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THE QUESTION OF MORTON BROWN AND THE DEVELOPABILITY OF $w\triangle$-SPACES

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Abstract

We have for a solution to the Question of Morton Brown the very weak property of being an *ooo*-space, a property common to all solutions recorded in the literature, except two. One of the exceptions is the property of being a $w\triangle$-space, so different in nature is it that it hardly intersects with that of being an *ooo*-space. Indeed, $w\triangle$-spaces are developable if they are T_3, θ-refinable and of barely coherent countable pseudocharacter, a property similar to that of being *ooo*-spaces in formulation, but much weaker.

In view of the observation of F.B. Jones that all the theorems (for developable spaces) in the first chapter of Moore's book, *Foundations of point set theory*, "turn out to be theorems for semi-metric spaces with few exceptions" (§1 of [12]), the question of the developability of semi-metric spaces, known in the literature as the Question of Morton Brown, is natural, interesting and important. Various topological properties have been advanced by way of an answer, starting with Heath's *point-countable base* [10] (Theorem 1.5 of [12]), the θ-*base* (=quasi-developability) of Worrell and Wicke [20] put forth by Bennett

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and Berney [4] ((9) of §6 of [12]) and the *δθ-base* of Aull [2] ((16) of §9 of [12]). There are also the *bases of countable order* (BCO's) of Arhangel'skiĭ[1] and the weaker *primitive bases* of Wicke and Worrell [19], the latter weaker than the θ-bases (Theorem 2.2 of [19]). Hodel in his famous unified approach defined *γ-spaces* and the weaker *θ-spaces* (Proposition 4.2 and Remarks 4.8 of [11]). (It should be noted that the θ in θ-spaces has nothing to do with the θ in θ-base, $\delta\theta$-base and θ-refinability, where it connotes some kind of point-finiteness of families of subsets). Fletcher and Lindgren, noting (Proposition 3.3 of [7]) that primitive bases (and therefore *a fortori* θ-bases) make θ-spaces of topological spaces, deemed θ-spaces "an adequate solution" to the Question of Morton Brown, when neither θ-bases nor γ-spaces are so deemed, they being non-comparable to each other. They had evidently forgotten Heath's point-countable bases and Aull's $\delta\theta$-bases. Indeed, Hodel had asked whether a point-countable base makes θ-spaces of topological spaces (Problem 4.11 of [11]) and received a negative answer [8], and (the property of being a) θ-space is thus *not* a common factor of *all* the proposed solutions above to the Question of Morton Brown. All of these is not to mention the *non-archimedean spaces*, the *proto-metrizable spaces* or spaces with *monotone ortho-bases*, all of which are solutions to the Question of Morton Brown (Theorem 2.3 of [18]). (Creede's solution, the $w\triangle$-space, is of a very different nature. See the second last of these Introductory paragraphs.)

Of late, I came upon the notions, in terms of shrinkings of open neighbourhoods, of ϵ- and *o-spaces* [14], [15], classes even more extensive than (that of) Hodel's θ-spaces and found the latter nonetheless good enough an answer to the Question of Morton Brown. But I was unable to show that a $\delta\theta$-base makes an o-space of a topological space. In this paper, we *formally* weaken the notion of o-spaces to that of *ooo-spaces* and have a property shared by *all but one* of the solutions to the Question of Morton Brown (besides Creede's) cited above (see Diagram 1), thus putting the topic in perspective (Theorem 1.1).

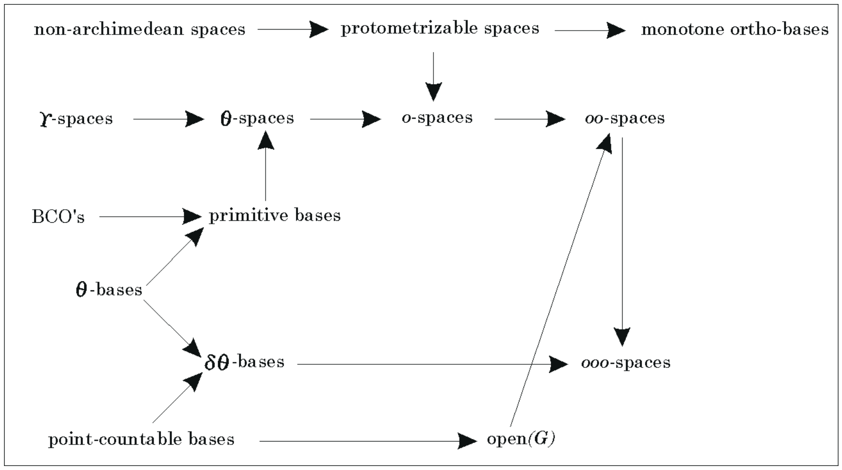

Diagram 1.

Indeed our solution to the Question of Morton Brown remains a solution if we ask for the properties that make developable spaces of first countable *θ-refinable $\iota\iota$-spaces*, $\iota\iota$-spaces being generalizations of monotonic β-spaces of Chaber [5] and quasi β spaces of Fletcher and Lindgren [7] (see Observation 0.2). And, semi-metrizability, relatively strong and factoring into first countability $+\theta$-refinability $+\beta$-space $+G^*_\delta$-diagonal, we can, by asking *instead* about the developability of *spaces with G^*_δ-diagonals*, have Borges' $w\triangle$-spaces for an answer (Theorem 3.3 of [9]) and thus *another type* of solutions to the Question of Morton Brown ((6) of §10 of [12]).

This last result amounting to: θ-refinable $w\triangle$-spaces with W_δ-diagonals are developable (Theorems 2.11 and 6.6 of [9]), the question of developability of θ-refinable $w\triangle$-spaces (or equivalently the metrizability of paracompact $w\triangle$-spaces) can be answered with some conditions, sufficient, under present circumstances, for the existence of W_δ-diagonals, akin to those for that of monotonic developability in the above, generalizing Chaber (Theorem 9.4 of [12]) and Hodel ((11) of §9 of [12]) (Theorem 2.1).

0. Definitions, Notations, Terminology and Simple Facts

1. Throughout this paper, $(X, \mathcal{T})$ denotes a T_1-space. Given a collection $\mathcal{U}$ of subsets on X. For each $x \in X$, we write $\mathcal{U}(x)$ for the sub-collection $\{U : x \in U \in \mathcal{U}\}$.

2. Given a topological space $(X, \mathcal{T})$. Let there be $A : \{(x, U) : x \in U \in \mathcal{T}\} \to \mathcal{T}$. A is said to be a *shrinking of open neighbourhoods* on X, if $x \in A(x, U) \subset U$, whenever $x \in U \in \mathcal{T}$. Given two shrinkings, A and B, of open neighbourhoods on X, if $B(x, U) \subset A(x, U)$, whenever $x \in U \in \mathcal{T}$, we write $B < A$. A property $\mathcal{P}$ on the shrinking A of open neighbourhoods on X is said to be *monotone* if, A has property $\mathcal{P} \Rightarrow B$ has property $\mathcal{P}$ whenever $B < A$. For monotone properties on A, we can assume $x \in A(x, U) \subset ClA(x, U) \subset U$, if X is regular. In the following, we define three monotone properties on the shrinking A of open neighbourhoods on X:

(o) given open neighbourhoods U and V of, respectively, x and y, $x \in A(y, V)$ and $y \in A(x, U)$ implies either $A(y, V) \subset U$ or $A(x, U) \subset V$;

(ϵ) given open neighbourhoods U_n of x, $x \in U_{n+1} \subset A(x, U_n)$ for all $n \in \omega$ implies $\bigcap\{U_n : n \in \omega\}$ is not a neighbourhood of x, unless x has a *smallest* neighbourhood, and

(ι) given open neighbourhoods U_n of $x_n, n \in \omega, x_{n+1} \in U_{n+1} \subset A(x_n, U_n)\backslash\{x_n\}$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} \neq \emptyset$ implies $\langle x_n \rangle$ has a cluster point.

We say X is an o-, an ϵ- and an ι-space, if on X is, respectively, a shrinking of open neighbourhoods with (o), with (ϵ) and with (ι). We can also have a *sequence* $\langle A_n \rangle$ of shrinkings of open neighbourhoods on X and define on it two properties, the second one monotone in the sense that if $B_n < A_n$ for all $n \in \omega$, then $\langle B_n \rangle$ has the same property:

(oo) given open neighbourhoods U and V of, respectively, x and y, $x \in A_o(y,V)$ and $y \in A_o(x,U)$ implies either $x \in A_l(y,V) \subset U$ for some $l \in \omega$ or $A_o(x,U) \subset V$, and

($\iota\iota$) given open neighbourhoods U_n of $x_n, n \in \omega, U_{n+1} \subset A_n(x_n,U_n)\backslash\{x_n\}$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} \neq \emptyset$ implies $\langle x_n \rangle$ has a cluster point.

We say X is an *oo*- and an $\iota\iota$-space, if on X is, respectively, a sequence of shrinkings of open neighbourhoods with (oo) and with ($\iota\iota$). β-spaces are $\iota\iota$-spaces [16]. (In [14], we made the assertion that β-spaces are ι-spaces. That is not quite right. In [16], the definition given of (oo) is a little more restrictive.)

Observation 0.1. Spaces of countable pseudocharacter are ϵ-spaces.

Observation 0.2. Monotone β-spaces (Definition 1.6 of [5]) and quasi β spaces (§4 of [7]) are $\iota\iota$-spaces. A *base of countable order* $\mathcal{B}$ on a topological space X makes it an ι-space. For, we can always let the *range* of the shrinking A of open neighbourhoods required of an ι-space be within $\mathcal{B}$.

Observation 0.3. i) θ-spaces (Definition 4.4 of[11]) are (first countable) o-spaces. (See Proposition 1.4 and third item of Remarks on it in [15].) The converse is not true and a counterexample can be found in the *non-archimedean space* of Gruenhage's construction (§3 of [8]), non-archimedean spaces being clearly o-spaces, as are proto-metrizable spaces.

Observation 0.4. A space $(X, \mathcal{T})$ is said to have (G) if, for every $x \in X$, there is a sequence $\langle W_n(x) \rangle$ of subsets, each containing x, so that, if $\xi \in U \in \mathcal{T}$, there is an open neighbourhood $V(\xi,U)$ of ξ so that, for every $y \in V(\xi,U)$, $\xi \in W_m(y) \subset U$ for some $m \in \omega$ (dependent on y). If $\langle W_n(x) \rangle$ is *decreasing*, X is said to have *decreasing* (G). If $W_n(x)$ is an *open* neighbourhood of x, X is said to have *open* (G) [6]. The notion of a decreasing (G) is equivalent to that of a *point-network* of Balogh [3]. Clearly, spaces $(X, \mathcal{T})$ with decreasing (G) are ι-spaces. For, if, for any $x \in U \in \mathcal{T}$, we let $A(x,U) = V(x,U)$, we see that, given the hypothesis of the condition of (ι) with $\xi \in \bigcap\{U_n : n \in \omega\}$ and therefore $m_n \in \omega$ such that $x_n \in W_{m_n}(\xi) \subset U_n$, for every $n \in \omega$, the sequence $\langle m_n \rangle$ has to be strictly increasing and $x_n \to \xi$. Spaces $(X, \mathcal{T})$ with open (G) are *oo*-spaces. For, we can always let $A_n(\xi, U) \equiv V(\xi, U) \cap W_n(\xi)$ for every $n \in \omega$.

3. Let there be a *relatively open refinement* $\hat{\mathcal{C}} = \{\hat{C}_o, \hat{C}_1, \ldots\}$ of a countable closed cover $\mathcal{C} = \{C_o, C_1, \ldots\}$ of X so that, for every $n \in \omega$, $\hat{C}_n$ is a relatively open subset of C_n. Let there be $\Gamma : \{(x,U) : x \in U \in \mathcal{T}\} \to \mathcal{C}$ such that $x \in \hat{\Gamma}(x,U)$. Given a sequence $\langle A_n \rangle$ of shrinkings of open neighbourhoods on X, we define on it the following property with respect to $\mathcal{C}, \hat{\mathcal{C}}$ and Γ:

(*ooo*) given open neighbourhoods U and V of, respectively, x and $y \in \hat{\Gamma}(x,U)$, $x \in A_o(y,V)$ and $y \in A_o(x,U)$ implies either $x \in A_l(y,V) \subset U$ for some $l \in \omega$ or $A_o(x,U) \subset V$.

We say X is an *ooo*-space, if on X is a sequence of shrinkings of open neighbourhoods with (*ooo*).

Remarks. For the purpose of Theorems 1.1 and 1.2 below, the open sets $A_n(x,U)$, $n > 0$, in the definition of the property (*ooo*), need not contain x. Indeed, instead of the *countable* family $\{A_n(x,U) : n > 0\}$, we can have a *point-countable* family $\mathcal{A}(x,U)$ of open subsets covering $A_o(x,U)$ so that wherever $A_l(y,V)$ suffices we can offer up a member of $\mathcal{A}(y,V)(x)$.

Observation 0.5. Spaces $(X, \mathcal{T})$ with $\delta\theta$-bases. There is on X a collection $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$ of open subsets such that, for every $x \in X, \bigcup\{\mathcal{U}_n(x) : 1 \leq |\mathcal{U}_n(x)| \leq \omega\}$ is a local base at x, which can be enumerated and becomes $\langle V_{x,n} \rangle$. Clearly, for every $n \in \omega$, $C_n \equiv \{x \in X : |\mathcal{U}_n(x)| \leq \omega\}$ is closed in X and $\hat{C}_n \equiv \{x \in X : 1 \leq \mathcal{U}_n(x)| \leq \omega\}$ is open in C_n. Given any $x \in U \in \mathcal{T}$. For $\Gamma(x,U)$, we pick C_n such that i) $x \in \hat{C}_n$ and ii) there is $\Xi \in \mathcal{U}_n(x)$ so that $x \in \Xi \subset U$, and let $B_m(x,U) \equiv \Xi \cap V_{x,m}$, for every $m \in \omega$. Clearly, if $y \in \hat{\Gamma}(x,U) \cap \Xi, \Xi = V_{y,l}$ for some $l \in \omega$. From this fact, it is not difficult to see that $(X, \mathcal{T})$ is an *ooo*-space.

4. Given a topological space $(X, \mathcal{T})$, we say X has *coherent countable pseudocharacter* if, for every $x \in X$, there is such a sequence $\langle g(x,l) \rangle$ of open neighbourhoods that

i) $\bigcap\{g(x,l) : l \in \omega\} = \{x\}$ and

ii) if $y \in g(x,m)$, then we have $x \in g(y,n) \subset g(x,m)$ for some $n \in \omega$.

Clearly, point-countable p-bases on X confer on X such coherent countable pseudocharacter. Note that if X has coherent countable pseudocharacter and is countably compact, then it is compact. For, to be otherwise, there is a countably complete, free, closed ultrafilter $\mathcal{U}$ on X and, if we write $G(x)$ for $\bigcup\{g(x,l) : l \in \omega, \backslash g(x,l) \in \mathcal{U}\}$ for every $x \in X$, we have $y \in G(x) \Leftrightarrow x \in G(y)$ for every $x, y \in X$ and some *finite* $\triangle \subset X$ such that $\bigcup\{G(x) : x \in \triangle\} = X$. $\backslash G(x)$ being a member of $\mathcal{U}$, we have a void finite intersection of members of $\mathcal{U}$ (cf. Cor. 3 of [13]).

5. Let there be a *relatively open refinement* $\hat{\mathcal{C}} = \{\hat{C}_o, \hat{C}_1, \ldots\}$ of a countable closed cover $\mathcal{C} = \{C_o, C_1, \ldots\}$ so that, for every $n \in \omega$, $\hat{C}_n$ is a relatively open subset of C_n. Let there be $\Gamma : X \to \mathcal{C}$ such that $x \in \hat{\Gamma}(x)$. We can define the concept of *barely coherent countable pseudocharacter* with respect to $\mathcal{C}, \hat{\mathcal{C}}$ and Γ by substituting ii) in the definition of the concept of coherent countable pseudocharacter with
ii)' for every $g(x,m)$, there is a $g'(x,m) \in \{g(x,l) : l \in \omega\}$ such that if $y \in g'(x,m) \cap \hat{\Gamma}(x)$ and $x \in g'(y,n)$ then we have $x \in g(y,p) \subset g(x,m)$ for some $p \in \omega$ or $g'(x,m) \subset g(y,n)$.

6. The structures in iii) of Theorem 6.3 in [9] that determine the existence of BCO's and W_δ-diagonals are referred to here respectively as BCO trees and W_δ-diagonal trees.

1. Factorization of Monotone Developability

Theorem 1.1. *T_3-spaces X that are simultaneously ϵ-, $\iota\iota$- and ooo-spaces have BCO's. Conversely, BCO's on a topological space make it an ϵ-, an ι- and an o-space.*

Proof. On X, let A be a shrinking of open neighbourhoods and $\langle A_n \rangle$ be a sequence of shrinkings of open neighbourhoods satisfying respectively (ϵ) and $(\iota\iota)$. Let there be $\mathcal{C}, \hat{\mathcal{C}}$ and Γ (0.3 above) with respect to which the sequence $\langle B_n \rangle$ of shrinkings of open neighbourhoods satisfies (*ooo*).

We are to construct a tree $\mathcal{V}$ of open neighbourhoods (of *specific* points) of height ω, each element of which, as an open neighbourhood of a specific point, is the union of the family of its immediate successors (as open subsets), each branch $\mathcal{B}$ of which constitutes a base at any $\xi \in \bigcap \mathcal{B}$, and the first level of which covers X. To construct such a tree, we need only indicate, in the following, how branches are constructed, i.e., how we form an element on the first level and how, given an initial segment of a branch, we form an immediate successor. Specifically, for $x_o \in X$, we let $W_o \equiv X$ if $x_o \in C_o$. Otherwise, let $W_o \equiv X \backslash C_o$. Let $V_o \equiv A(x_o, W_o) \cap A_o(x_o, W_o) \cap B_o(x_o, W_o)$. Clearly, V_o is an open neighbourhood of x_o. Suppose we have constructed open neighbourhoods $V_o, V_1, \ldots, V_n$ of, respectively, $x_o, x_1, \ldots, x_n$ such that $V_o \supset V_1 \supset \ldots \supset V_n$. For $x_{n+1} \in V_n$, if i) $x_{n+1} \neq x_n$, we let

$$W_{n+1} \equiv [V_n \backslash \{x_n\}] \cap \bigcap \{B_i(x_j, W_j) : x_{n+1} \in B_i(x_j, W_j); i, j \leq n\} \backslash \bigcup \{C_i : x_{n+1} \notin C_i, i \leq n+1\}, \text{ and}$$

$$V_{n+1} \equiv A_\nu(x_{n+1}, W_{n+1}) \cap B_o(x_{n+1}, W_{n+1}), \quad \text{where } \nu = |\{x_o, x_1, \ldots, x_n\}|;$$

and if ii) $x_{n+1} = x_n$, we let

$$W_{n+1} \equiv V_n \cap \bigcap \{B_i(x_j, W_j) : x_{n+1} \in B_i(x_j, W_j); i, j \leq n\} \backslash \bigcup \{C_i : x_{n+1} \notin C_i, i \leq n+1\}, \text{ and}$$

$$V_{n+1} \equiv A(x_{n+1}, W_{n+1}).$$

We have constructed a BCO tree (0.6), provided we can show that if $\xi \in \bigcap\{V_n : n \in \omega\}$, then $\{V_n : n \in \omega\}$ is a local base at ξ. The case of its alternative being almost trivial, we assume that the sequence $\langle x_n \rangle$ consists of infinitely many distinct points and property $(\iota\iota)$ manifesting itself via $\langle A_n \rangle$ ensures that $\langle x_n \rangle$ clusters to some $\eta \in \bigcap\{V_n : n \in \omega\}$. Let U be an arbitrary open neighbourhood of η. There are arbitrarily large $i, j \in \omega$ such that $x_i, x_j \in B_o(\eta, U) \cap \hat{\Gamma}(\eta, U)$ and such that $x_j \notin W_i$. Of course, i can be so chosen that $\eta \in B_o(x_i, W_i)$. Then, because of (*ooo*), either $B_o(\eta, U) \subset W_i$ or $\eta \in B_l(x_i, W_i) \subset U$ for some $l \in \omega$. But then, we cannot have the first alternative, which implies that $x_j \notin B_o(\eta, U)$, a contradiction. Therefore we have to have the second alternative. The open set $B_l(x_i, W_i)$, being a neighbourhood of η, contains x_k, for arbitrarily large k, and, so long as $k > l$, W_k and V_k are its subsets, yielding the result that $V_k \subset U$, for some $k \in \omega$, i.e., $\{V_i : i \in \omega\}$ is a base at η. X being T_1, $\xi \notin \bigcap\{V_i : i \in \omega\}$ unless $\xi = \eta$. We therefore have a BCO tree on X and the monotone developability of X.

The converse follows from Observations 0.1, 0.2 and 0.3. □

Remarks. Indeed, if we weaken the property $(\iota\iota)$ by weakening the conclusion, in the definition of $(\iota\iota)$, to the requirement that $\bigcap\{U_n : n \in \omega\}$ is not open (for the same hypothesis), arriving at property (†) (cf. Remarks 1 in [14]), and bring in a *complementary* property $(*)$ that demands, from the same hypothesis, the conclusion that either $\bigcap\{U_n : n \in \omega\}$ is open or the family $\{U_n : n \in \omega\}$ is a base at some point, we can almost immediately see that a first countable T_1-space has a BCO if and only if it has sequences of shrinkings of open neighbourhoods with $(*)$ and with (†). Noting that on spaces with monotone ortho-bases there are sequences of shrinkings of open neighbourhoods with $(*)$, we see strengthenings of the Theorem of Phillips (Theorem 2.3 of [18]) in the above.

Corollary. *An ooo-space is developable if (and only if) it is semi-stratifiable (cf. Cor. 2.2 of [15]).*

Theorem 1.2. *T_3-spaces X that are simultaneously ϵ- and ooo-spaces are D_o-spaces (see [7]) and, in particular, first countable.*

Proof. Let A be a shrinking of open neighbourhoods and $\langle B_n\rangle$ be a sequence of shrinkings of open neighbourhoods on X, satisfying respectively (ϵ) and (ooo) with respect to some $\mathcal{C}, \hat{\mathcal{C}}$ and Γ. Let K be a compact subset of X. We are to construct a tree $\mathcal{V}$ of open neighbourhoods of specific points *on* K of height ω, each level $\mathcal{V}_n$ of which is *finite* and *covers* the compact subset K, each element V on levels other than the 0^{th} has its closure ClV contained in its immediate predecessor and each branch $\mathcal{B}$ of which constitutes a base at any $\xi \in \bigcap \mathcal{B}$. The construction is *mutatis mutandis* that in the Proof of Theorem 1.1 above, the compactness of K playing the role of the $(\iota\iota)$ of $\langle A_n\rangle$ in its securing of the convergence of $\langle x_i\rangle$. Then the family $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a base of K. For otherwise there is an open set $U \supset K$ and, according to König [17], a branch $\mathcal{B} \equiv \{V_n : n \in \omega\}$ of $\mathcal{V}$ such that $V_n \backslash U \neq \emptyset$ for every n which is impossible if $\mathcal{B}$ is a local base at ξ. □

Remarks. That essentially the same argument is used to construct a BCO in Theorem 1.1 and to establish the countable character of compact subsets in Theorem 1.2 shows that, at least for this type of construction, what is sufficient to bring forth a BCO in T_3 $\iota\iota$-spaces is also sufficient to establish the countable character of compact subsets in T_3-spaces. To that extent, the D_o-space requirement is necessary as a solution to the Question of Morton Brown and F.B. Jones' conjecture is not without basis.

2. Conditions for W_δ-diagonals

Theorem 2.1. *T_3 $\iota\iota$-spaces X of* barely coherent countable pseudocharacter *have W_δ-diagonals.*

Proof. On X, let $\langle A_n\rangle$ be a sequence of shrinkings of open neighbourhoods satisfying $(\iota\iota)$. Let there be $\mathcal{C}, \hat{\mathcal{C}}$ and Γ with respect to which X is of barely coherent countable pseudocharacter.

We are to construct a tree $\mathcal{V}$ of open neighbourhoods (of *specific* points) of height ω, each element of which, as an open neighbourhood of a specific point, is the union of the family of its immediate successors (as open subsets), for each branch $\mathcal{B}$ of which we have $|\bigcap \mathcal{B}| \leq 1$, and the first level of which covers X. To construct such a tree, we need only indicate, in the following, how branches are constructed, i.e., how we form an element on the first level and how, given an initial segment of a branch, we form an immediate successor. Specifically, for $x_o \in X$, we let $W_o \equiv X$ if $x_o \in C_o$. Otherwise, let $W_o \equiv X \backslash C_o$. Let $V_o \equiv A_o(x_o, W_o)$. Clearly, V_o is an open neighbourhood of x_o. Suppose we have constructed open neighbourhoods $V_o, V_1, \ldots, V_n$ of, respectively, $x_o, x_1, \ldots, x_n$ such that $V_o \supset V_1 \supset \ldots \supset V_n$. For $x_{n+1} \in V_n$,

if i) $x_{n+1} \neq x_n$, we let
$W_{n+1} \equiv [V_n \backslash \{x_n\}] \cap \bigcap \{g(x_j, i) : x_{n+1} \in g(x_j, i); i, j \leq n\}$
$\backslash \bigcup \{C_i : x_{n+1} \notin C_i, i \leq n+1\}$, and
$V_{n+1} \equiv A_\nu(x_{n+1}, W_{n+1}) \cap \bigcap \{g'(x_{n+1}, \mu^i_{n+1}) : i \leq n\}$,
where $\nu = |\{x_o, x_1, \ldots, x_n\}|$ and μ^i_{n+1} is such that $x_i \notin g(x_{n+1}, \mu^i_{n+1})$ for all $i \leq n$;
and if ii) $x_{n+1} = x_n$, we let
$W_{n+1} \equiv V_n$, and
$V_{n+1} \equiv W_{n+1} \cap \bigcap \{g(x_{n+1}, i) : i \leq n+1\}$.

We have constructed a W_δ-diagonal (0.6), provided we can show that $|\bigcap \{V_n : n \in \omega\}| \leq 1$. The case of its alternative being almost trivial, we assume that the sequence $\langle x_n \rangle$ consists of infinitely many distinct points and property $(\iota\iota)$ manifesting itself via $\langle A_n \rangle$ ensures that $\langle x_n \rangle$ clusters to some $\eta \in \bigcap \{V_n : n \in \omega\}$, unless $\bigcap \{V_n : n \in w\} = \emptyset$. There are (for any $m \in \omega$) arbitrarily large $i, j \in \omega, j < i$, such that $x_i, x_j \in g'(\eta, m) \cap \hat{\Gamma}(\eta)$.

Of course, i can be so chosen that $\eta \in g'(x_i, \mu^j_i)$. Then, either $g'(\eta, m) \subset g(x_i, \mu^j_i)$ or $\eta \in g(x_i, l) \subset g(\eta, m)$ for some $l \in \omega$. But then, we cannot have the first alternative, which implies that $x_j \notin g'(\eta, m)$, a contradiction. Therefore we have to have the second alternative. The open set $g(x_i, l)$, being a neighbourhood of η, contains x_k, for arbitrarily large k, and, so long as $k > l, i$, and $x_{k-1} \neq x_k$, W_k and V_k are its subsets, yielding the result that $V_k \subset g(\eta, m)$, for some $k \in \omega$, i.e., $|\bigcap\{V_i : i \in \omega\}| = 1$. We therefore have on X a W_δ-diagonal. □

Corollary 2.2. *T_3 θ-refinable $w\triangle$-spaces of barely coherent countable pseudocharacter are developable. T_3 θ-refinable M-spaces of barely coherent countable pseudocharacter are metrizable. Hausdorff M-spaces of coherent countable pseudo-character are metrizable.*

Questions. 1. While it is obvious that point-countable p-bases confer on the space coherent countable pseudocharacter, does (barely) coherent countable pseudocharacter on X confer on X point-countable p-bases?

2. While $(ooo), (\epsilon)$ and $(\iota\iota)$ add up to a BCO, do combinations involving less than all three of them add up to anything of significance in the literature? For example, are $\iota\iota$-spaces countably metacompact (as β-spaces are)? Are ϵ-spaces necessarily of countable pseudocharacter? Do they add up to anything under some special circumstances?

3. Does a point-countable base make o-spaces of topological spaces (cf. Problem 4.11 of Hodel [11] in the light of Observation 3)? Do $\delta\theta$-bases? Do open (G)'s? Negative answers would imply that the properties (o) and (ooo) are different. Are they different?

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