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SEVERAL OLD AND NEW PROBLEMS IN CONTINUUM THEORY

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Abstract

We present some old and new questions in continuum theory. Introductory comments and references are included to make the paper understandable for non-specialists also.

Properties of continua (i.e., compact connected Hausdorff spaces) have been concentrating much attention since the very beginning of topology studies. Now, when foundations of general topology are already established, a great number of natural questions about continua remain open. Many of them are easy to formulate and understand even for beginners. Nevertheless, they turned out to be difficult and they are still a great challenge and inspiration to current research. Below we present a sample of these questions. For other collections of continuum theory problems see historically the first such set [9], and also [8], [20] and [21].

The presented questions are divided into two parts. First, we list five old and well known problems that should be reminded whenever important question in topology are discussed. However, assuming that the Poincaré conjecture and the fixed point problem for nonseparating plane continua are widely known to non-continuum theory specialists, we skip these two most famous ones in this presentation. Second, we recall seven newer questions that are connected with authors' recent research. All problems presented below concern metric spaces only.

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Five Classical Problems

1. Hereditary equivalence. Assume that a nondegenerate continuum X is homeomorphic to each of its proper nondegenerate subcontinua. Must then X be either an arc or a pseudo-arc?

Such continua X are named *hereditarily equivalent*. As early as 1921 S. Mazurkiewicz posed a question as to whether every hereditarily equivalent continuum is an arc, [25]. In 1948 E. E. Moise constructed the pseudo-arc which is hereditarily equivalent and hereditarily indecomposable, [28], and thus answered Mazurkiewicz's question in the negative. Later G. W. Henderson showed that a hereditarily equivalent decomposable continuum is an arc, [12]. H. Cook proved that a hereditarily equivalent continuum is tree-like, [7]. Compare [38, Section 2, p. 307].

Research directed to classify homogeneous continua was initiated by the question of B. Knaster and K. Kuratowski in 1920, [17], whether the simple closed curve is the only homogeneous nondegenerate plane continuum. A continuum X is said to be *homogeneous* provided that for every two points x and y of X there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. The next two questions concern the study of homogeneous continua.

2. Homogeneous tree-like continua. Is each nondegenerate homogeneous tree-like (planar, weakly chainable) continuum a pseudo-arc?

A *weakly chainable* continuum is meant a continuous image of the pseudo-arc. J. T. Rogers, Jr., proved in [36] that a hereditarily indecomposable homogeneous continuum is tree-like. Answering an old question of R. H. Bing, the second named author showed (the proof is presented in the joint paper [18]) that tree-like homogeneous continua are hereditarily indecomposable. A positive answer to any of these questions would finally classify, after eight decades of study, all nondegenerate homogeneous plane continua as: the circle, the pseudo-arc and the circle of pseudo-arcs. For more detailed information on classifications of homogeneous continua, see Chapter 8 of [3], [22] and [39].

3. Homogeneous indecomposable continua. Is each nondegenerate homogeneous indecomposable (cell-like) continuum one-dimensional?

The pseudo-arc, solenoids and solenoids of pseudo-arcs are the only known nondegenerate homogeneous indecomposable continua, and all they are one-dimensional. If the answer to any of these questions is yes, then an essential progress in the study of the structure of homogeneous higher dimensional continua would be obtained, namely the completely regular decompositions described in [14], [37] and [24, Theorem 7.1, p. 18] would be trivial (in particular such continua would be aposyndetic and they would contain no proper nondegenerate terminal subcontinua). On the other hand an example of a higher dimensional homogeneous indecomposable continuum would be of a great importance in this area.

4. Property of Kelley. Assume that a continuum X has the property of Kelley. Does the product $X \times [0,1]$ necessarily have this property?

A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each sequence of points $x_n \in X$ converging to x and for each continuum K such that $x \in K \subset X$ there exists a sequence of continua $K_n \subset X$ such that $x_n \in K_n$ and $\operatorname{Lim} K_n = K$. The property is a one of the most extensively studied and useful in continuum theory. All hereditarily indecomposable, all (openly) homogeneous continua, all locally connected continua and all absolute retracts for hereditarily unicoherent continua have this property (see [13, p. 167-175, 277-279 and 405-406]; [2] and [6, Corollary 3.7]).

The recalled problem arose from the original question of S. B. Nadler, Jr., [29, 16.37, p. 558], whether the property of Kelley of a continuum X implies the property of the hyperspace $C(X)$

of all nonempty subcontinua of X with the Hausdorff metric. In [15, Corollary 3.3, p. 1147] H. Kato proved that Nadler's question is equivalent to the considered problem. Since Kato's variant of the problem is more intuitive for non-specialists, we have chosen it here.

5. Dendroids and small retractions onto dendrites. Let X be a dendroid. Do there exist, for each $\varepsilon > 0$, a tree $T \subset X$ and a retraction $r : X \to T$ with $d(x, r(x)) < \varepsilon$ for each point $x \in X$?

Dendroids appear as the intersection of the two important classes of continua: of arcwise connected continua and of hereditarily unicoherent ones. Their structure seems to be relatively simple and intuitive. They are tree-like, and are approximated from within by dendrites (so AR's). A positive answer to the question, which seems to be likely, would imply that every dendroid is an approximative absolute retract (see [11] for the definition). Some partial positive answer can be found in [10].

When in 1958 several ways to define dendroids were discussed, B. Knaster saw this class of arcwise connected curves as ones that can be retracted onto their subdendrites under small retractions, i.e., retractions that move points a little. Later the contemporary definition has been formulated as much more convenient to work with. But the problem if the two classes are equal is still open. The problem is very challenging because evidently some important information is missing about dendroids.

Some New Questions

Problems 6, 7 and 8 below are related to each other. They deal with a more general question:

Given continua X and Y, does there exist a continuous surjection of X onto Y?

Among initial famous results in this area there is the construction of a continuous surjection of $[0,1]$ onto $[0,1]^2$ by G. Peano and its generalization, the Hahn-Mazurkiewicz theorem saying that each locally connected continuum is a continuous image of $[0,1]$.

In this area we study invariants and inverse invariants of continuity for continua (sometimes called generalized continuous invariants). The study of generalized continuous invariants (e.g. local connectedness, uniform pathwise connectedness, various types of so called "indices of local disconnectivity", see e.g. [35], [31], [4], [16], and compare also σ-local connectedness in [19]), did not allow yet to exclude the existence of continuous surjections questioned in Problems 6, 7 and 8.

6. Mappings onto hyperspaces of subcontinua. Does there exist a continuum X admitting no continuous surjection onto its hyperspace $C(X)$ of all nonempty subcontinua?

Originally, a related problem was considered by S. B. Nadler, Jr. in [29, Question 4.6, p. 243]. No tools are known to prove nonexistence of a continuous surjection from any continuum X onto $C(X)$. On the other hand, no natural tools promising to construct such mappings for all continua are developed either. A (possible) continuum X with no such mapping must be non-locally connected, and each of its open subsets must have countably many components only, see a remark in [29, Question 4.6, p. 243].

7. Mappings between hyperspaces of subcontinua. Assume that there exists a continuous surjection $f : X \to Y$ between continua X and Y. Does there exist a continuous surjection $g : C(X) \to C(Y)$ between their hyperspaces $C(X)$ and $C(Y)$?

If the mapping f is weakly confluent, then the induced mapping $A \mapsto f(A)$ betwen $C(X)$ and $C(Y)$ is surjective, [29, Theorem 0.49.1, p. 24]. However, there are pairs of continua X and

Y admitting a continuous surjection f and such that there is no weakly confluent mapping from X onto Y.

8. Mappings between Cartesian squares. Does there exist a pair of continua X and Y with a continuous surjection $f : X^2 \to Y^2$ that admits no continuous surjection from X onto Y?

An example of such a pair for locally compact, noncompact, connected spaces was found by M. Morayne (an oral communication).

In the recent paper [5] an extensive study of absolute retracts for hereditarily unicoherent continua was presented. The following two problems seem to be the most important among those that arose from this research.

9. Tree-likeness of absolute retracts. Is every absolute retract X for the class of all hereditarily unicoherent continua a tree-like continuum?

Such a continuum X has the property of Kelley, and each of its arc components is dense in X (in particular X is approximated from within by trees). Proofs of these properties, together with many other ones, are presented in [5].

10. Absolute retracts and inverse limits. Does there exist an absolute retract X for tree-like continua such that X cannot be represented as an inverse limit of trees with confluent bonding mappings?

The arc-like continuum having exactly three end points as constructed in [30, 1.10, p. 7, and Figure 1.10, p. 8] is our candidate for such a continuum X. It is proved in [5, Theorem 3.6] that the inverse limit of trees with confluent bonding mappings is an absolute retract for hereditarily unicoherent continua.

11. Continuous decomposition of a 3-book. Let T be a simple triod. Does there exist a continuous decomposition of the product $T \times [0, 1]$ into pseudo-arcs?

For motivation of studying continuous decompositions into pseudo-arcs see the introduction of [33]. In [23] and in the recent papers [33] and [34] it was shown that the plane and each locally connected continuum in a 2-manifold with no local separating point, as well as the Menger curve, admit a continuous decomposition into pseudo-arcs (compare also [40] and [41]). Among Peano continua local separating point is the only known true obstacle to construct such a decomposition, [33, Proposition 15, p. 34]. The methods developed in the above quoted papers cannot be directly extended to the 3-book case.

12. Homogeneous Peano continua in the 3-space. Does there exist a homogeneous locally connected 2-dimensional continuum in the Euclidean 3-space that is neither a surface nor the Pontryagin sphere?

We can define the Pontryagin sphere as the quotient space of the standard Sierpiński universal plane curve S in $[0,1] \times [0,1]$. Namely we identify each pair of points belonging to the boundary of one component of $\mathbb{R}^2 \setminus S$ having either x-coordinates or y-coordinates equal. The Pontryagin sphere can also be seen as the quotient space of the disjoint union of two Pontryagin discs $\mathbb{D}^2$ (see [27, Section 3, p. 608-609]) with each pair of the corresponding points in the boundary $\partial\mathbb{D}^2$ identified.

S. Mazurkiewicz had shown that the only nondegenerate locally connected homogeneous plane continuum is the simple closed curve, [26]. Locally connected 1-dimensional homogeneous continua are characterized as the simple closed curve and the Menger universal curve (see e.g. [24, 12.2, p. 29]). Therefore, a negative answer to this question would provide a complete classification of locally connected homogeneous continua in 3-space. A continuum in question could not contain a 2-cell, see [32], and it would not be an ANR, see [1, Theorem 16.10, p. 194].

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