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COUNTABLY COMPLETE TOPOLOGICAL ALGEBRAS

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Abstract

For a completely regular Hausdorff space X, denote by $C(X)$ the topological algebra, with the compact-open topology, of all complex valued continuous functions on X. There are spaces X for which neither $C(X)$ nor any extension of $C(X)$, is simultaneously bornological and complete. To overcome this difficulty, this paper defines the concept of *countable completeness* and proves that every $C(X)$ admits a minimal extension that is both bornological and countably complete.

1. Introduction

In [5], W. W. Comfort asked the question: *If X is a k-space must the Hewitt realcompactification υX of X be a k-space?* This general topology question was promptly and elegantly answered in the negative by N. Noble [18] and Comfort [6]. Interest in this question has been rekindled by M. Fragoulopoulou and N. C. Phillips in the context of automatic continuity of $*$-homomorphisms on *lmc* $*$-algebras [8] and commutative Pro-C^*-algebras [21]. The connection between the topology problem of [5] and the functional analysis problems of [8] and [21] is displayed in Theorems A and B below.

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Throughout this paper, X will denote a completely regular Hausdorff space and $C(X)$, the topological $*$-algebra of complex valued continuous functions on X with the compact-open topology. For definitions not stated in sections 3 and 4 below, see [7], [10], [14] or [22].

Theorem A. *For any completely regular Hausdorff space X, the following are equivalent:*

(a) X is realcompact.

(b) $C(X)$ is bornological (Nachbin [17] and Shirota [26]).

(c) $C(X)$ is iseomorphic as a topological vector space to a direct limit of Banach spaces (Warner [30; Th. 5] or [32]).

(d) Every positive linear functional on $C(X)$ is continuous (Hewitt [11; Th. 21]).

(e) Every multiplicative linear functional on $C(X)$ is continuous (Michael [15; 6.1]).

(f) Every order bounded linear functional on $C(X)$ is continuous (Hewitt [11; Th. 23]).

(g) Every $*$-homomorphism from $C(X)$ to a C^*-algebra is continuous.

Proof. The proof that statements (a) through (f) are equivalent may be found in the references noted. Since each multiplicative linear functional on $C(X)$ is a $*$-homomorphism (Kaplansky [12]), it follows that (g) implies (e).

(a) implies (g). For a $*$-homomorphism ϕ of $C(X)$ into a C^*-algebra, the image $\phi[C(X)] = B$ is a commutative C^*-algebra with unit; whence, by the Gelfand Theorem ([7;1.4.1]), B is isometrically isomorphic with $C(Y)$ for some compact Hausdorff space Y. The composition ϕ_1 of this isometry with ϕ is a homomorphism of $C(X)$ onto $C(Y)$. By 10.8 and 10.3(b) of [10], ϕ_1 induces a homeomorphism from Y onto a compact subspace K of X. It is now clear that ϕ is continuous since it is a composition of the restriction map $f \longrightarrow f|_K$ with with an iseomorphism. $\square$

Definition 1.1. A function f on a space X is *k-continuous* if $f|_K$ is continuous on K for each compactum in X. $KC(X)$

will denote the $*$-algebra of all k-continuous, complex valued functions on X. For a space X, $k_r X$ will denote the set X with the $KC(X)$-weak topology. Finally, X is a k_r-*space* if it is completely regular Hausdorff and $KC(X) = C(X)$.

Theorem B. *For any completely regular Hausdorff space X, $C(X)$ is complete with respect to the compact-open topology if and only if X is a k_r-space (Pták [23]).*

2. The Question

From the point of view of functional analysis, the best possible scenario is for $C(X)$ to have the automatic continuity property (i.e., X is realcompact) and to be complete (i.e., X is a k_r-space). This is the case when X is locally compact and realcompact.

Definition 2.1. A topological algebra $(B, \mathcal{E})$ is an *extension* of the topological algebra $(A, \mathcal{T})$ with *extendor* θ provided θ is an injective homomorphism from A to B that is also an open map from $(A, \mathcal{T})$ to $\theta[A]$ with the relative $\mathcal{E}$ topology.

Question 2.2. *For a completely regular space X, does $C(X)$ with the compact-open topology, admit an extension $C(Y)$ that is both complete and bornological?*

Answer: No. In [16], Mrówka displays an example that gives an emphatically negative answer to this question.

Mrówka's Example 2.3. This space is a locally compact, non-realcompact space X which is the union of two closed realcompact subspaces. By [3], this space is a μ-space for which both $K \cap X$ and $K \setminus X$ are compact for every compact $K \subset \upsilon X$. (Note: υX is the Hewitt realcompactification of X [10; 8.4]). Thus X is both open and closed in $k_r(\upsilon X)$. This means that if the operators υ and k_r are alternately applied to X, one obtains an increasing transfinite sequence of distinct spaces. If one forms $C(T)$ for each of these spaces T $(= \upsilon X, k_r \upsilon X, \cdots)$, then one gets a strictly increasing transfinite sequence of commutative $*$-algebras. The algebras in this sequence are alternately bornological or complete but no single algebra has both properties.

The Mrówka Example shows, in general, that it is not possible to achieve both completeness and the automatic continuity properties listed in Theorem A. To avoid this pathology, functional analysts typically assume that the algebra $C(X)$ satisfies a strong countability condition, such as, requiring that the compact-open topology be separable, or sequentially complete [21; 1.4.5] or that it be first countable [8; 3.4]. This suggests that by adding an appropriate countability condition, a positive answer to the above question is achievable. It is the purpose of the next section to introduce a countability condition that is weaker (in the present context) than both first countability and separability. (See 3.3 and 3.4 below.)

3. Countable Completeness

Definition 3.1. A uniform space is *countably complete* if each Cauchy net which assumes only countably many values, is convergent.

Theorem 3.2 *For any uniform space (X, μ), the following are equivalent:*

(a) (X, μ) is countably complete.
(b) Each Cauchy filter, containing a countable set, converges.
(c) The closure of any countable set is complete.
(d) Each closed separable subspace is complete.

Proof. That (a) $\Leftrightarrow$ (b) follows from the standard translation from nets to filters.

(a) $\Rightarrow$ (c). Let E be the closure of a countable set D and let $\{x_\alpha\}$ be a Cauchy net in E. Then this net $\{x_\alpha\}$ converges to a point s in the closure of E in the completion (X^*, μ^*). For each index α, there is a net $\{y_{\alpha,j}\}$ in D which converges in X

to x_α. By the Theorem on Iterated Limits (See [13], page 69), there exists a subnet of $\{y_{\alpha,j}\}$ which converges in X^* to s. Since this subnet is countably valued, condition (a) implies that it converges to a point in X. So $s \in E$; whence E is complete.

Clearly (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (b). Let $\mathcal{F}$ be a Cauchy filter containing a countable set D and set $E = \mathrm{cl}D$. Then by (d), the restriction of $\mathcal{F}$ to E converges; whence $\mathcal{F}$ converges. $\square$

Corollary 3.3. *Each countably complete, separable uniform space is complete.*

Theorem 3.4. *If $C(X)$ is first countable, then its completion is bornological.*

Proof. Recall that $C(k_rX)$ is the completion of $C(X)$. If $C(X)$ is first countable then both X and k_rX are σ-compact and thus realcompact. The desired conclusion now follows from Theorem A. $\square$

Remark 3.5. Complete $\Rightarrow$ countably complete $\Rightarrow$ sequentially complete.

If X is the space described in Mrówka's Example above, then $C(\upsilon X)$ is countably complete, but not complete, with respect to the compact-open topology. The following example describes a uniform space that is sequentially complete (i.e., each Cauchy sequence converges) but is not countably complete.

Example. Let $X = \beta\mathbf{N} \setminus \{p\}$ where p is a non-isolated point of the Stone-Čech compactification $\beta\mathbf{N}$ of the countably infinite discrete space $\mathbf{N}$. This space X admits a unique Hausdorff uniformity μ [10; 15R, p.238]. The completion of (X, μ) is $\beta\mathbf{N}$. Since (X, μ) is separable but not complete, it is not countably complete. Every convergent sequence in $\beta\mathbf{N}$ is eventually constant [10; 9N.2, p.139]. This implies that every μ-Cauchy sequence in X converges.

Definition. (a) A topological space is $\aleph_0$-*bounded* if it is completely regular Hausdorff and each countable subset has compact closure [22; p.381].
(b) A uniform space is $\aleph_0$-*totally bounded* if each filter containing a countable set is contained in a Cauchy filter.

Theorem 3.6. *For any completely regular Hausdorff space X, the following are equivalent:*

(a) X is $\aleph_0$-bounded.

(b) Every filter $\mathcal{F}$ on X, containing a countable set, clusters.

Proof. Clearly (a) implies (b). Assume that (a) fails. Then X contains a countable set D whose closure in X is non-compact. Then there is a point p in the βX-closure of D that is not in X. If $\mathcal{F}$ is the filter on X generated by D and the trace on X of the βX neighborhood system of p, then $\mathcal{F}$ contains the countable set D but fails to have a cluster point in X. □

Corollary 3.7. *A uniform space is $\aleph_0$-bounded if and only if it is countably complete and $\aleph_0$-totally bounded.*

Note. In the proof of the next theorem, we will repeatedly refer to [22]. Although Porter and Woods in their book [22] work solely within the category of Hausdorff spaces and continuous maps, their categorical approach permits ready application to the category $UNIF$ of Hausdorff uniform spaces and uniformly continuous maps.

Theorem 3.8. *(a) Countable completeness is preserved by products, closed subspaces and intersections.*

(b) Countable completeness is a uniform extension property (in the sense of [22]).

(c) If $f : (X, \mu) \longrightarrow (Y, \nu)$ is uniformly continuous, then $f^{\leftarrow}[Z]$ is countably complete whenever both $Z \subset Y$ and X are countably complete.

(d) Each uniform space (X, μ) admits a minimal countable completion (X^c, μ^c) that is unique up to a uniform isomorphism.

(e) Each uniformly continuous map $f : (X, \mu) \longrightarrow (Y, \nu)$ with countably complete codomain (Y, ν) admits a unique uniformly continuous extension $f^c : (X^c, \mu^c) \longrightarrow (Y, \nu)$.

Proof. The proof that countable completeness is productive and closed hereditary is the same as it is for completeness. See pages 194 and 192 in [13]. That countable completeness is preserved by intersections follows as in 5.9(d) of [22]. Parts (b), (c), (d) and (e) follow from 5.3(c), p.369; 5.9(c), p.385; 5.3(d), p.371 and 5.3(f), p.372 in [22], respectively. □

The following theorem gives an explicit characterization of the countable completion of a uniform space.

Theorem 3.9. *If (X^*, μ^*) is the completion of (X, μ), then*

$$X^c = \bigcup \{X^*\text{-}cl\ D : D \subset X,\ D \text{ is countable}\}$$

and μ^c is the restriction of μ^* to X^c.

Proof. Since any closed separable subspace E of X^c lies in the closure of a countable subset of X, it follows that E is closed in X^*; therefore E is complete. □

Remark 3.10. Let $UNIF$ be the category of Hausdorff uniform spaces and uniformly continuous maps. The full subcategory *COUNT COMP* of $UNIF$, whose objects are the countably complete uniform spaces, is an epireflective subcategory. [22; 9.6(b)(4), p.717].

The question in Section 2 can now be restated.

Revised Question 3.11. *For a completely regular space X, does $C(X)$ with the compact-open topology, admit an extension $C(Y)$ that is both countably complete and bornological?*

To achieve an answer (see 5.2) to the revised question, we obtain the analogue of Theorem B, namely Theorem 4.3 below, that characterizes countable completeness.

4. k_r^ω-spaces

Definition 4.1. A map $f : X \longrightarrow Y$ is an *ω-map* if it factors through a separable metric space; i.e., there exists a separable metric space M, a map $g : M \longrightarrow Y$ and a continuous surjection $\phi : X \longrightarrow M$ so that f is the composition $g \circ \phi$. A map is *k^ω-continuous* provided it is both k-continuous and an ω-map. $K^\omega C(X, Y)$ will denote the set of all k^ω-continuous maps from X to Y while $K^\omega C(X)$ will denote the topological $*$-algebra of complex valued k^ω-continuous functions, with the compact-open topology. Finally, X is a *k_r^ω-space* if it is completely regular Hausdorff and $K^\omega C(X) = C(X)$.

Remark 4.2. (a) If X is separable metrizable, then every map defined on X is an ω-map.

(b) If Y is separable metrizable, then every continuous map from X to Y is an ω-map.

Theorem 4.3. *A completely regular Hausdorff space X is a k_r^ω-space if and only if $C(X)$, with the compact-open topology, is countably complete.*

Proof. It suffices to prove that $C(X)$ is countably complete if and only if $K^\omega C(X) = C(X)$. This is done in the next two Propositions. □

Proposition 4.4. *If $C(X)$ is countably complete, then $K^\omega C(X) = C(X)$.*

Proof. By 4.2(b) above, $C(X) \subset K^\omega C(X)$. Let $f \in K^\omega C(X)$. Then f is k-continuous and there are a separable metric space (M, d), a continuous surjection $\phi : X \longrightarrow M$ and a map $g : M \longrightarrow \mathbf{C}$ so that $f = g \circ \phi$. For a countable dense subspace $\{t_n : n \in \omega\}$ of M, define $h_n(t) = d(t, t_n)$; then $\{h_n : n \in \omega\}$ separates points of M. Let B be the $Q[i]$-linear ring generated by $\{h_n : n \in \omega\}$. (Here $Q[i]$ denotes the field of Gaussian rationals.) Note that the restriction of g to $\phi[K]$ is continuous for each compactum K in X. By the Stone-Weierstrass Theorem (Theorem 11, p.58 in [28]), each $g \in KC(M)$ can be approximated on any compact subset of M by a sequence in B. Therefore f is the compact-open limit of a net in the countable subset $\{h \circ \phi : h \in B\}$ of $C(X)$. Since $C(X)$ is countably complete, $f \in C(X)$. $\square$

Proposition 4.5. *For any completely regular space X, the equality $K^\omega C(X) = C(X)$ implies that $C(X)$ is countably complete.*

Proof. Let $\{f_\alpha : \alpha \in D\}$ countably valued Cauchy net in $C(X)$. Since this net converges uniformly on compacta, the point-wise limit f of $\{f_\alpha : \alpha \in D\}$ is k-continuous. To show that f is an ω map, enumerate the set underlying $\{f_\alpha : \alpha \in D\}$ by ω to get $\{g_n : n \in \omega\} = \{f_\alpha : \alpha \in D\}$; define ϕ to be the evaluation map, i.e., $\phi(x) = (g_n(x))_{n\in\omega} \subset \mathbf{C}^\omega$ and $M = \phi[X]$. Then M is separable metrizable and ϕ is a continuous surjection. For $t \in M$, each g_n is constant on $\phi^{\leftarrow}(t)$; therefore f is constant on these sets. This means $g = f \circ \phi^{\leftarrow}$ is a well-defined map from M to $\mathbf{C}$ and that f is a k^ω-map. The hypothesis now insures that f is continuous. So $\{f_\alpha : \alpha \in D\}$ converges to f in $C(X)$. $\square$

Theorem 4.6. *(a) Each open subspace of a k_r^ω-space is k_r^ω.*
(b) If X is a k_r^ω-space, then the Hewitt realcompactification υX is k_r^ω.

Proof. (a) Let Y be an open subspace of the k_r^ω-space X and $\{f_\alpha\}$ be a countably valued Cauchy net in $C(Y)$; denote by f the point-wise limit of $\{f_\alpha\}$. For $y_0 \in Y$, let $h \in C(X, [0,1])$ be so that $h(y_0) = 1$ while $h^{\leftarrow}(0)$ is a neighborhood of $X \setminus Y$. For each α, let F_α vanish on $X \setminus Y$ and be equal to the product $f_\alpha \cdot h|_Y$ on Y. Then $F_\alpha \in C(X)$. Since each compactum K in X is the union of a compact subset of Y and $K \cap h^{\leftarrow}(0)$, it follows that $\{F_\alpha\}$ is a countably valued Cauchy net in $C(X)$. The fact that X is a k_r^ω-space, implies $\{F_\alpha\}$ converges to some $F \in C(X)$. Since $f(y) = F(y)/h(y)$ at all points $y \in Y$ at which $h(y) \neq 0$, it follows that f is continuous on a neighborhood of y_0. So $f \in C(Y)$. That Y is a k_r^ω-space, is now a consequence of Theorem 4.3.

(b) Let $f \in K^{\omega}C(\upsilon X)$. Since f factors through a separable metric space M, there is a continuous surjection $\phi : \upsilon X \longrightarrow M$ and a map $g : M \longrightarrow \mathbf{C}$, so that $f = g \circ \phi$. Since X is a k_r^{ω}-space, $f|_X$ is continuous and has an extension $f_1 \in C(\upsilon X)$ [10; 8.7(II)]. For $p \in \upsilon X$, $\phi^{\leftarrow}[\phi(p)]$ is a zero-set in υX; therefore by [10; 8.8(b)], $\phi^{\leftarrow}[\phi(p)]$ is the closure of its intersection with X. This implies that both f and f_1 are constant on $\phi^{\leftarrow}[\phi(p)]$. So $f = f_1 \in C(\upsilon X)$ and thus $K^{\omega}C(\upsilon X) = C(\upsilon X)$. □

Definition 4.7. Recall that for completely regular Hausdorff spaces X and Y, a continuous surjection $\tau : X \longrightarrow Y$ is a *completely regular quotient mapping* if the topology on Y is the coarsest one for which all functions in $\{g : g \in \mathbf{C}^Y,\ g \circ \tau \in C(X)\}$ are continuous. See 10.11 in [10].

Theorem 4.8. *Each completely regular quotient of a k_r^{ω}-space is k_r^{ω}.*

Proof. Let Y be a completely regular quotient of the the k_r^{ω}-space X with respect to the continuous surjection τ. If $\{g_\alpha\}$ is a countably valued Cauchy net in $C(Y)$, then $\{g_\alpha \circ \tau\}$ converges to some $f \in C(X)$. Clearly, $g = f \circ \tau^{\leftarrow}$ is a well defined map on Y. Then $g \in C(Y)$ since τ is a completely regular quotient map. Since $\{g_\alpha\}$ is Cauchy and converges uniformly to g on $\tau[K]$, for each compactum K in X, it converges to g with respect to the compact-open topology in $C(Y)$. Therefore $C(Y)$ is countably complete and Y is a k_r^{ω}-space. □

Notation. Let *CReg* be the category of completely regular Hausdorff spaces and continuous maps and let K_r^{ω} be the full subcategory of *CReg* whose objects are k_r^{ω}-spaces.

Theorem 4.9. *K_r^{ω} is a coreflective subcategory of CReg.*

Proof. It is clear that K_r^{ω} is closed with respect to the formation of topological sums. That K_r^{ω} is coreflective now follows from 4.8 above and [22; 9.7(f), p.728]. □

Theorem 4.10. *Let $(Y, \mathcal{T})$ be a completely regular Hausdorff space.*

(a) There is a coarsest completely regular topology $\mathcal{T}_r^\omega$ on the set Y, finer than the original topology, for which $(Y, \mathcal{T}_r^\omega)$ is a k_r^ω-space.

(b) If $f \in C(X, (Y, \mathcal{T}))$ for a k_r^ω-space X, then f is also continuous as a map from X to $(Y, \mathcal{T}_r^\omega)$.

Proof. Part (a) follows immediately from 4.9 above. See the comment in [22] that follows the the proof of 9.7(c) on p.727. Part (b) is intrinsic in the definition of coreflection. See [22; 9.7(a)]. □

5. Main Result

Recall that the definition of extension is stated in 2.1 above.

Definition 5.1. For extensions $(B_1, \mathcal{E}_1)$ and $(B_2, \mathcal{E}_2)$ of $(A, \mathcal{T})$ with extendors θ_1 and θ_2, respectively, define $(B_1, \mathcal{E}_1) \leq (B_2, \mathcal{E}_2)$ to mean that there is an injective homomorphism $h : B_1 \longrightarrow B_2$ so that $h \circ \theta_1 = \theta_2$.

Notation. Let $f \in C(X, Y)$.

(a) Then f_r^ω denotes the continuous map from $k_r^\omega X$ to $k_r^\omega Y$ which equals f as a set mapping.

(b) Also, f^υ denotes the continuous map from υX to υY which extends f. [10; 8.7(I), p.118]

Theorem 5.2. *For any completely regular Hausdorff space X, the algebra $C(\upsilon[k_r^\omega(\upsilon X)])$ is an extension of $C(X)$ which is minimal among the countably complete, bornological extensions of $C(X)$.*

Proof. Observe that $C(\upsilon[k_r(\upsilon X)])$ is a countably complete, bornological extension of $C(X)$ with extendor ψ defined by $\psi(f) = [(f^\upsilon)_r^\omega]^\upsilon$. That ψ is an open map follows the fact that the υX-compact-open topology on $C(\upsilon X) \cong C(X)$ is finer than the X-compact-open topology. Let Y be completely regular so that $C(Y)$ is a countably complete, bornological extension of $C(X)$ with extendor θ. By [10; 10.8, p.143], there is an open-closed subspace E of Y and a continuous mapping $\tau : E \longrightarrow \upsilon X$ so that $f^{\upsilon} \circ \tau$ is the restriction of $\theta(f)$ to E. Since $C(Y)$ is countably complete and bornological, the open-closed subspace E of Y is a realcompact k_r^{ω}-space. By 4.10(b), the map τ is continuous from E into $k_r^{\omega}(\upsilon X) \subset \upsilon[k_r^{\omega}(\upsilon X)]$. Define a homomorphism $h : C(\upsilon[k_r(\upsilon X)]) \longrightarrow C(Y)$ so that $h(g)$ vanishes on $Y \setminus E$ and equals $g \circ (\tau_r^{\omega})^{\upsilon}$ on E. Since θ is an open map, there exists, for each compact $K \subset X$, a compact subset H of Y for which $\tau[H \cap E] = K$; hence $\tau[E]$ contains X. Thus the image of E, under $(\tau_r^{\omega})^{\upsilon}$, is dense in its codomain. It now follows from [10; 10.3, p.141] that h is an injective homomorphism. Clearly $\theta = h \circ \psi$. Hence $C(\upsilon[k_r^{\omega}(\upsilon X)])$ is minimal among the countably complete, bornological extensions of $C(X)$. □

Corollary 5.3. *For any completely regular space X, the algebra $C(\upsilon[k_r X])$ is a countably complete, bornological extension of $C(X)$ which is 'larger' than $C(\upsilon[k_r(\upsilon X)])$.*

Proof. Note that the homomorphism θ from $C(X)$ to $C(\upsilon[k_r X])$ defined by $\theta(f) = (f_r^{\omega})^{\upsilon}$ is an extendor. The rest of 5.3 follows from 4.6(b) and 5.2. □

Added in proof.
Theorem 5.4. *The space $k_r^{\omega} X$ is realcompact for each realcompact X.*

Proof. Available at mack@ms.uky.edu □

In 5.2, this means that the desired extension of $C(X)$ is $C([k_r^{\omega}(\upsilon X)])$.

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