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A SUFFICIENT CONDITION THAT THE HIGSON CORONA OF THE HALF OPEN INTERVAL $[0, \infty)$ IS A DECOMPOSABLE CONTINUUM

KAZUO TOMOYASU

ABSTRACT. The Higson compactification (cf. [5]) is a metric dependent compactification. In this paper, we will give a sufficient condition that the Higson corona of the half open interval is a decomposable continuum.

1. INTRODUCTION

All spaces considered in this paper are assumed to be locally compact and Hausdorff. By $C^*(X)$ (resp. $C(X)$), we denote the ring of all bounded real-valued (resp. real valued) continuous functions on X. It is well-known that there is a one-to-one correspondence between the compactifications of a space X and the closed subrings of $C^*(X)$ containing the constants and generating the topology of X. Let $f : X \to Y$ be a continuous function between metric spaces (X, d) and (Y, ρ). We say that the function f satisfies *the* $(*)_d$-*condition* provided that

$$(*)_d \qquad \lim_{x\to\infty} \operatorname{diam}_\rho f\big(B_d(x,r)\big) = 0 \text{ for each } r > 0,$$

that is, for each $r > 0$ and each $\varepsilon > 0$, there is a compact set $K = K_{r,\varepsilon}$ in X such that $\operatorname{diam}_\rho f\big(B_d(x,r)\big) < \varepsilon$ for each $x \in X \setminus K$.

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Let $C^*_d(X) = \{f \in C^*(X) \mid f \text{ satisfies } (*)_d\}$. Then $C^*_d(X)$ is a closed subring of $C^*(X)$.

The Higson compactification $\overline{X}^d$ of a proper metric space (X, d) is the compactification associated with the closed subring $C^*_d(X)$ of $C^*(X)$ [5], where a metric d on a space X is said to be *proper* provided that every bounded subset in X has the compact closure. The remainder $\overline{X}^d \setminus X$ is called the Higson corona and we denote the Higson corona of X by $\nu_d X$. For undefined notations and terminologies, see [1] and [3].

In ([4], Theorem 1.6), we showed the following theorem: Let d be a proper metric on $J = [0, \infty)$ satisfying the following condition (†): $d(x, y) + d(y, z) = d(x, z)$ for each $x,\ y,\ z \in J$ with $x < y < z$. Then the Higson corona $\nu_d J$ is an indecomposable continuum. Of course, the usual metric on $[0, \infty)$ satisfies the condition (†). This condition says that the metric d is induced by a homeomorphism between half-open intervals $[0, \infty)$.

In fact, if a proper metric d satisfies (†), then the map $h : [0, \infty) \to [0, \infty)$ defined by $h(x) = d(0, x)$ is a homeomorphism satisfying $d(x, y) = |h(x) - h(y)|$ for each $x, y \in [0, \infty)$. On the contrary, for any given homeomorphism $h : [0, \infty) \to [0, \infty)$, define $d(x, y) = |h(x) - h(y)|$ for $x, y \in [0, \infty)$. Then we obtain a proper metric d satisfying (†).

In the above argument, the Higson corona of the half-open interval with a proper metric induced by a homeomorphism between half-open intervals $[0, \infty)$ is an indecomposable continuum. Then we are interested in the Higson corona of the half-open interval with a proper metric induced by a subspace metric of the Euclidean plane $\mathbb{R}^2$. In this paper, we give a sufficient condition that the Higson corona of the half-open interval with a proper metric induced by a subspace metric of $\mathbb{R}^2$ is a decomposable continuum.

2. Indecomposable continua and decomposable continua

Given our theorem, we may recall some basic properties concerning the Higson compactification.

Proposition 2.1 ([2], Theorem 1.4)**.** *Let X be a proper metric space with a proper metric d and let Y be a closed subset with* *the induced metric* d_Y*. Then* $\mathrm{cl}_{\overline{X}^d} Y$ *is homeomorphic to* $\overline{Y}^{d_Y}$ ($\mathrm{cl}_{\overline{X}^d} Y \cong \overline{Y}^{d_Y}$).

Proposition 2.2 (cf. [4], Lemma 1.5)**.** *Let* (X,d) *be a non-compact proper metric space and let* N_r *be an* r*-dense* [1] *closed subspace of* X*, where* $r>0$*. Then the Higson corona* $\nu_d X$ *is equal to* $\mathrm{cl}_{\overline{X}^d} N_r \setminus N_r$.

Now, a finite system $\{E_1,\ldots,E_n\}$ of subsets of a proper metric space (X,d) *diverges* if, for each $R>0$ the intersection of the R-neighborhoods of the sets E_i, $i=1,\ldots,n$, is a bounded subset of X. Equivalently, a system $\{E_1,\ldots,E_n\}$ diverges if and only if the function $F: X \to J$ defined by $F(x)=\sum_{i=1}^n d(x,E_i)$ satisfies the condition $\lim_{x\to\infty} F(x)=+\infty$. From Taĭmanov theorem (cf. [3], Theorem 3.5.5) we can state the following characterization which was essentially proved by A.N. Dranishnikov, J. Keesling and V.V. Uspenskij.

Proposition 2.3 (cf. [2], Proposition 2.3)**.** *Let* X *be a non-compact metric space with a proper metric* d*. Then the following conditions are equivalent*:

(1) *A compactification* αX *of* X *is equivalent to* $\overline{X}^d$*, and*
(2) *For disjoint closed subsets* A*,* $B\subset X$*, the system* $\{A,B\}$ *diverges if and only if* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$

Definition 2.4. A topological space is said to be *generalized continuum* (resp. *strongly generalized continuum*) if it is a locally compact connected separable space (resp. connected proper metric space [2]). A connected space is said to have the *complementation property* if the complement of every compact subset has at most one non-relatively compact component.

Lemma 2.5. *Let* X *be a non-compact locally connected strongly generalized continuum with a proper metric* d *having the complementation property. Then* $\nu_d X$ *is a non-metric continuum.*

Proof: Recall that a compact subset K of a locally connected generalized continuum X is contained in a compact subset C such

[1] A subset A of a metric space (X,d) is r-dense provided that for any $x\in X$, $B_r(x,d)\cap A\neq\emptyset$.

[2] Every proper metric space is always locally compact σ-compact.

that $X \setminus C$ has only finitely many components (cf. [6], page 237 9.26). Since X is σ-compact, there exists a compact cover $\{K_n\}_{n<\omega}$ of X such that $K_n \subset \mathrm{int}_X K_{n+1}$ for each $n < \omega$. Using the above fact, for each K_n, there exists a compact subset C_n containing K_n such that $X \setminus C_n$ has only finitely many components. Since X has the complementation property, $X \setminus C_n$ has exactly one non-relatively compact component V_n. Thus, $\mathrm{cl}_{\overline{X}^d} V_n$ is connected and contains $\nu_d X$. Then note that $\nu_d X = \bigcap_{n<\omega} \mathrm{cl}_{\overline{X}^d} V_n$ and therefore connected. □

Proposition 2.6. *Let (X, d) be a locally connected strongly generalized continuum having the complementation property. If $X = Y \cup Z$ such that Y and Z are locally connected strongly generalized continua having the complementation property. Then if there exist non-compact closed subsets $A \subset Y \setminus Z$ and $B \subset Z \setminus Y$ such that $\{A, Z\}$ and $\{Y, B\}$ diverge, then $(\mathrm{cl}_{\overline{X}^d} Y \setminus Y) \setminus (\mathrm{cl}_{\overline{X}^d} Z \setminus Z) \neq \emptyset$, $(\mathrm{cl}_{\overline{X}^d} Z \setminus Z) \setminus (\mathrm{cl}_{\overline{X}^d} Y \setminus Y) \neq \emptyset$, and thus the Higson corona $\nu_d X$ is a non-metric decomposable continuum.*

Proof: Note that

$$\nu_d X = (\mathrm{cl}_{\overline{X}^d} Y \setminus Y) \cup (\mathrm{cl}_{\overline{X}^d} Z \setminus Z)$$

Put $\Sigma_0 = \mathrm{cl}_{\overline{X}^d} Y \setminus Y$ and $\Sigma_1 = \mathrm{cl}_{\overline{X}^d} Z \setminus Z$. Note that $\Sigma_0 \supset \mathrm{cl}_{\overline{X}^d} A \setminus A$ and $\Sigma_1 \supset \mathrm{cl}_{\overline{X}^d} B \setminus B$. From Proposition 2.3 $\Sigma_0 \setminus \Sigma_1 \neq \emptyset$ and $\Sigma_1 \setminus \Sigma_0 \neq \emptyset$. From Proposition 2.1 we note that $\Sigma_0 \cong \nu_{d_Y} Y$ and $\Sigma_1 \cong \nu_{d_Z} Z$, where d_Y and d_Z are subspace metrics induced by d in X and Y, respectively. From Lemma 2.5 $\nu_{d_Y} Y$ and $\nu_{d_Z} Z$ are non-metric continua. Then we have shown that $\nu_d X$ is a non-metric decomposable continuum, and the proof is complete. □

Here, it is natural to ask a question whether such subsets A and B exist as in the above Proposition 2.6. In the following Lemma 2.9, we will give a sufficient condition guaranteeing that such subsets A and B exist.

Definition 2.7. Let (Z, σ) be a connected metric space. A non-compact closed system $\{X, Y\}$ of Z satisfies the *condition* ($\sharp$) provided that there exists a compact conneced cover $\{K_n\}_{n<\omega}$ of Z with $K_n \subset \mathrm{int}_Z K_{n+1}$ and $K_{n+1} \setminus \mathrm{int}_Z K_n$ is connected and $X \cap$

$Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n) \neq \emptyset$ for each $n < \omega$ satisfies the following conditions :

($\sharp$1) $\sup_{n<\omega} \mathrm{diam}(X \cap Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n)) < +\infty$,

($\sharp$2) If $x \in (K_{n+1} \setminus \mathrm{int}_Z K_n) \cap X$ (resp. $y \in (K_{n+1} \setminus \mathrm{int}_Z K_n) \cap Y$), then $\sigma(x, Y) = \sigma(x, X \cap Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n))$ (resp. $\sigma(y, X) = \sigma(y, X \cap Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n))$.

($\sharp$3) $\sup_{n<\omega} \mathrm{diam}(X \cap (K_{n+1} \setminus \mathrm{int}_Z K_n)) = +\infty$ and $\sup_{n<\omega} \mathrm{diam}$ $(Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n)) = +\infty$, and

($\sharp$4) $X \cap (K_{n+1} \setminus \mathrm{int}_Z K_n)$ and $Y \cap (K_{n+1} \setminus \mathrm{int}_Z K_n)$ are connected.

Example 2.8. *Put $Z = [0, \infty) \times \mathbb{R}$ and σ is a subspace metric of $\mathbb{R}^2$.* Let X and Y be defined as below:

$$
\begin{aligned}
X &= \{(x, y) : x \geq 0 \text{ and } 0 \leq y \leq x\} \\
Y &= \{(x, y) : x \geq 0 \text{ and } -x \leq y \leq 0\}
\end{aligned}
$$

Put $K_n = \{(x, y) : x \leq n\}$ for each $n < \omega$. Then we can easily verify that a non-compact closed system $\{X, Y\}$ of Z satisfies the condition ($\sharp$).

Lemma 2.9. *Let (Z, σ) be a proper metric space and $\{X, Y\}$ a non-compact closed system of Z satisfying the condition ($\sharp$). Then there exist sequences $\{x_k\}_{k<\omega}$ and $\{y_k\}_{k<\omega}$ with $x_k \in X \setminus Y$ and $y_k \in Y \setminus X$ for each $k < \omega$ such that $\{\{x_k\}_{k<\omega},\ Y\}$ and $\{X,\ \{y_k\}_{k<\omega}\}$ diverge.*

Proof: Let $\{K_n\}_{n<\omega}$ be as in Definition 2.7. Put

$$
\begin{aligned}
L_n &= K_{n+1} \setminus \mathrm{int}_Z K_n, \\
a_n &= \mathrm{diam}(X \cap L_n), \\
b_n &= \mathrm{diam}(Y \cap L_n), \\
c_n &= \mathrm{diam}(X \cap Y \cap L_n), \\
A_n &= (X \cap L_n) \setminus Y, \\
B_n &= (Y \cap L_n) \setminus X, \text{ and} \\
C_n &= X \cap Y \cap L_n
\end{aligned}
$$

for each $n < \omega$. From the condition ($\sharp$1) $c = \sup_{n<\omega} c_n$ is bounded. From conditions ($\sharp$1) and ($\sharp$3) we can take a natural number n_0 and choose a point $x_0 \in A_{n_0}$ with $\sigma(x_0, Y) > 0$. By a similar argument, there exists an $n_1 > n_0$ such that $a_{n_1} > \max\{\sigma(x_0, Y), 3 + 2c(= 1 + 2(c+1))\}$. Then we show the following fact:

Fact. There exists an $x_1 \in A_{n_1}$ such that $\sigma(x_1, Y) > 1$.

Assume the contrary that for each $x \in A_{n_1}$ with $\sigma(x, Y) \leq 1$. Note that $A_{n_1} \subset B_{1+\varepsilon}(Y, \sigma)$ for some $\varepsilon > 0$ with $\varepsilon < 1/3$. Now, $a_{n_1} = \mathrm{diam}(X \cap L_{n_1}) = \mathrm{diam}(A_{n_1} \cup C_{n_1}) \leq \mathrm{diam}A_{n_1} + \sigma(A_{n_1}, C_{n_1}) + \mathrm{diam}C_{n_1}$. At first, we will verify that $\sigma(A_{n_1}, C_{n_1}) = 0$. Assume the contrary that $\sigma(A_{n_1}, C_{n_1}) > 0$. Put $\delta = \sigma(A_{n_1}, C_{n_1})/2$. Choose arbitrarily points $x \in A_{n_1}$ and $y \in C_{n_1}$. From the condition ($\sharp 4$) $A_{n_1} \cup C_{n_1}$ is compact connected. Then there exist a sequence $\{u_k\}_{k \leq m} \subset A_{n_1} \cup C_{n_1}$ such that $x = u_0, u_1, \ldots, u_m = y$ and $\sigma(u_i, u_j) < \delta$ for each $i, j \in \{0, \ldots, m\}$ (See [3], page 359, 6.1.D). Thus, we can choose points u_k and u_{k+1} such that $u_k \in A_{n_1}$ and $u_{k+1} \in C_{n_1}$. Note that $\sigma(u_k, u_{k+1}) \geq \sigma(A_{n_1}, C_{n_1}) > \delta$ and then we obtain a contradiction. Secondly, we will verify that $\mathrm{diam}A_{n_1} < 3 + c$. In fact, there exists α_{n_1}, $\beta_{n_1} \in A_{n_1}$ such that $\sigma(\alpha_{n_1}, \beta_{n_1}) > \mathrm{diam}A_{n_1} - \varepsilon$. From the condition ($\sharp 2$) there exist points $y(\alpha_{n_1})$, $y(\beta_{n_1}) \in C_{n_1}$ such that $\mathrm{diam}A_{n_1} < \sigma(\alpha_{n_1}, \beta_{n_1}) + \varepsilon \leq \sigma(\alpha_{n_1}, y(\alpha_{n_1})) + \sigma(y(\alpha_{n_1}), y(\beta_{n_1})) + \sigma(y(\beta_{n_1}), \beta_{n_1}) + \varepsilon$. From this estimation, note that $\mathrm{diam}A_{n_1} < 3 + c$. By the above arguments, we can verify that $a_{n_1} < 3 + 2c$. This is a contradiction.

Then continuing in this fashion, we can obtain a sequence $\{x_k\}_{k<\omega}$ satisfying the following conditions:

(1) $n_k < n_{k+1}$ and $x_k \in A_{n_k}$,
(2) $a_{n_{k+1}} > \max\{\sigma(x_k, Y), 1 + 2(c + k)\}$, and
(3) $\sigma(x_k, Y) > k$

for each $k < \omega$. Then, finally, we will prove the follwing claim:

Claim. $\{\{x_k\}_{k<\omega}, Y\}$ diverges.

In fact, fix a natural number $k < \omega$ and take an element $x \in Z - B_k(K_{n_k}, \sigma)$. Note that there exist $l < \omega$ and $y_x \in Y$ such that $\sigma(x, \{x_k\}_{k<\omega}) + \sigma(x, Y) = \sigma(x, x_l) + \sigma(x, y_x) \geq \sigma(x_l, y_x) \geq \sigma(x_l, Y) > l$. Here, without loss of generality, we may assume that $x_l \notin B_k(K_{n_k}, \sigma)$. Then note that $l > k$. This implies that $\sigma(x, \{x_k\}_{k<\omega}) + \sigma(x, Y) > k$.

Mimicking the proof above, we can obtain a sequence $\{y_k\}_{k<\omega} \subset Y$ satisfying the following conditions:

(4) $m_k < m_{k+1}$ and $y_k \in B_{m_k}$,

(5) $b_{m_{k+1}} > \max\{\sigma(y_k, Y), 1 + 2(k+c)\}$, and
(6) $\sigma(y_k, X) > k$

for each $k < \omega$. In particular, $\{X, \{y_k\}_{k<\omega}\}$ diverges. □

Now, we will write J_f as $\{(x, f(x)) : x \in J\}$ for $f \in C(J)$. In the rest of this section, J_f is equippted with a subspace metric of σ defined by $\sigma((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}$ for each $(x,y),(x',y') \in J \times \mathbb{R}$.

Theorem 2.10. *If $f \in C^*(J)$, then the Higson corona $\nu_\sigma J_f$ is an indecomposable continuum.*

Proof: Put $X = J \times [\inf_{x\in J} f(x), \sup_{x\in J} f(x)]$. From Lemma 2.5 $\nu_d X$ is an non-metric continuum. From Proposition 2.1 and 2.2 $\nu_\sigma J_f \cong \nu_\sigma X \cong \nu_\sigma J_{\overline{0}}$, where $\overline{0}$ is the constant function taking value 0. From Theorem 1.6 in [4] $\nu_\sigma J_{\overline{0}}$ is an indecomposable continuum. Thus, from these arguments above we conclude that $\nu_\sigma J_f$ is an indecomposable continuum and then the proof is complete. □

Theorem 2.11. *Let X, Y, and Z be non-compact locally connected closed strongly generalized continuum of $J \times \mathbb{R}$ having the complementation property with $Z = X \cup Y$ and a system $\{X, Y\}$ satisfy the condition ($\sharp$), and let J_f be as in the above with $J_f \subset Z$. If $J_f \cap X$ and $J_f \cap Y$ are r-dense in X and Y, respectively, for some $r > 0$, then $\nu_\sigma J_f$ is a decomposable continuum.*

Proof: From Propositions 2.1 and 2.2 $\nu_\sigma J_f \cong \mathrm{cl}_{\overline{Z}^\sigma} J_f \setminus J_f \cong \nu_\sigma Z$. By Lemma 2.5 and the last argument $\nu_\sigma J_f$ is a non-metric continuum. Here, $\mathrm{cl}_{\overline{Z}^\sigma} J_f \setminus J_f = \mathrm{cl}_{\overline{Z}^\sigma}(J_f \cap X) \setminus (J_f \cap X) \cup \mathrm{cl}_{\overline{Z}^\sigma}(J_f \cap Y) \setminus (J_f \cap Y)$. Put $\Sigma_0 = \mathrm{cl}_{\overline{Z}^\sigma}(J_f \cap X) \setminus (J_f \cap X)$ and $\Sigma_1 = \mathrm{cl}_{\overline{Z}^\sigma}(J_f \cap Y) \setminus (J_f \cap Y)$. By Proposition 2.2 $\Sigma_0 \cong \nu_\sigma X$ and $\Sigma_1 \cong \nu_\sigma Y$. Using Lemma 2.5, Σ_0 and Σ_1 are non-metric continua. From Propositions 2.1 and 2.2 $\Sigma_0 \cong \mathrm{cl}_{\overline{Z}^\sigma} X \setminus X$ and $\Sigma_1 \cong \mathrm{cl}_{\overline{Z}^\sigma} Y \setminus Y$. By Proposition 2.6 and Lemma 2.9 we note that $\Sigma_0 \setminus \Sigma_1 \neq \emptyset$ and $\Sigma_1 \setminus \Sigma_0 \neq \emptyset$. Then $\nu_\sigma J_f$ is a decomposable continuum. Thus, the proof is complete. □

Example 2.12. Let X and Y be as in the above Example 2.8. Put $Z = X \cup Y$ and $f(x) = x \sin x$ for each $x \in J$. By Theorem 2.11 the Higson corona of J_f with a subspace metric of $\mathbb{R}^2$ is a decomposable continuum.

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Miyakonojo National College of Technology, Miyakonojo-shi Miyazaki 885-8567, Japan

E-mail address: tomoyasu@cc.miyakonojo-nct.ac.jp