Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 25, Summer 2000

Pages 73–77

ADDENDUM TO LIUSTERNIK, SCHNIRELMAN FOR SUBSPACES

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Abstract

The purpose of this note is to present, for a given sequence of numbers $2 \leq k \leq \ell \leq n$, an example of a subspace A in a sphere S^m with relative closed coloring number k, with relative open coloring number ℓ, and with coloring number n.

Introduction

In this text all spaces are assumed to be separable metric and all mappings are assumed to be continuous. For the necessary background of this note we refer to [1].

There it was asked to construct for a given sequence

$$2 \leq k \leq \ell \leq n$$

a subset A with the property

$$\text{r.c.col}(A) = k, \quad \text{r.o.col}(A) = \ell \quad \text{and } \text{col}(A) = n.$$

In [1] such examples were presented under the additional condition that $\ell = n$. These examples were subspaces of S^{n-2}, the sphere of smallest possible dimension. In this note we present the required examples, again subspaces of S^{n-2}.

Mathematics Subject Classification: Primary 55M30; Secondary 47H10, 54B05, 54F45, 54F99, 54H25, 55M10, 55M20.

Key words: Liusternik–Schnirelman, coloring, dimension, involution.

1. Two Constructions

The first construction is the construction of the "uppersuspension". We recall this construction from [1]. For a space A we define

$$S^*(A) = S(A) \setminus \{\text{southpole}\}$$

where $S(A)$ is the suspension of A. Note that if $A \subset S^n$ then $S^*(A) \subset S(S^n) = S^{n+1}$. We restate lemma 13 of [1] for the case $X = S^n$.

Lemma 1. *If $A \subset S^n$ then:*

(1) *If A is a dense G_δ subset in S^n with $A \cup \alpha(A) = S^n$, then $S^*(A)$ is a dense G_δ subset in S^{n+1} with $S^*(A) \cup \alpha(S^*(A)) = S^{n+1}$,*

(2) $\text{r.c.col}(A, S^n) = \text{r.c.col}(S^*(A), S^{n+1})$.

By this lemma this first construction does not raise the relative closed coloring number, but by theorem 6 of [1] the relative open coloring number will (in the case that A is a dense G_δ with $A \cup \alpha(A) = S^n$).

The second construction is a type of suspension which in special cases does not raise the relative open coloring number. Let A be a space and $D \subset A$. We define the *weak uppersuspension with respect to D* as

$$S_D(A) = (A \times \{0\}) \cup S^*(D).$$

Note that $S_D(A) \subset S^*(A)$. The following lemma is easy to verify.

Lemma 2. *If both A and D are dense in S^n and $D \cup \alpha(D) = S^n$ then $S_D(A)$ is dense in S^{n+1} and $S_D(A) \cup \alpha(S_D(A)) = S^{n+1}$.*

The following lemma describes the situation in which we want to use the weak uppersuspension.

Lemma 3. *Let A be a dense set in S^n with $A \cup \alpha(A) = S^n$. If $D \subset A$ is dense with $D \cup \alpha(D) = S^n$ and $D \cap \alpha(D) = \emptyset$ then*

(1) $\text{r.c.col}(A, S^n) = \text{r.c.col}(S_D(A), S^{n+1})$,

(2) $\text{r.o.col}(A, S^n) = \text{r.o.col}(S_D(A), S^{n+1})$.

Proof. To prove (1), note $A \times \{0\} \subset S_D(A) \subset S^*(A)$ and so the statement follows from the second part of lemma 1.

For (3) let $U_1, \ldots, U_k$ be a relative open coloring of $A \subset S^n$. For each subset U_i we define:

$$U_i^o = (U_i \times \{0\}) \cup \big(S^*(D) \backslash (D \times \{0\})\big).$$

Then U_i^o is open in $S_D(A)$ and $S_D(A) = U_1^o \cup \ldots \cup U_k^o$, and the property $D \cap \alpha(D) = \emptyset$ easily implies that U_i^o is a color of $S_D(A)$. □

Remark 1. If A is a dense subset of S^n with $A \cup \alpha(A) = \emptyset$ then such a set D always exists.

2. The Examples

Example 1. Assume a sequence

$$2 \leq k \leq \ell \leq n$$

is given. Consider the sequence

$$A_1 = S^{k-2} \subset S^{k-2}, \quad A_2 = S^*(A_1) \subset S^{k-1}, \quad \ldots$$

Lemma 1 part (1) and theorem 6 part (5) from [1] imply that each time we take an uppersuspension the relative open coloring number is raised with 1 (so $\text{r.o.col}(A_i) = 1 + \text{r.o.col}(A_{i-1})$), while lemma 1 part (2) implies that the relative closed coloring number remains unchanged, so remains equal to k. So $A_{\ell-k+1}$ is a subset of $S^{\ell-2}$ with

$$\text{r.c.col}(A_{\ell-k+1}) = k \quad \text{and} \quad \text{r.o.col}(A_{\ell-k+1}) = \text{col}(A_{\ell-k+1}) = \ell.$$

Now we start taking weak uppersuspensions

$$B_1 = A_{\ell-k+1} \subset S^{\ell-2}, \quad B_2 = S_{D_1}(B_1) \subset S^{\ell-1}, \quad \ldots$$

We continue this process until we are in S^{n-2}. By lemma 2 and the remark the sets D_i with the required properties always exist. By lemma 3 the relative closed and the relative open coloring do not change anymore. Moreover, lemma 2 and theorem 6 part (4) from [1] imply that $\mathrm{col}(B_i) = 1 + \mathrm{col}(B_{i-1})$. So $B_{n-\ell+1} \subset S^{n-2}$ has the required coloring numbers:

$$\mathrm{r.c.col}(B_{n-\ell+1}) = k, \quad \mathrm{r.o.col}(B_{n-\ell+1}) = \ell, \quad \mathrm{col}(B_{n-\ell+1}) = n.$$

The reason why we require $k \geq 2$ in our sequence $2 \leq k \leq \ell \leq n$ is that the property $\mathrm{r.c.col}(A) = 1$ implies that $\mathrm{r.o.col}(A) = 1$. For the sake of completeness we mention the following.

Example 2. (1) If A is a point in S^m, then $\mathrm{r.c.col}(A) = 1$, $\mathrm{r.o.col}(A) = 1$ and $\mathrm{col}(A) = 1$,

(2) For a subset A with $\mathrm{r.c.col}(A) = 1$, $\mathrm{r.o.col}(A) = 1$ and $\mathrm{col}(A) = 2$, see example 1 in [1],

(3) If A is dense in S^{n-2} with $A \cup \alpha(A) = S^{n-2}$ and $A \cap \alpha(A) = \emptyset$ then $\mathrm{r.c.col}(A) = 1$, $\mathrm{r.o.col}(A) = 1$ and $\mathrm{col}(A) = n$ (for $n > 2$).

We end this note with the following example.

Example 3. The numbers $2 \leq k \leq \ell$ are given. For each $n > \ell$ there exists a subset A_n of S^n with $\mathrm{r.c.col}(A_n) = k$, $\mathrm{r.o.col}(A_n) = \ell$ and $\mathrm{col}(A_n) = n$. Now by putting $A = \oplus_n A_n \subset S = \oplus_n S^n$ we have obtained a subset A of S with $\mathrm{r.c.col}(A) = k$, $\mathrm{r.o.col}(A) = \ell$ and $\mathrm{col}(A) = \infty$.

References

[1] J.A.M. de Groot and J. Vermeer, *Liusternick Schnirelman for subspaces*, Topology and Applic **115** (2001), 343–354.

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