only if there is $\gamma < \delta$ such that $f(\beta) = 0$, for every $\beta > \gamma$.

For each $n \in \omega$, we have $\delta_n \in M$ and therefore $2^{\delta_n} \in M$. Also, $\delta_n < \omega_2$, so 2^{δ_n} is separable. By elementarity, for each $n \in \omega$, we can then take $A_n \in M$ such that A_n is a countable dense subset of 2^{δ_n}. Define

$$D_n = \{f \in X \cap M : f \restriction \delta_n \in A_n \text{ and } f(\beta) = 0 \text{ for every } \beta \geq \delta_n\}.$$

Let $D = \bigcup_{n\in\omega} D_n$. Note that each D_n is countable (since A_n is countable) and therefore D is countable.

It just remains to show that D is dense in X_M. Let $V_p = \{f \in X \cap M : p \subseteq f\}$, p finite, be a basic open set of X_M. If $n \in \omega$ is such that $\delta_n > sup\{\beta \in dom(p) : p(\beta) \neq 0\}$, then $D_n \cap V_p \neq \emptyset$, and therefore $D \cap V_p \neq \emptyset$.

In what follows, unless mentioned otherwise, X will always be a topological space and M an elementary submodel with $X \in M$. The next result will show that, in the previous examples, it was essential that we always took elementary submodels M such that $M \cap \kappa$ had countable cofinality, for a certain κ:

Theorem 4.7. *Any of the following implies $cf(\kappa \cap M) = \omega$:*
(a) X_M is second countable and $w(X) = \kappa$;
(b) X_M is first countable and $\chi(X) = \kappa$;
(c) X_M is separable, $d(X) = \kappa$ and $|X| = \kappa$.

Proof: (a) Since $w(X) = \kappa$, there is $\mathcal{B} \in \mathcal{P}(\mathcal{T})$ such that $\mathcal{B}$ is a base of X of size κ (and therefore for every $\mathcal{B}' \subseteq \mathcal{B}$ of size $< \kappa$, $\mathcal{B}'$ is not a base of X). By elementarity we can take $\mathcal{B} \in M$. Also, $\mathcal{B} \cap M$ is a base for X_M. So X_M second countable implies that there is $\mathcal{B}' \subseteq \mathcal{B} \cap M$ countable such that $\mathcal{B}'$ is a base of X_M.

Now, $|\mathcal{B}| = \kappa$, so there is a bijection f from κ onto $\mathcal{B}$. Again, by elementarity, we can take $f \in M$ and we have that

$$M \models f : \kappa \longrightarrow \mathcal{B} \text{ is a bijection.}$$

Therefore,

$$f : \kappa \cap M \longrightarrow \mathcal{B} \cap M \text{ is a bijection.}$$

Note that $B \in M$, for every $B \in \mathcal{B}' \subseteq \mathcal{B} \cap M$. Since $f \in M$, this implies that $f^{-1}(B) \in M$. Thus the set $I = \{f^{-1}(B) : B \in \mathcal{B}'\} \subseteq M \cap \kappa$ and it is countable.

Suppose $cf(\kappa \cap M) > \omega$. Then I is bounded in $\kappa \cap M$ and we can fix $\gamma \in \kappa \cap M$ such that $\gamma > \alpha$, for every $\alpha \in I$. Thus $I \subseteq \gamma$.

Now, define in M, $\mathcal{B}''$ as the image of γ by the function f. Since γ and f are in M, we have that $\mathcal{B}'' \in M$. Also note that $\mathcal{B}' \subseteq \mathcal{B}''$, because $I \subseteq \gamma$. Thus, $\mathcal{B}''$ is a base of X_M.

We then have that

$$M \models \mathcal{B}'' \text{ is a base of } X.$$

Thus, by elementarity, $\mathcal{B}''$ is a base of X. But $\mathcal{B}'' \subseteq \mathcal{B}'$ and $|\mathcal{B}''| \leq |\gamma| < \kappa$, a contradiction.

(b) Similar to (a), but work with a point $x \in X \cap M$ and a base $\mathcal{B}_x$ at x instead of $\mathcal{B}$.

(c) Since $|X| = \kappa$, there is a bijection $f : \kappa \longrightarrow X$ and, using elementarity, we can take f in M. Separability of X_M gives us a countable set $D \subseteq X \cap M$ dense in X_M.

Suppose $cf(\kappa \cap M) > \omega$. Then, since D is countable, $f^{-1}(D)$ is bounded in $\kappa \cap M$. Therefore we can pick $\alpha \in \kappa \cap M$ such that $\alpha > sup(f^{-1}(D))$. Since $\alpha \in M$, we have that $E = f"\alpha \in M$. But also $D \subseteq E$ and thus E is dense in X_M. We then have that there is a set E in M which is dense in X_M and $f^{-1}(D)$ is bounded in $\kappa \cap M$. This implies that $M \models X$ has a dense subset of size $< \kappa$. Thus, by elementarity, $d(X) < \kappa$, a contradiction.

The following corollaries of 4.7 (b) and (c) will be used later:

Corollary 4.8. *If X_M is first countable, $\chi(X) \leq \aleph_1$ and $\omega_1 \subseteq M$, then X is first countable.*

Corollary 4.9. *If $|X| = \aleph_1$, X_M is separable and $\omega_1 \subseteq M$, then X is separable.*

In [19], F. D. Tall showed that hereditary separability and hereditary Lindelöfness are upwards preserved whenever $\omega_1 \subseteq M$.

The assumption $|X| = \kappa$ in 4.7 (c) seems, at a first glance, to be extra. But the following modification of example 4.6 shows that the hypotheses $|X| = \aleph_1$ cannot be weakened to $d(X) = \aleph_1$ in the previous corollary:

Example 4.10. *There is a space X and an elementary submodel M such that $\omega_1 \subseteq M$ and $d(X) = \aleph_1$, but X_M is separable.*

Proof: Let $D = \{f_\delta : \delta < \omega_1\}$ be any dense subset of 2^{ω_2} of size $\aleph_1$. Fix $\gamma < \omega_2$ of cofinality ω_1 and $\{\gamma_\alpha : \alpha < \omega_1\}$ a cofinal increasing sequence converging to γ. For each $\delta < \omega_1$ and $\alpha < \omega_1$ define:

$$g_\alpha^\delta(\beta) = \begin{cases} 0 & \text{if } \beta \in [\gamma_\alpha, \gamma) \\ f_\delta(\beta) & \text{otherwise} \end{cases}$$

Then $E = \{g_\alpha^\beta : \alpha < \omega_1,\ \delta < \omega_1\}$ is a dense subset of 2^{ω_2} of size $\aleph_1$ without countable dense subsets (in 2^{ω_2}).

We now take X as in example 4.6 and $Y = X \cup E$. Then $d(Y) = \aleph_1$, but we can take again M an elementary submodel such that $M \cap \omega_2$ is an ordinal of countable cofinality and proceed as in 4.6 to show that Y_M is separable.

We have the following corollaries of 4.7 (b):

Corollary 4.11. *Suppose M and X are such that $\omega_1 \subseteq M$, X_M is separable metrizable and $w(X) \leq \omega_1$. Then X is separable metrizable.*

Corollary 4.12. *(CH) If X_M is separable metrizable and $\omega_1 \subseteq M$, then X is separable metrizable.*

It is natural to ask if CH is necessary in the previous corollary. We make this discussion below, but first we investigate how a counterexample should look.

Lemma 4.13. *Let M be an elementary submodel with $\omega_1 \subseteq M$. Suppose there is a space $X \in M$ such that X is not separable metrizable but X_M is. Then there is a space $Y \in M$ with $Y = \omega \cup \{x\}$ such that $\chi(x, Y) > \aleph_1$ but Y_M is first countable (and therefore separable metrizable).*

Proof: Suppose there is $X \in M$ not separable metrizable. Then X is not second countable (note that since X_M is metrizable, it is regular, so X is also regular), so there is $Z \subset X$ of cardinality $\aleph_1$ such that Z is not second countable (see e.g. [9]). By elementarity we can take $Z \in M$. Since we are assuming $\omega_1 \subseteq M$, this implies that $Z \subseteq M$. Also, Z_M is separable and $|Z| = \omega_1$, so corollary 4.9 give us that Z is separable.

Now, Z_M is separable metrizable but Z is not and $Z \cap M = Z$. So the topologies of Z_M and Z are obviously different. Thus, Z cannot be first countable (otherwise we would have that Z_M is a subspace of Z [10], and therefore they would have the same topology). Note that since $\omega_1 \subseteq M$, by corollary 4.8, we actually have that $\chi(Z) > \aleph_1$ and therefore we can fix $x \in Z$ such that $\chi(x, Z) > \aleph_1$.

Using the fact that Z is separable, we can now take D a countable dense subset of Z. By elementarity (since $Z \in M$), we can take $D \in M$. Then $D \cup \{x\}$ is a subspace of Z satisfying the conditions we want for Y (note that $\chi(x, Z) = \chi(x, D \cup \{x\})$ because D is dense in Z). So we can take $Y = \omega \cup \{x\}$ homeomorphic to $D \cup \{x\}$.

The previous lemma gives one of the implications of the following equivalence, which was formulated by G. Gruenhage (he proved this implication independently; the proof of the other direction is also due to him).

Lemma 4.14. *The following are equivalent:*

(a) there is a non metrizable space X and an elementary submodel M satisfying $\omega_1 \subseteq M$ and $X \in M$ such that X_M is an uncountable separable metrizable space;

(b) there is a filter $\mathcal{F}$ on ω which is not countably generated (of size $\geq \aleph_2$) and an uncountable elementary submodel M such that $\omega_1 \subseteq M$, $\mathcal{F} \in M$ and $\mathcal{F}_M = \mathcal{F} \cap M$ is countably generated.

Proof: Suppose we have (a) and take $Y = \omega \cup \{x\}$ as in the previous lemma. Look at the filter $\mathcal{F}$ on ω determined by the neighborhoods of x. Then $\chi(x, Y) > \aleph_1$ implies that $\mathcal{F}$ is not countably generated, but Y_M first countable implies that $\mathcal{F}_M$ is.

For the other implication, let $\mathcal{F}$ and M be as in (b) and define $X = \omega \cup \{\mathcal{F}\}$, with the points of ω being isolated and the neighborhoods of $\mathcal{F}$ being given by $\{\mathcal{F}\} \cup F$, for $F \in \mathcal{F}$. Clearly X is not first countable since $\mathcal{F}$ does not have a countable base, so X is not metrizable. But $\mathcal{F}_M$ is countably generated, which implies that X_M is first countable and therefore it is separable metrizable. To get an uncountable space we can replace each isolated point by a copy of the real line, i.e., we can take Y as the topological sum of countably many copies of the real line plus the point $\mathcal{F}$, where a basic neighborhood of $\mathcal{F}$ is $\mathcal{F}$ with F many copies of the real line for $F \in \mathcal{F}$. Then Y_M is second countable because it is first countable at $\mathcal{F}$ and therefore it is metrizable, but, as before, Y is not metrizable.

The following example is also due to G. Gruenhage:

Example 4.15. *($\mathfrak{t} = \aleph_2$) There is a space X such that X_M is separable metric but X is not metrizable.*

Proof: Suppose $\mathcal{T}$ is a tower of size $\aleph_2$ and let $\mathcal{F}$ be the filter generated by $\mathcal{T}$. Since we are assuming $\mathfrak{t} = \aleph_2$, $\mathcal{F}$ is not countably generated. Take M any elementary submodel of size $\aleph_1$ such that $\delta = M \cap \omega_2$ has countable cofinality. We then have that $\mathcal{F}_M$ is countably generated, so we can now use the previous lemma to get the example.

In view of corollary 4.12 and the previous example, it is natural to ask if $\neg CH$ is consistent with "for any M with $\omega_1 \subseteq M$, X_M separable metrizable implies X separable metrizable". This is equivalent to ask if the negation of (b) in lemma 4.14 is consistent with $\neg CH$. To answer this question we will use the following result from [5]. I would like to thank A. Dow and I. Juhász for pointing it out to me.

Lemma 4.16. *Let V be a model of CH and G be a $Fn(\kappa, 2)$-generic filter. If, in $V[G]$, $\mathcal{F}$ is a filter on ω which is not countably generated, then there are filters $\mathcal{F}_\alpha$ ($\alpha < \omega_1$) such that every member of $\mathcal{F}$ is a member of all but countably many of the $\mathcal{F}_\alpha$'s and for every $\alpha < \omega_1$ there is some $a \in \mathcal{F}_\alpha$ such that $\omega \setminus a \in \mathcal{F}$.*

Using this lemma we can prove the following:

Theorem 4.17. *If V, G and $\mathcal{F}$ are as in the previous lemma and (in $V[G]$) M is an elementary submodel with $\omega_1 \subseteq M$ and $\mathcal{F} \in M$, then $\mathcal{F} \cap M$ is not countably generated.*

Proof: Suppose $\mathcal{F} \cap M$ is countably generated and take $\{A_n : n \in \omega\}$ generating $\mathcal{F} \cap M$. By the lemma above and by elementarity, there is $\{\mathcal{F}_\alpha : \alpha < \omega_1\} \in M$ as in the conclusion of the lemma. Since $\omega_1 \subseteq M$ we have that $\mathcal{F}_\alpha \in M$ for every $\alpha < \omega_1$.

Now, for each $n \in \omega$, $A_n \in \mathcal{F} \cap M$ and therefore $A_n \in \mathcal{F}_\alpha$, for every $\alpha \in \omega_1 \setminus I_n$, for some countable set I_n. Thus there is $\alpha \in \omega_1$ such that $A_n \in \mathcal{F}_\alpha$ for every $n \in \omega$. Then $\mathcal{F} \cap M \subseteq \mathcal{F}_\alpha$, which implies that $\mathcal{F} \cap M \subseteq \mathcal{F}_\alpha \cap M$. Now, $\mathcal{F}_\alpha \in M$, so M models that $\mathcal{F} \subseteq \mathcal{F}_\alpha$ and, by elementarity, we have that $\mathcal{F} \subseteq \mathcal{F}_\alpha$. But there is $a \in \mathcal{F}_\alpha$ such that $\omega \setminus a \in \mathcal{F}$, a contradiction.

This and lemma 4.14 then give us the following:

Corollary 4.18. *Let V be a model of CH and G be a $Fn(\kappa, 2)$-generic filter. In $V[G]$, if X_M is separable metrizable and M is an elementary submodel with $X \in M$ and $\omega_1 \subseteq M$, then X is separable metrizable.*

We finish this section proving that for the Lindelöf property and pointwise countable type we have results similar to theorem 4.7:

Theorem 4.19. *If X_M is Lindelöf and $L(X) = \kappa$, for some successor cardinal κ, then $cf(\kappa \cap M) = \omega$.*

Proof: Suppose $cf(\kappa \cap M) > \omega$. Since $L(X) = \kappa$ and κ is a successor cardinal, there is an open cover $\mathcal{C}$ of X with no subcover of size $< \kappa$ (otherwise we would have $L(X) \leq \lambda$, for λ such that $\kappa = \lambda^+$). Since $L(X) = \kappa$, we can suppose that $\mathcal{C}$ has size κ. By elementarity, we can take $\mathcal{C} \in M$ and we have that $\mathcal{C} \cap M$ is an open cover of X_M. Thus it has a countable subcover $\mathcal{D}$.

Fix a bijection $f : \kappa \longrightarrow \mathcal{C}$. We can take $f \in M$, by elementarity, and we have that $f : \kappa \cap M \longrightarrow \mathcal{C} \cap M$ is a bijection.

Since $f \in M$, $f^{-1}(B) \in M$ for every $B \in \mathcal{D}$. By assumption, $cf(\kappa \cap M) > \omega$ and $\mathcal{D}$ is countable. So there is $\alpha \in \kappa \cap M$ such that $\alpha > sup\{\beta \in \kappa \cap M : \beta = f^{-1}(B),\ B \in \mathcal{D}\}$. Let (in M) $\mathcal{D}' = f"\alpha$. Then $\mathcal{D}'$ is a subcover of $\mathcal{C} \cap M$ and it is in M. But this implies that

$$M \models \mathcal{C} \text{ has a subcover of size } < \kappa$$

which, by elementarity implies that $\mathcal{C}$ has a subcover of size $< \kappa$, a contradiction.

The following corollary follows immediately from the previous result:

Corollary 4.20. *X_M Lindelöf, $L(X) \leq \aleph_1$ and $\omega_1 \subseteq M$ imply that X is Lindelöf.*

For pointwise countable type, we have the following:

Theorem 4.21. *If X is a regular space with $w(X) = \aleph_1$ and M is such that $\omega_1 \subseteq M$ and X_M has pointwise countable type, then X has pointwise countable type.*

Proof: Let $\mathcal{B}_0 \in M$ (by elementarity) be a base of size $\aleph_1$ of X, and we can suppose that $\mathcal{B}_0$ is closed under finite unions. For $x \in X \cap M$, there is a compact set K (in X_M) such that $x \in K$ and $\chi(K, X_M) = \aleph_0$. As in the proof of theorem 3.8, we now have to work to get a compact set $K' \in M$ satisfying the same properties.

Using that K is compact, we can argue as in the proof of theorem 3.8, and show that, without loss of generality, we can suppose that there is $\mathcal{B} = \{U_n : n \in \omega\} \subseteq M$ an outer base for K such that $U_n \in \mathcal{B}_0$ for every $n \in \omega$ (this last condition is possible because we are assuming that $\mathcal{B}_0$ is closed under finite unions).

Now, there is a bijection $f : \omega_1 \longrightarrow \mathcal{B}_0$ and we can take it in M. Take $\alpha < \omega_1 \subseteq M$, $\alpha > sup\{f^{-1}(U_n) : n \in \omega$ and $\mathcal{B}' = f''\alpha\}$. Then $\mathcal{B}'$ is countable, $\mathcal{B}' \in M$ and $\mathcal{B}' \supseteq \mathcal{B}$. We can now finish the proof as in theorem 3.8.

5. Preservation of non-metrizability and Hamburger's problem

We saw that if M is ω-covering, then the property of being "separable metrizable" is upwards preserved (corollary 3.6). What about only metrizability? If the question is for any M and any X, the examples given in the previous section for non-upwards preservation of first countability, can also be used here (e.g. example 4.2). So the more interesting cases are when first countability is upwards preserved, for instance if M is ω-covering (theorem 3.5), or when we simply assume that X is first countable. In [19], F. D. Tall gives a consistent example of a non-metrizable first countable space of size $\aleph_2$ such that X_M is metrizable for every M of size $\aleph_1$. This example is a first countable non-metrizable space with all subspaces of size $\aleph_1$ metrizable, i.e., it is an example for Hamburger's problem (see e.g. [8]). We next show that any such example has to be of this form. We first show:

Theorem 5.1. *If X is a first countable space such that there is an elementary submodel M with $X \in M$, X_M metrizable and $\omega_1 \subseteq M$, then all subspaces of size $\aleph_1$ of X are metrizable.*

Proof: Since X is first countable, X_M is a subspace of X [10]. Suppose there is $Y \subseteq X$ such that $|Y| = \aleph_1$ and Y is not metrizable. We then have that

$H_\theta \models \exists Y \in \mathcal{P}(X)$ such that $|Y| = \aleph_1$ and $(Y, \mathcal{T})$ is not metrizable.

By elementarity, we can take $Y \in \mathcal{P}(X) \cap M$ such that

$$H_\theta \models |Y| = \aleph_1 \text{ and } (Y, \mathcal{T}) \text{ is not metrizable.}$$

Now, $Y \in M$ and $|Y| = \aleph_1$. Since $\omega_1 \subseteq M$, this implies that $Y \subseteq X \cap M$ and therefore Y is a subspace of X_M (because X_M is a subspace of X). But X_M is metrizable and Y is not, a contradiction.

We have the following corollaries:

Corollary 5.2. *Suppose X and M are such that M is an ω-covering elementary submodel with $X \in M$ and X_M is metrizable. Then X must have all subspaces of size $\aleph_1$ metrizable.*

Proof: Just note that, by theorem 3.5, M ω-covering implies that X is first countable (since X_M metrizable implies X_M first countable) and also implies that $\omega_1 \subseteq M$ (by lemma 3.3).

Corollary 5.3. *Assume M is an elementary submodel with $\omega_1 \subseteq M$ and $|M| = \aleph_1$. There is a first countable non-metrizable space X such that $X \in M$ and X_M is metrizable if and only if there is a first countable non-metrizable space X with all subspaces of size $\aleph_1$ metrizable.*

Proof: One direction is just the previous theorem. For the other direction, suppose there is a first countable non-metrizable space X with all subspaces of size $\aleph_1$ metrizable. Then, if M is an elementary submodel of cardinality $\aleph_1$ with $X \in M$, then X_M is a subspace of X (since X is first countable [10]) of size $\leq \aleph_1$, and therefore it is metrizable.

The previous result connected the question of preservation of non-metrizability by elementary submodels to Hamburger's problem. It is interesting to notice that the question of preservation of non-metrizability by countably closed forcing is also related to Hamburger's question:

Theorem 5.4. *Suppose that X is a topological space with $|X| = \aleph_1$ and that $\mathbb{P}$ is a countably closed partial order. If X is metrizable in $V^{\mathbb{P}}$, then X is metrizable in V.*

Proof: First note that X must be first countable: since X is metrizable in $V^{\mathbb{P}}$, X is first countable in $V^{\mathbb{P}}$; but countably closed forcing does not add new countable subsets (see e.g. [13]), so X was already first countable in V.

Suppose $X = \{x_\alpha : \alpha < \omega_1\}$ and let $\mathcal{V}_\alpha = \{V_n(\alpha) : n \in \omega\}$ be a base at x_α for each $\alpha < \omega_1$. Let $r \in \mathbb{P}$ be such that $r \Vdash X$ is metrizable. Then $r \Vdash X$ has a σ-discrete base. Fix names $\dot{\mathcal{B}}$, $\dot{\mathcal{B}}_n$ and $s \leq r$ such that

$$s \Vdash \dot{\mathcal{B}} \text{ is a base of } X \text{ such that } \dot{\mathcal{B}} = \bigcup_{n\in\omega} \dot{\mathcal{B}}_n \text{ and each } \dot{\mathcal{B}}_n \text{ is discrete.}$$

Also, fix $\dot{\varphi}$ a name for a function in the extension that chooses the open sets of $\dot{\mathcal{B}}$, more precisely, fix a name $\dot{\varphi}$ and $p \leq s$ such that

$$p \Vdash \dot{\varphi} : \omega_1 \times \omega \longrightarrow \mathcal{T} \text{ with } \dot{\varphi}''\omega_1 \times \omega = \dot{\mathcal{B}} \text{ and } \dot{\varphi}''\omega_1 \times \{n\} = \dot{\mathcal{B}}_n.$$

Let $h : \omega_1 \longrightarrow \omega_1 \times \omega$ be any bijection. For each $\alpha < \omega_1$ and each $n \in \omega$ define:

$$D_{\alpha,n} = \{q \in \mathbb{P} : \text{ there is } U \in \mathcal{T} \text{ and } \gamma < \omega_1 \text{ such that } (q \Vdash x_\alpha \in U \subseteq V_n(\alpha) \text{ and } U = \dot{\varphi}(h(\gamma))\ )\}.$$

A standard density argument shows that $D_{\alpha,n}$ is dense below p for every $\alpha \in \omega_1$ and $n \in \omega$.

Define a decreasing sequence $\{p_\beta : \beta < \omega_1\}$ such that $p_0 \leq p$ and $p_\beta \in D_{h(\beta)}$. This is possible because $\mathbb{P}$ is ω_1-closed and the sets are dense below p. For each $\beta < \omega_1$ we can fix an open set $U_{h(\beta)}$ and $\gamma_\beta < \omega_1$ witnessing $p_\beta \in D_{h(\beta)}$.

Define

$$\mathcal{U}_k = \{U_{h(\beta)} : \beta \text{ is such that } h(\gamma_\beta) \in \omega_1 \times \{k\}\,\}$$

and let $\mathcal{U} = \bigcup_{k\in\omega} \mathcal{U}_k$.

We first show that $\mathcal{U}$ is a base for X. Fix $\alpha < \omega_1$ and $n \in \omega$. Let $\gamma = h^{-1}(\alpha, n)$. Then $p_\gamma \Vdash x_\alpha \in U_{h(\gamma)} \subseteq V_n(\alpha)$ which implies that $x_\alpha \in U_{h(\gamma)} \subseteq V_n(\alpha)$.

It just remains to show that $\mathcal{U}_k$ is discrete, for each $k \in \omega$. Since X is first countable, it is enough to show that $\mathcal{V}$ is discrete for every countable $\mathcal{V} \subseteq \mathcal{U}_k$. Suppose $\mathcal{V} = \{U_{h(\alpha_n)} : n \in \omega\}$. Since $\mathbb{P}$ is ω_1-closed, we can take $q \leq p_{\alpha_n}$, for each $n \in \omega$. Then, we have

$$q \Vdash U_{h(\alpha_n)} = \dot{\varphi}(h(\gamma_{\alpha_n})) \text{ and } h(\gamma_{\alpha_n}) \in \omega_1 \times \{k\},$$

which implies $q \Vdash \mathcal{V} = \{U_{h(\alpha_n)} : n \in \omega\} \subseteq \dot{\mathcal{B}}_k$. But $q \leq p$ and $p \Vdash \dot{\mathcal{B}}_k$ is discrete, therefore, $q \Vdash \mathcal{V}$ is discrete. Since to be discrete just depends on basic open sets, it is easy to see that this implies $\mathcal{V}$ discrete and we are done.

As a consequence we have:

Corollary 5.5. *Suppose X is a non-metrizable space such that for some countably closed forcing $\mathbb{P}$, X is metrizable in $V^{\mathbb{P}}$. Then X has all subspaces of size $\aleph_1$ metrizable.*

Here, however, we do not have an equivalent result as we had in the elementary submodel case. This can be shown, for instance, by the next result, which is from [15]. In [15], the result and the proof are given using some notations defined there. The proof below is the same proof, but here we present it in a more topological form and a slightly stronger version. This result shows, for instance, that the ladder system on an E-set example (due to Fleissner [7]) for Hamburger's problem cannot become metrizable after any countably closed forcing. We need the following definition:

Definition 5.6. We say that a topological space X is stationarily κ-collectionwise Hausdorff if for every closed discrete subspace $\{x_\alpha : \alpha < S\}$ indexed by a stationary $S \subseteq \kappa$, there is a stationary $T \subseteq S$ such that $\{x_\alpha : \alpha \in T\}$ can be separated by disjoint open sets. X is stationarily collectionwise Hausdorff if it is stationarily κ-collectionwise Hausdorff for every κ.

Note the same proof that shows that the ladder system on an E-set is not collectionwise Hausdorff, also shows it is not stationarily collectionwise Hausdorff. So by the next result, it cannot be collectionwise Hausdorff after a countably closed forcing, and therefore cannot be made metrizable.

Theorem 5.7. *Suppose that $\mathbb{P}$ is countably closed and that θ is a regular cardinal $\geq \aleph_2$. If X is a first countable regular not stationarily θ-collectionwise Hausdorff space, then X is not θ-collectionwise Hausdorff in $\mathbf{V}^{\mathbb{P}}$.*

Proof: Suppose X is not stationarily θ-collectionwise Hausdorff and take a discrete set $Y \subseteq X$ indexed by a stationary $S \subseteq \theta$ witnessing it. We identify $Y = S$. Suppose S can be separated in $\mathbf{V}^{\mathbb{P}}$ and we will work for a contradiction.

For every $\alpha \in S$, fix $\mathcal{U}_\alpha = \{U_n(\alpha) : n \in \omega\}$ a decreasing base at α and let $\mathcal{U} = \bigcup_{\alpha \in S} \mathcal{U}_\alpha$. Because X is regular, we can suppose that $\mathcal{U}$ is such that the set $H_{\alpha,\beta} = \{(n,m) : U_n(\alpha) \cap U_m(\beta) \neq \emptyset\}$ is finite for each $\alpha, \beta \in S$. To see this, just note that, since S is discrete, we can take, for each $\alpha \in S$, $U_0(\alpha)$ such that $\overline{U_0(\alpha)} \cap S = \{\alpha\}$.

Now, we are assuming that S is separated in $\mathbf{V}^{\mathbb{P}}$, so there is $p_0 \in \mathbb{P}$ and a name for a function $\dot{g}$ such that

$$p_0 \Vdash \dot{g} : S \longrightarrow \omega \text{ and } \{U_{g(\alpha)}(\alpha) : \alpha \in S\} \text{ is a separation of } S.$$

Let M be an elementary submodel of H_κ (for κ large enough) such that we have $\theta, S, \mathcal{U}, \mathbb{P}, \Vdash_{\mathbb{P}}, p_0, \dot{g} \in M$ and such that $M \cap \theta = \beta \in S$ (possible since X is stationary in θ).

If possible, choose $p_1 \leq p_0$, $\alpha_0 < \beta$, $p_1 \in M$ such that

$$p_1 \Vdash U_{g(\alpha_0)}(\alpha_0) \cap U_0(\beta) \neq \emptyset.$$

Note that this would imply $p_1 \Vdash g(\beta) > 0$.

Similarly, by induction, choose, if possible, $p_0 \geq p_1 \geq \ldots \geq p_{n+1}$, $\alpha_n < \beta$, $p_{n+1} \in M$ such that

$$p_{n+1} \Vdash U_{\dot{g}(\alpha_n)}(\alpha_n) \cap U_n(\beta) \neq \emptyset$$

(and therefore $p_{n+1} \Vdash \dot{g}(\beta) > n$).

Note that the process must stop: otherwise, since $\mathbb{P}$ is ω_1-closed, this would imply that there is $s \in \mathbb{P}$ such that $s \Vdash \dot{g}(\beta) > n$, for every $n \in \omega$, a contradiction. Thus, there is $\bar{n} \in \omega$ such that $p_{\bar{n}} \in M$ and for every $q \leq p_{\bar{n}}$, for every $\alpha < \beta$, $q \in M$ implies that $q \nVdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\beta) \neq \emptyset$.

Claim 5.8. *For every $\alpha < \beta$, $p_{\bar{n}} \Vdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\beta) = \emptyset$.*

Proof: Suppose not. Then there is $\alpha < \beta$ and there is $q \leq p_{\bar{n}}$ such that $q \Vdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\beta) \neq \emptyset$. Then there is $\alpha < \beta$ and there is $q \leq p_{\bar{n}}$ such that $q \Vdash (\dot{g}(\alpha), \bar{n}) \in H_{\alpha,\beta}$.

By our assumption on $\mathcal{U}$, $H_{\alpha,\beta}$ is a finite subset of $\omega \times \omega$ and therefore it is in M. Fix $\alpha < \beta$. So

$$H_\lambda \models \text{ there is } q \leq p_{\bar{n}} \text{ such that } q \Vdash (\dot{g}(\alpha), \bar{n}) \in H_{\alpha,\beta},$$

and $p_{\bar{n}}, \dot{g}, \alpha, H_{\alpha,\beta}$ are all in M. Therefore, by elementarity,

$$M \models \text{ there is } q \leq p_{\bar{n}} \text{ such that } q \Vdash (\dot{g}(\alpha), \bar{n}) \in H_{\alpha,\beta}.$$

Thus, there is $q \in M$, $q \leq p_{\bar{n}}$, such that $q \Vdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\beta) \neq \emptyset$, a contradiction.

Let

$$B = \{\gamma \in S : \text{ for every } \alpha < \gamma\, p_{\bar{n}} \Vdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\gamma) = \emptyset\}.$$

Since $\beta = \theta \cap M \in B$ (by the claim), a standard elementary submodel argument shows that B is stationary. Indeed, first note that B has all parameters in M, so

$$M \models \text{ for every club } C \subseteq \theta \text{ there is } \gamma \in C \text{ such that } \gamma \in B$$

(since, for every club C in M, $\beta = M \cap \theta \in C$ and by elementarity). We then have that $M \models B$ is stationary in θ, and therefore, by elementarity again, B is stationary in θ.

For each $\alpha \in S$, let $q_\alpha \leq p_{\bar{n}}$ and $m_\alpha \in \omega$ such that $q_\alpha \Vdash \dot{g}(\alpha) = m_\alpha$. Define (in $\mathbf{V}$), $f : S \longrightarrow \omega$ by $f(\alpha) = m_\alpha + \bar{n}$.

It just remains to show that f gives a separation of B, and we will get a contradiction, by our choice of Y. Fix α, β in B. Without loss of generality, we can suppose $\alpha < \beta$. Since $\beta \in B$, $p_{\bar{n}} \Vdash U_{\dot{g}(\alpha)}(\alpha) \cap U_{\bar{n}}(\beta) = \emptyset$. Therefore, by the definition of f we have

$$q_\alpha \Vdash U_{f(\alpha)}(\alpha) \cap U_{f(\beta)}(\beta) = \emptyset,$$

which implies that the same is true in $\mathbf{V}$ and we are done.

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