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SEQUENTIAL ORDER OF FINITE PRODUCTS OF TOPOLOGIES

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Abstract

Precise estimates of the sequential order of the finite products of topological spaces are provided in terms of the listing of the nodalities of the component spaces. The optimal listing is a transfinite combinatorial function. In the special case of Lašnev topologies of finite nodality an exact formula is given.

1. Introduction

The *sequential order* $\sigma(x)$ of a point x of a topological space X is the least ordinal α such that whenever x belongs to the sequential closure of a set, then it belongs to the α-iteration of its sequential adherence [6]. The sequential order of X is equal to $\sup_{x \in X} \sigma(x)$. Some authors define the sequential order on replacing, in the definition above, "sequential closure" by "closure" [7]; The latter definition is meaningful only in the case of *sequential* topologies (that are the topologies in which every sequentially closed set is closed), the case in which the two definitions coincide. Recall that a topology is *Fréchet* whenever it is sequential of order less than or equal to 1; a topology is *Lašnev* if it is a closed continuous image of a metrizable topology. In [7, 8] T. Nogura and A. Shibakov provided a series of results on the sequential order of products of two sequential topologies.

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Multisequences have been conceived for the study of sequential order. It was proved in [5] that for every $\alpha < \sigma(x)$, there exists a free distransverse multisequence [1] of rank α that converges to x; on the other, if a free transversally closed distransverse multisequence of rank α converges to x, then $\alpha \geq \sigma(x)$.

A multisequence approach is particularly insightful in the investigations of the sequential order of product topologies. It enabled S. Dolecki and S. Sitou to obtain in [5] an exact formula for the sequential order of the product of two Lašnev topologies in terms of the fascicularities and the sagittalities of the component spaces.

In this paper we provide a pattern for the composition of multisequences that leads to estimates for the sequential order of the product of finitely many topologies. This pattern provides lower bounds in the case of regular Fréchet topologies, and, upper bounds in the case of dychotomic topologies, but only if the sequential order is finite. Lašnev topologies belong to both these classes. We do not know if our listing formula holds for infinite nodalities.

Similar bounds do not hold for infinite products; in fact, the sequential fan is a Lašnev topological space, the sequential order of the n-th power of which is n, while that of the countable power is ω_1 [4]. On the other hand, there exists a compact (hence non dychotomic) sequential space such that all its countable powers are of order 2 [4].

All the topologies considered in this paper are supposed to be Hausdorff.

2. Multisequences

A *(sequential) cascade* T is a tree with a least element $\varnothing = \varnothing_{-T}$ such that each non empty subset has a maximal element and every non maximal element of T is the cofinite filter of the infinite countable of its immediate successors. The elements of

[1] Distransverse multisequences were originally called *admissible*.

a cascade are called *indices*. It follows that every non maximal $t \in T$, we can represent by $\{(t,n) \in \omega\}$ the set of the immediate successors of t.

A *multisequence* on a set X is a mapping from a cascade T to X. [2] Each cascade admits its natural topology: the finest topology for which each sequence of the form (t,n) converges to t. A subset B of a cascade is *eventual* (at $\varnothing$) whenever it is a neighborhood of $\varnothing$ in the natural topology.

A multisequence $g : S \to X$ is a *transmultisequence* of $f : T \to X$ if there exists a mapping h : from an eventual subcascade $\tilde{S}$ of S to T such that $g = f \circ h$ and

$$h\varnothing = \varnothing, \tag{2.1}$$

$$\underset{s \in \tilde{S}}{\forall} \quad h(s,n) \sqsupseteq (h(s), m_n) \text{ with } \lim_n m_n = \infty, \tag{2.2}$$

$$h(\max \tilde{S}) \subset \max T. \tag{2.3}$$

A *submultisequence* g of $f : T \to X$ is a transmultisequence such that

$$\underset{s \in \tilde{S}}{\forall} \quad h(s,n) = (h(s), m_n) \text{ with } \lim_n m_n = \infty. \tag{2.4}$$

We say that a cascade W is *almost included* in a cascade T (in symbols, $W \subset_0 T$) if there exists an eventual subcascade (that is, a subset with the induced natural topology) $\tilde{W}$ of W which is a subcascade of T.

If $g : S \to X$ is a submultisequence of $f : T \to X$, and $h : \tilde{S} \to T$ such that $g \circ h$ coincides with f on $\tilde{S}$, then $h(S)$ is almost included in T. A submultisequence g is *eventual* if for every s in (2.4) the sequence $(m_n)_n$ takes all but finitely many values.

[2] In fact, the definition of multisequence that has been recently used is that of a mapping from the set $\max T$ (of maximal elements of a cascade T) to X, while multisequence in our present sense has been recently called *extended multisequence*. We drop here *extended* for the sake of brevity.

It follows from the definition that every totally ordered subset of a sequential cascade is finite; thus each element of a sequential cascade is representable as a finite sequence of natural numbers. The *level* $l(t)$ of an element of a cascade is the length of the finite sequence t. The *rank* $r(t) = r(t;T)$ of an element t of a cascade T is defined by

$$\begin{aligned} t \in \max T &\Longrightarrow\ r(t) = 0 \\ t \notin \max T &\Longrightarrow\ r(t) = \sup_{n\in\mathbb{N}}(r(t,n)+1). \end{aligned} \tag{2.5}$$

A multisequence of rank 2 is called *bisequence*. A cascade is said to be *monotone* if for every $t \notin \max T$, the sequence $r(t,n)$ is increasing. If a cascade is monotone, then

$$r(t) = \lim_n (r(t,n)+1). \tag{2.6}$$

A multisequence $f : T \to X$, valued in a (Hausdorff) topological space X, *converges* to a point x if for every $t \in T \setminus \max T$, $\lim_n f(t,n) = f(t)$ and $x = f(\varnothing)$.

We denote by $\mathrm{adh}_{\mathrm{Seq}}\, A$ the sequential adherence of A (the union of the limits of sequences valued in A) and by $\mathrm{cl}_{\mathrm{Seq}}\, A$ the sequential closure (the least sequentially closed set that includes A).

3. Freedom-classifiable Multisequences

From the point of view of convergence, two sequences are equivalent if they generate the same filter. More generally, multisequences can be seen as special multifilters [3].

Let us call a filter *sequential* if it is generated by a sequence. It is known that every sequential filter is the infimum of two (possibly degenerate) sequential filters that do not mesh, one of which is free (that is the intersection of its elements is empty), and the other is principal [2]. Among the sequences that generate a free sequential filter there is always a *free sequence* (all terms are distinct). In Hausdorff spaces, each converging principal sequential filter is an ultrafilter, and thus is generated by a *stationary sequence* (all terms are equal). Accordingly, every converging sequence in a Hausdorff space admits either a free or a stationary subsequence. These facts motivate the forthcoming formalism.

By induction on the rank, we can prove that every converging multisequence in a Hausdorff space X admits a submultisequence $f : T \to X$ such that for every (non maximal) $t \in T$ the sequence $f(t, n)$ is either free or stationary; call such t a *free* (resp., *stationary*) index of f. An index is *freedom-classifiable* if it is either free or stationary. We shall call a multisequence *freedom-classifiable* if its every (non maximal) [3] index is freedom-classifiable. We can now reformulate the observation made above:

Proposition 3.1. *Each converging multisequence in a Hausdorff space has a freedom-classifiable submultisequence.*

If f is a multisequence, then one can define, the *freedom function* $\varphi_f = \varphi$ by

$$\varphi(t) = \begin{cases} 0 & t \text{ stationary;} \\ 1 & t \text{ free;} \\ * & \text{otherwise.} \end{cases} \tag{3.1}$$

If t is of rank 0, then it has no successors, so that it is both free and stationary. This is why the freedom function is not defined for maximal indices. We will also write $\varphi(t) = *$ when the value exists but is unknown or irrelevant for our purposes.

[3] Maximal indices are simultaneously free and stationary.

4. Transversality

A convergent multisequence f valued in X is called *transverse at* t (t is a *transverse* index of f) if for every sequence (t_n),

$$t_n \sqsupset (t,n) \Longrightarrow \lim_n f(t_n) = f(t); \tag{4.1}$$

antitransverse at t (t is a *antitransverse* index of f) if for every sequence (t_n),

$$t_n \sqsupset (t,n) \Longrightarrow \lim_n f(t_n) \neq f(t). \tag{4.2}$$

A convergent multisequence $f : T \to X$ is *distransverse at* t (t is a *distransverse* index of f) [5] if for every sequence (t_n),

$$t_n \sqsupset (t,n),\ \limsup_n (r(t_n)+1) < r(t) \Longrightarrow \lim_n f(t_n) \neq f(t). \tag{4.3}$$

The properties above do not depend on the immediate successors of t. Therefore the points t of rank 0 or 1 have all these properties at once. In particular, a convergent sequence (x_n) is both transverse and antitransverse.

An index is *transversally classifiable* if it is either transverse or antitransverse. A multisequence is said to be *transversally classifiable* if its every index is transversally classifiable. A *transversality function* $\zeta_f = \zeta$ of a multisequence f is defined by

$$\zeta(t) = \begin{cases} - & t \text{ antitransverse;} \\ + & t \text{ transverse;} \\ * & \text{otherwise.} \end{cases} \tag{4.4}$$

If the trasversality of t is irrelevant, then we also write $\zeta(t) = *$. We shall use the abbreviation

$$c(t) = \varphi(t)_{\zeta(t)}.$$

We say that a multisequence is *classifiable* if it is both freedom- and transversally classifiable. Every element of a classifiable multisequence is of one of the following four types: $0_-, 0_+, 1_-, 1_+$, that is, stationary/antitransverse, stationary/transverse, free /antitransverse, free/transverse.

Proposition 4.1. *For a classifiable multisequence, almost all the immediate successors of an antitransverse index are free; for a classifiable multisequence in a Fréchet space, almost all the immediate successors of an antitransverse stationary index are free and transverse.*

Proof. If t is antitransverse (that is, of the type $*_-$) for a multisequence f, then no sequence of the form $(f(t, n_p, k_p))_p$ converges to $f(t)$. Therefore if (t, n) were of the type 0_*, then, by definition, $f(t, n) = f(t, n, k)$ for almost all k, yielding a contradiction. In the case of Fréchet topologies, no index of the rank greater than 1 is free antitransverse (type 1_-), as every convergent free bisequence admits a transverse transsequence. □

We shall also consider *conditionally antitransverse* multisequences, that is, such that (4.2) holds provided that (t_n) is free. There follows a definition of *conditionally transversally classifiable* multisequence.

A convergent multisequence $f : T \to X$ is called a *multifan* if for each t of even level in $T \setminus \max T$, one has $f(t, n) = f(t)$ for each $n \in \mathbb{N}$. A convergent multisequence $f : T \to X$ is said to be an *arrow* if for every t in $T \setminus \max T$ of odd level, one has $f(t, n) = f(t)$ for each $n \in \mathbb{N}$. In other words, f is an arrow if for each n, the restriction of f to $T_{\restriction n} := \{s : (n, s) \in T\}$ is a multifan.

We say that a multifan or an arrow is *correct* if all its stationary points are antitransverse [4] and its all free points are transverse.

[4] Such multifans and arrows were called *untraversable* in [5].

In other words, the types of successive indices of a correct multifan are

$$0_- \ 1_+ \ 0_- \ 1_+ \ 0_- \ 1_+ \ldots ,$$

and of a correct arrow are

$$1_+ \ 0_- \ 1_+ \ 0_- \ 1_+ \ 0_- \ldots$$

A multisequence is *transversally closed at* t if for every sequence (t_n),

$$t_n \sqsupseteq (t,n) \Longrightarrow \lim f(t_n) \subset \{f(t)\}. \tag{4.5}$$

If a multisequence $f : T \to X$ fulfills (4.5) for every t, then in particular $f(T)$ is a closed subset of X.

Proposition 4.2. *A multisequence is antitransverse at t and transversally closed at t if and only if*

$$t_n \sqsupset (t,n) \Longrightarrow \operatorname{adh}_{\mathrm{Seq}} f(t_n) = \varnothing. \tag{4.6}$$

An element x of a topological space is *Fréchet* if $x \in \operatorname{cl} A$ implies the existence of a sequence on A that converges to x; *regular* if it admits a base of closed sets.

In [5, Theorem 3.1] S. Dolecki and S. Sitou extended [9, Theorem 3.8] of T. Nogura and Y. Tanaka from bisequences to multisequences to the effect that

Proposition 4.3. *If a multisequence is antitransverse at a regular Fréchet point, then it admits at that point transversally closed eventual submultisequence.*

Fréchetness cannot be dropped in Proposition 4.3. Take a compact sequential topology that is not Fréchet (for example, a compact MAD topology). This space contains a convergent antitransverse bisequence. This bisequence must not admit a transversally closed subbisequence, because the topology is sequentially compact.

5. Level Classification

Let $\mathfrak{P}$ be a finite class of properties of indices. A multisequence $f : T \to X$ is said to be *classifiable* with respect to $\mathfrak{P}$ if for every index t, there is $\mathcal{P} \in \mathfrak{P}$ such that $t \in \mathcal{P}$; it is said to be *level-classifiable* if for every $m \in \omega$, there is $\mathcal{P} \in \mathfrak{P}$ such that if $l(t) = m$ then $t \in \mathcal{P}$.

Proposition 1. *Every multisequence of finite rank which is classifiable with respect to $\mathfrak{P}$ has a level-classifiable submultisequence with respect to $\mathfrak{P}$.*

Proof. We proceed by induction on the rank $r(f)$. For $r(f) = 0$ the fact is obvious. Suppose that it holds for the rank less than m and that $r(f) = m + 1$. By the inductive assumption for each $n \in \omega$ the restriction f_n of f to $\{s : (n, s) \in T\}$ is level-classifiable. As it has $m + 1$ levels, it can be characterized by $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_m)$, where $\mathcal{P}_i \in \mathfrak{P}$ is the property of the indices of level i. Therefore every index n is classifiable with respect to the finite class of properties consisting of all $(\mathcal{P}_{-1,}\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_m)$ with $\mathcal{P}_i \in \mathfrak{P}$ for $-1 \le i \le m$. It follows that there is $(\mathcal{P}_{-1,}\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_m)$ such that infinitely many n have the property $(\mathcal{P}_{-1,}\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_m)$. We select these n to construct the submultisequence. □

6. Sequential Order in Terms of the Ranks of Multisequences

The *sequential order of x with respect to A* is given by

$$\sigma(x; A) = \min\{\alpha : x \in \operatorname{adh}^{\alpha}_{\mathrm{Seq}} A\}.$$

The sequential order $\sigma(f)$ of a convergent multisequence f is, by definition, $\sigma(f(\varnothing); f(\max T))$. An easy induction shows that $x = \lim f$ implies that $\sigma(x, A) \le r(f)$ for every multisequence f on A . It is easy to see that a converging everywhere free multisequence in a Hausdorff space has an injective submultisequence. It was shown in [5, Theorem 1.3] that

Theorem 6.1. *If $x \in \mathrm{cl}_{\mathrm{Seq}} A$, then there exists on A an injective (everywhere) free, distransverse, monotone multisequence of rank $\sigma(x;A)$ that converges to x.*

[5, Example 1.2] shows that "distransverse" cannot be replaced by "antitransverse". On the other hand, D. Fremlin defined in [6] a *sequentially regular embedding* to be an injective map from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ such that $\lim_n f(t,n) = f(t)$ and $x = f(\varnothing)$ for every t and which fulfills (4.6). He proved what in our terminology amounts to the fact that each injective (hence free) multisequence that fulfills (4.6) has the sequential order equal to the rank.

Theorem 6.2. *The sequential order of x is greater than or equal to the rank of each (everywhere) free, distransverse, transversally closed multisequence that converges to x.*

Consequently, converging, free, distransverse, transversally closed multisequences can be used to provide lower bounds for the sequential order. However the lower bounds obtained in this way need not be exact; for example, in a sequentially compact sequential space no multisequence of rank greater than 1 is transversally closed. Therefore in this case the method of Theorem 6.2 yields the lower bound for the sequential order 1. On the other hand, there exists a Hausdorff, sequentially compact, sequential topology of rank ω_1 (for example, the ω_1-iterated MAD topology [10]). There exists as well a (Hausdorff) regular, sequentially compact, sequential topology of rank 2 (for example, the compact MAD topology and its countable powers [4])

7. Products of Multisequences

In this section we study properties (like being free, stationary, transverse, and so on) of finite diagonal products of multisequences. In particular, we are interested in sufficient conditions, in terms of component multisequences, for the diagonal product to be free, antitransverse and transversally closed. Let $X_1, X_2, \dots, X_m$ be spaces and let $f_i : T \to X_i$ be multisequences for $i = 1, \dots, m$. Let

$$\bigotimes_{1\le i\le m} f_i(t) = (f_1(t), \dots, f_m(t)).$$

Theorem 7.1. *If there exists i such that f_i is free (resp., antitransverse, distransverse) at t, then $\bigotimes_i f_i$ is free (resp., antitransverse, distransverse) at t.*

If for every i the multisequence f_i is stationary (resp., transverse) at t, then $\bigotimes_i f_i$ is stationary (resp., transverse) at t.

Theorem 7.2. *If f_i is transversally closed at t for every i, then $\bigotimes_i f_i$ is transversally closed at t.*

The rules established in the theorems above enable us to construct free, antitransverse product multisequences out of multifans and arrows in component spaces. In the examples below we illustrate an algorithm based on successive use of *nodes* (that is, stationary antitransverse indices) from different component multisequences. Notice that multifans and arrows are used to construct multisequences in which some stationary indices are split to several successive stationary indices. Although correct multifans and arrows are classifiable, the so constructed multisequences are no longer transversally classifiable.

Example 7.3. Consider three spaces X, Y and Z. The first one consists of a free convergent sequence, each of the two others consists of an antitransverse fan. Then we can build up an antitransverse multisequence of rank 3 by the following procedure. In the table below, the first three rows correspond the component multisequences and the last row corresponds to their diagonal product; the columns correspond to levels, the first on the left to the level 0.

1_+	$*$	$*$	$*$
0_-	1_+	$*$	$*$
0_*	0_-	1_+	$*$
1_-	1_-	1_+	$*$

Observe that the rank of all the considered multisequences is 3.

Example 7.4. Consider now four spaces, in which we take respectively, an arrow of rank 3, two fans, and a multifan of rank 4. Then we can build up an antitransverse multisequence of rank 6 by the following procedure:

1_+	0_*	0_*	0_-	1_+	$*$	$*$
0_-	1_+	0_*	0_*	0_-	1_+	$*$
0_*	0_-	1_+	$*$	$*$	$*$	$*$
0_*	0_*	0_-	1_+	$*$	$*$	$*$
1_-	1_-	1_-	1_-	1_-	1_+	$*$

We notice that the adopted algorithm passes from a fan to another (in another space!) and that one of the multisequences must be an arrow. Therefore, in order to analyze the procedure, it is enough to look only at a starting 1_+ in the first column, and at the segments $(0_-, 1_+)$ of the components. Moreover, because the components are Fréchet spaces, 1_+ automatically follows 0_-, so that it is enough to make sure that in every column (except for the last two) there exists one 0_-, and that there exists 1_+ in the first column. For instance, in the case of Example 7.4, one may use only the following table (on dropping the two last columns entirely):

1_+			0_-		
0_-				0_-	
	0_-				

0_- corresponds to a node, while 1_+ in the first column corresponds to an initializing free sequence.

8. Nodality, Fascicularity and Sagittality

In the case of a product of two Lašnev spaces, two kinds of multisequences (multifans and arrows) in component spaces were used to determine the sequential order of the product [5]. Multifans and arrows are also decisive for the sequential order of the product of finitely many Lašnev spaces.

Let $f : T \to X$ be a multifan and let R be the subtree of T obtained by removing all maximal indices of odd level. If $f : R \to X$ is correct then we define its *fascicularity* [5] $\lambda(f)$ as the rank $r(\varnothing; R)$. Similarly, if $f : T \to X$ is a correct arrow and if R is a subtree of T obtained by removing all maximal indices of even level, then we define its *sagittality* $\mu(f)$ as the rank $r(\varnothing; R)$. If $R = \varnothing$, then we convene that $\mu(f) = -1$. The *fascicularity* $\lambda(x)$ of a point x is the least upper bound of $\lambda(f)$ of all the correct multifans f converging to x. The *sagittality* $\mu(x)$ is the least upper bound of $\mu(g)$ of all the correct arrows g converging to x. Remember that always $\mu(x) + 1 \geq \lambda(x)$.

Multisequences of different rank can have the same image. This is already the case of a stationary sequence (rank 1) and the corresponding multisequence of rank 0. A multifan is composed of fans, that is, of the multisequences of rank 2 of the form

$$x = x_n \leftarrow_k x_{n,k},$$

where for each n the sequence $(x_{n,k})_k$ is free. We are interested in antitransverse fans, that is, of the type $(0_-, 1_+, *)$. Arrows are sequences of multifans. Arrows of rank 1 are the free sequences, of order 3 are the form

$$x \leftarrow_n x_n = x_{n,k} \leftarrow_p x_{n,k,p},$$

hence of the type $(1_+, 0_-, 1_+, *)$. In the case of products of more than 2 spaces the fans constituting multifans and arrows will be used as images of multisequences of rank higher than 2.

[5] The terms *fascicularity* and *sagittality* are derived from respective Latin terms for *bundle* and *arrow*.

Recall that a *node* of a multisequence is its stationary antitransverse index. It follows from the definition that maximal indices of a multisequence are nodes. The set of the nodes of a multisequence is inversely well founded, so that the following definition makes sense. The *nodality* $\nu(f)$ of a multisequence f is the rank of the set of its nodes. The *nodality* $\nu(x)$ of an element x of a space X is the supremum of the nodalities of the correct multifans and the correct arrows that converge to x. An element x is *active* (or that the nodality is *active at* x) provided that that supremum is attained by the nodalities of arrows. The *nodality* $\nu(X)$ of a space X is the supremum of the nodalities of the points of X; the *active nodality* $\nu(X)$ of a space X is the supremum of the nodalities of the active points of X; If f is a multifan, then $\nu(f) = \lambda_0(f) + \frac{\lambda_1(f)}{2}$, where $\lambda(f) = \lambda_0(f) + \lambda_1(f)$ is the decomposition of the ordinal to its transfinite and its finite parts. Alike, if f is an arrow, then $\nu(f) = \mu_0(f) + \frac{\mu_1 - 1}{2}$, where $\mu(f) = \mu_0(f) + \mu_1$ is the decomposition.

9. Products of Regular Fréchet Spaces: Lower Bounds for Sequential Order

By Proposition 4.3, correct multifans and arrows in regular Fréchet spaces admit transversally closed submultisequences. Therefore the ranks of finite diagonal compositions of such multisequences constitute lower bounds for the sequential order, provided they are free and distransverse everywhere. We restrict our attention to free and antitransverse multisequences (easier to handle than distransverse multisequences).

It follows from Theorem 7.1 that a product multisequence is (everywhere) free and antitransverse if for every index t, there exist a component which is antitransverse at t and another component that is free at t. In Fréchet spaces antitransverse indices of classifiable multisequences must be stationary. Therefore in order to obtain a free and antitransverse multisequence as a diagonal product of classifiable multisequences $\bigotimes_i f_i : T \to \prod_i X_i$, every index $t \in T$ must be a node of one of the component multisequence f_i. As a successor of a node in a classifiable multisequence is not a node, a construction of a free and antitransverse product multisequence consists in choosing nodes in a different component at every new step. This leads to the notion of listing.

The *listing* of a set of ordinals $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is an injection g from an ordinal γ to $\bigcup_{i=1}^{k}(\nu_i \times \{i\})$, with the property that no two successive values of g belong to the same $\nu_i \times \{i\}$ and such that its restriction to $g^{-1}(\nu_i \times \{i\})$ is increasing for every i. The supremum of the ordinals γ such that there exists a listing $g : \gamma \to \bigcup_{i=1}^{k}(\nu_i \times \{i\})$ is called the *listing number* and is denoted by $\kappa(\nu_1, \nu_2, \ldots, \nu_k)$. A listing for which the listing number is attained is referred to as a *maximal listing.*

Theorem 9.1. *Let X_i for $1 < i \leq p$ be regular Fréchet spaces. If one among $x_i \in X_i$ is active, then*

$$\sigma(x_1, x_2, \ldots, x_p) \geq \kappa(\nu(x_1), \nu(x_2), \ldots, \nu(x_p)).$$

Proof. Suppose that $g_i : S_i \to X_i$ are multisequences that converge to x_i respectively, so that all but one, say g_1, are multifans, and g_1 is an arrow.

Let ν_i be the nodality of g_i. As the considered spaces are regular and Fréchet, all the correct multifans and arrows admit transversally closed submultisequences, so that we can assume without loss of generality that g_i are already transversally closed.

Let us induce on the rank of the listing number

$$\kappa = \kappa(\nu_1, \nu_2, \ldots, \nu_p).$$

If $\kappa = 1$, then there is $i_0 \neq 1$ and a listing $h : 1 = \{0\} \to (\nu_1 \times \{1\}) \cup (\nu_2 \times \{2\}) \cup \ldots \cup (\nu_p \times \{p\})$ such that $h(0) \in \nu_{i_0} \times \{i_0\}$. This means that the nodality of g_{i_0} is (at least) 1. Then $\bigotimes_{1 \leq i \leq p} g_i$ is of the type $1_-, 1_+, *$, proving that the sequential order of $(x_1, x_2, \ldots, x_p)$ is at least 2.

Assume that $\kappa > 1$, and that the inductive hypothesis holds for the listing numbers less than κ. Let $h : \kappa \to (\nu_1 \times \{1\}) \cup (\nu_2 \times \{2\}) \cup \ldots \cup (\nu_p \times \{p\})$ be a listing. Then there is an increasing sequence of ordinals $(\kappa_n)_n$ so that $\kappa = \sup_{n\in\omega}(\kappa_n + 1)$, and for every n there exists $i(n) \neq i_0$ for which $h(\kappa_n) \in \nu_{i(n)} \times \{i(n)\}$. This is possible, for otherwise there would be $\alpha < \kappa$ so that $h(\beta) \in \nu_{i_0} \times \{i_0\}$ for each $\beta > \alpha$. Then there exists $i_1 \neq i_0$ such that $i(n) = i_1$ for infinitely many n.

For $1 \leq i \leq p$, let $f_i(\varnothing) = x_i = g_i(\varnothing)$ for every i, and $f_1(n) = x_n^{(i)} = g_1(n)$ for each n. Then by inductive assumption,

$$\sigma(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)}) \geq \kappa(\nu(x_n^{(1)}), \nu(x_n^{(2)}), \ldots, \nu(x_n^{(p)})) = \kappa_n.$$

Therefore there exist a cascade T and multifunctions $F_i : T \to X_i$ so that they coincide on $\{\varnothing\} \cup \omega$ with the convergent sequences defined above, and such that $\bigotimes_{1\leq i\leq p} F_i$ is a free anti-transverse transversally closed of the rank κ. □

Because of Theorem 7.2 and Proposition 4.3,

Corollary 9.2. *If X_i are regular Fréchet topological spaces for $1 < i \leq p$, then*

$$\sigma(\prod_{1\leq i\leq p} X_i) \geq \kappa(\nu(X_1), \nu(X_2), \ldots, \nu(X_p))$$

provided that one of the nodalities is active.

10. Products of Dychotomic Fréchet Topologies: Search for Upper Bounds on Sequential Order

A topology is *dychotomic* if every converging multisequence which either free at $\varnothing$ or of the type $(0_*, 1_*)$ at $\varnothing$, has a submultisequence classifiable at $\varnothing$. The *upper Kuratowski limit* $\mathrm{Ls}_n A_n$ of a sequence (A_n) of a topological space is defined by

$$\mathrm{Ls}_n A_n = \bigcap_{m<\omega} \mathrm{cl}(\bigcup_{n\geq m} A_n).$$

Theorem 10.1. *Lašnev spaces are dychotomic.*

Proof. Let f be a converging multisequence of rank ≥ 2 valued in a Lašnev space X. Denote $x = f(\varnothing)$, $x_n = f(n)$ and $x_{n,k} = f(n,k)$. For free indices the fact follows from [5, Proposition 4.3]. So suppose that $\varnothing$ is stationary but its immediate successors are not. Therefore $x_{n,k} \to_k x_n = x$. Let $h : W \to X$ be a continuous closed map from a metrizable space W onto X. By the active boundary theorem [1, Theorem 1.1], for every n there is a compact subset $K_n = \mathrm{Ls}_k\, h^-(x_{n,k})$ of the boundary of $h^-(x)$ such that for every $O \in \mathcal{N}(K_n)$, there exists $k(n)$ such that $h^-(x_{n,k}) \subset O$ for $k \geq k(n)$.

Consider now $\mathrm{Ls}_n K_n$. This is a subset of $h^-(x)$, because the latter set is closed. If there is $w \in \mathrm{Ls}_n K_n$, then there exists a subset B of $\omega \times \omega$ such that there are infinitely many n for which there are infinitely many k with $(n,k) \in B$, and for every $(n,k) \in B$ there is $w_{n,k} \in h^-(x_{n,k})$ and $w_n \in h^-(x)$ so that the following bisequence converges to w:

$$w_{n,k} \to_k w_n \to_n w.$$

Because W is metrizable, there is $h : \omega \to \omega$ such that for every sequence $(k_n)_n$ with $k_n \geq h(n)$, the sequence $(w_{n,k_n})_n$ converges to w. By the continuity of h, for every sequence $(k_n)_n$ with $k_n \geq h(n)$, the sequence $(x_{n,k_n})_n$ converges to x.

If $\mathrm{Ls}_n K_n = \varnothing$, then $\{K_n : n \in \omega\}$ is discrete and thus for every m, the set $F_m = \bigcup_{n \geq m} K_n$ is closed; by countable paracompactness, there is a sequence of open sets (O_m) such that $K_m \subset O_m$ for every m, and $\bigcap_{m<\omega} \mathrm{cl}\, O_m = \varnothing$. It follows that there exists a map $g : \omega \to \omega$ such that for every sequence $(k_n)_n$ with $k_n \geq g(n)$, the family $\{O_n : n \in \omega\}$ is discrete. Therefore there exists a submultisequence f_0 of f such that for every $t_n \sqsupseteq (n, k_n)$, every subset of $\{f_0(n, k_n) : k \in \omega\}$ is closed, so that its limit is empty. □

Theorem 10.2. *If $f_1, f_2, \ldots, f_p$ are multisequences from T of finite rank to dychotomic Fréchet spaces such that $\bigotimes_i f_i$ is* *everywhere free and antitransverse, then there is a subcascade S of T such that for the restrictions of f_i to S,*

$$r(\bigotimes_i f_i) \leq \kappa(\nu(f_1), \nu(f_2), \ldots, \nu(f_p)). \tag{10.1}$$

Proof. Without loss of generality we can assume that all the multisequences are isotone level freedom-classifiable and conditionally level transversally classifiable. Therefore for every $m \leq k = r(\bigotimes_i f_i) - 2$, there exist i and j such that $c_{f_i}(t) = 0_*$ and $c_{f_j}(t) = 1_*$ for every t of level m. Among all such i, there exists at least one $i(m)$ such that $c_{f_{i(m)}}(t) = 0_-$ for each t of level m (what implies that $c_{f_{i(m)}}(s) = 1_+$ for every immediate successor s of such t). Therefore $i(m+1) \neq i(m)$. It follows that the constructed function $i : \{0, 1, \ldots, k\} \to (\nu(f_1) \times \{1\}) \cup (\nu(f_2) \times \{2\}) \cup \ldots \cup (\nu(f_p) \times \{p\})$ is a listing, and thus (10.1) holds. $\square$

Because the finite sequential order of a space is attained at some point, we conclude that

Corollary 10.3. *If $X_1, X_2, \ldots, X_p$ are dychotomic Fréchet spaces of finite nodality, then*

$$\sigma(\prod_{1 \leq i \leq p} X_i) \leq \kappa(\nu(X_1), \nu(X_2), \ldots, \nu(X_p)),$$

provided that one of the nodalities is active.

Corollary 10.4. *If $X_1, X_2, \ldots, X_p$ are Lašnev spaces of finite nodality, then*

$$\sigma(\prod_{1 \leq i \leq p} X_i) = \kappa(\nu(X_1), \nu(X_2), \ldots, \nu(X_p)),$$

provided that one of the nodalities is active.

Whether the above holds for arbitrary (infinite) nodalities remains an open question.

11. Listings and Decomposition of Ordinals

An ordinal number π is said to be *decomposable* if there exist $\alpha, \beta < \pi$ such that $\alpha + \beta = \pi$. We write $\alpha \ll \beta$ if $\alpha + \beta = \beta$.

Lemma 11.1. *An ordinal π is non decomposable if and only if $\alpha + \pi = \pi$ for every ordinal $\alpha < \pi$.*

Proof. Suppose that π is decomposable: there exist $\alpha, \beta < \pi$ such that $\alpha + \beta = \pi$. If the condition of Lemma 11.1 held, then $\pi + \pi = (\alpha + \beta) + \pi = \alpha + (\beta + \pi) = \alpha + \pi = \pi$: a contradiction. Conversely, if π is non decomposable, then $\alpha + \beta < \pi$ for every $\alpha, \beta < \pi$. Hence $\pi \geq \sup_{\beta<\pi}(\alpha + \beta) = \alpha + \sup_{\beta<\pi} = \alpha + \pi \geq \pi$, that is, the condition of Lemma 11.1 holds. □

Notice that if $\alpha_0 > \alpha_1 > \ldots > \alpha_l$ are non decomposable ordinals, then $\alpha_0 > \alpha_1 m_1 + \ldots + \alpha_l m_l$ for every choice $m_1, \ldots, m_l$ of natural numbers. It is known that for every ordinal β, there exist a unique finite sequence $\beta_0 > \beta_1 > \ldots > \beta_k$ of non decomposable ordinals and a unique finite sequence $n_0, n_1, \ldots, n_k$ of natural numbers such that

$$\beta = \beta_0 n_0 + \beta_1 n_1 + \ldots + \beta_k n_k. \tag{11.1}$$

The non decomposable ordinals of the decomposition (11.1) are called *components* of β.

Every non decomposable ordinal is limit, for if $\pi = \alpha + 1$, then $\alpha + \pi = \alpha + \alpha + 1 > \alpha$. Let us say that π is an *ordinary non decomposable ordinal* if there exists the greatest non decomposable ordinal less than π, and a *limit non decomposable ordinal* otherwise. For example, ω^m is ordinary non decomposable for every m, while $\sup_{m\in\omega} \omega^m$ is limit non decomposable.

Lemma 11.2. *Let π be a countable non decomposable ordinal and let $\nu_1, \nu_2, \ldots, \nu_p$ be nonzero finite multiples of π. Then there is $m_0 \in \omega$ such that πm_0 is the (optimal) listing number of $\{\nu_1, \nu_2, \ldots, \nu_p\}$, and there is an optimal listing h such that*

$$|\{i : \forall_{\alpha<\pi}\ h(\pi m + \alpha) \in \nu_i \times \{i\}\}| = 2 \tag{11.2}$$

for every $0 \leq m < m_0$.

Proof. Suppose that $h : \kappa \to (\nu_1 \times \{1\}) \cup (\nu_2 \times \{2\}) \cup \ldots \cup (\nu_p \times \{p\})$ is an optimal listing. Of course, $\pi \le \kappa \le \nu_1 + \nu_2 + \ldots + \nu_p = \pi l$ for some $l \ge 2$.

The claim holds for $\pi = \omega$. Indeed, if $\omega(m+1) \le \kappa$, then by the definition of listing, the set $\{i : |\{n : h(\omega m + n) \in (\nu_i \times \{i\})\}| = \infty\}$ contains at least two elements, say i_0 and i_1. Thus there exists an increasing sequence (n_k) such that $h(\omega m + n_{2k}) \in \nu_{i_0} \times \{i_0\}$ and $h(\omega m + n_{2k+1}) \in \nu_{i_1} \times \{i_1\}$ for every k. On redefining $h_m(\omega m + k) = h(\omega m + n_k)$ for every $k \in \omega$, we obtain another optimal listing; on doing this for every m such that $0 \le \omega m < \kappa$, we construct a listing such that (11.2) holds for every m. It follows that if m is the greatest natural number for which $\omega m < \kappa$, then $\omega(m+1) = \kappa$.

Suppose that the claim holds for all non decomposable ordinals less than ξ. If ξ is ordinary non decomposable and π is the greatest non decomposable ordinal less than ξ, then $\xi = \sup_{m\in\omega} \pi m$. Let $f : \kappa \to (\nu_1 \times \{1\}) \cup (\nu_2 \times \{2\}) \cup \ldots \cup (\nu_p \times \{p\})$ be an optimal listing; then by the inductive assumption, for every m there exists another optimal listing h that fulfills (11.2). There certainly exist i_0 and i_1 and a sequence (m_k) such that $h(\pi m_k + \alpha) \in (\nu_{i_0} \times \{i_0\}) \cup (\nu_{i_1} \times \{i_1\})$ for each $k \in \omega$ and every $0 \le \alpha < \pi$, so that $\hat{h}(\pi k + \alpha) = h(\pi m_k + \alpha)$ is a sought optimal listing.

If $\xi = \sup_{m\in\omega} \xi_m$ is limit non decomposable, and then the same argument holds on replacing πm by ξ_m. ☐

Theorem 11.3. *Let $0 < n_1 \le n_2 \le \ldots \le n_k$ be finite and $k \ge 2$. Then*

(11.3)
$$l = \kappa(n_1, n_2, \ldots, n_k) = \sum_{i=1}^{k-1} n_i + (n_k \wedge \sum_{i=1}^{k-1} n_i + 1).$$

Proof. If $g : \gamma \to \bigcup_{i=1}^{k}(n_i \times \{i\})$ is a listing, then on one hand, $|g(\gamma)| \leq \sum_{i=1}^{k} n_i$, and on the other, after every element a of $(n_k \times \{k\}) \cap g(\gamma)$ which is not the last element of $g(\gamma)$, there must be at least one element of $\bigcup_{i \neq k}(n_i \times \{i\})$ that follows a, hence $|\gamma| \leq 2\sum_{i=1}^{k-1} n_i + 1$ Therefore, $l \leq \sum_{i=1}^{k-1} n_i + (n_k \wedge \sum_{i=1}^{k-1} n_i + 1)$.

We shall prove by induction that the opposite inequality also holds. For $k = 2$, then $l = 2n_1$ if $n_1 = n_2$, and $l = 2n_1 + 1$ otherwise. Suppose that the formula is proved for some $k \geq 2$, and consider $n_1 \leq n_2 \leq \ldots \leq n_k \leq n_{k+1}$. Let $g : \gamma \to \bigcup_{i=1}^{k+1}(n_i \times \{i\})$ be a listing. As after every element a of $(n_{k+1} \times \{k+1\}) \cap g(\gamma)$ which is not the last element of $g(\gamma)$, there exists at least one element of $\bigcup_{ik}(n_i \times \{i\})$ that follows a, the cardinality of $(n_{k+1} \times \{k+1\}) \cap g(\gamma)$ is not greater than $\sum_{i=1}^{k} n_i + 1$. Therefore, it is enough to assume that $n_{k+1} \leq \sum_{i=1}^{k} n_i + 1$ and to prove that $l \geq \sum_{i=1}^{k+1} n_i$.

For a maximal listing $g_* : \gamma_* \to \bigcup_{1 \leq i \leq k}(n_i \times \{i\})$, let $\rho_k = \bigcup_{1 \leq i \leq k}(n_i \times \{i\}) \setminus g_*(\gamma_*)$. Define a listing g from γ to $\bigcup_{1 \leq i \leq k+1}(n_i \times \{i\})$ by $g(1) \in (n_{k+1} \times \{k+1\})$, and if $g(1), g(2), \ldots, g(j)$ have been already defined, and $g(j) \in n_{k+1} \times \{k+1\}$, then if $g(j)$ is the last element of $n_{k+1} \times \{k+1\}$ and $n_{k+1} = \sum_{i=1}^{k} n_i + 1$, then the listing stops; otherwise either $g(j+1) \in \rho_k$ if still there exist non listed elements of ρ_k, or $g(j+1)$ is the first element of $g_*(\gamma_*)$ that has not yet been listed; if $g(1), g(2), \ldots, g(j)$ have been already defined, and $g(j) \in g_*(\gamma_*)$, then $g(j+1)$ is either the first non listed element of $n_{k+1} \times \{k+1\}$ if any, or the element of $g_*(\gamma_*)$ that follows $g(j)$ otherwise.

Since $n_{k+1} \leq \sum_{i=1}^{k} n_i + 1$, this procedure exhausts the elements of $n_{k+1} \times \{k+1\}$, and since $|\rho_k| \leq n_{k+1}$, it exhausts the elements of $\bigcup_{1 \leq i \leq k}(n_i \times \{i\})$. We can conclude that in the case where $n_{k+1} \leq \sum_{i=1}^{k} n_i + 1$, the cardinality of a listing is $\sum_{i=1}^{k+1} n_i$, hence the constructed listing is maximal in view of the preliminary observation. □

Consider $\nu_1, \nu_2, \ldots, \nu_p$, such that $\nu_i = \nu_0^i n_0^i + \nu_1^i n_1^i + \ldots + \nu_{k(i)}^i n_{k(i)}^i$ is the decomposition for each $1 \leq i \leq p$. Let

$$A_0 \cup A_1 \cup \ldots \cup A_q = \{\nu_j^i n_j^i : j \leq j(i), 1 \leq i \leq p\}$$

be such that every A_j consists of natural multiples of a non decomposable ordinal π_j, and $\pi_0 \gg \pi_1 \gg \ldots \gg \pi_q$.

Theorem 11.4. *If $\kappa = \pi_0 m_0 + \pi_1 m_1 + \ldots + \pi_q m_q$ is the listing number of $\{\nu_1, \nu_2, \ldots, \nu_p\}$, then there exists an optimal listing $h : \kappa \to (\nu_1 \times \{1\}) \cup (\nu_2 \times \{2\}) \cup \ldots \cup (\nu_p \times \{p\})$ such that the restriction of h to $\pi_j m_j$ is the optimal listing of A_j if the A_i have been exhausted for $i < j$, and of $A_j \cup \{\pi_j \omega\}$ otherwise.*

Proof. The function h ranges first A_0 if it has more than one element, then ranges A_0 and A_1, and so on, because otherwise κ would have components of the type $\alpha + \beta$, where $\alpha + \beta = \beta$, contrary to the optimality. □

12. The Case of 2 Components

From Theorems 9.1, 10.2, 11.3 and 11.4, we recover the following estimate of [5]: if f is a correct multifan and g is a correct arrow in Lašnev spaces, then

$$\sigma(f \otimes g) = 1 + \min\{\lambda(f), \mu(g)\}.$$

Indeed, it is enough to show that in the case of $k = 2$, the listing (11.3) becomes $\min\{\lambda(f), \mu(g)\}$. For $k = 2$, (11.3) becomes

$$l = \nu_1 + \min\{\nu_2, 1 + \nu_1\}. \tag{12.1}$$

Hence if $\lambda > \mu$, then $\nu_1 = \frac{\mu-1}{2}$ and $\nu_2 = \frac{\lambda}{2}$. Thus $l = \min\{\frac{\lambda+\mu-1}{2}, \mu\}$. Since $\lambda \geq 1 + \mu$, we have $\frac{\lambda+\mu-1}{2} \geq \frac{1+\mu+\mu-1}{2} = \mu$, so that $l = \mu = \min\{\lambda, \mu\}$. If now $\mu > \lambda$, then $\mu \geq \lambda + 1$, hence $\nu_2 = \frac{\mu-1}{2} \geq \frac{\lambda}{2} = \nu_1$. Therefore (12.1) becomes $l = \min\{\frac{\lambda+\mu-1}{2}, 1 + \lambda\}$. But $\frac{\lambda+\mu-1}{2} \geq \frac{\lambda+\lambda+1-1}{2} = \lambda$, so that $l = \lambda = \min\{\lambda, \mu\}$.

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