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REFLECTING ON COMPACT SPACES

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Abstract

We consider whether, if a topological space reflects via an elementary submodel to a generalized Cantor discontinuum it must in fact equal its reflection. The answers involve large cardinals.

Given an elementary submodel M of some $H(\theta)$ (see [JW, Chapter 24] for a careful elucidation of the implications of this) and a topological space $\langle X, \mathcal{T} \rangle \in M$, we define X_M to be $X \cap M$ with topology generated by $\mathcal{T}_M = \{U \cap M : U \in \mathcal{T} \cap M\}$. In [JT$_1$] we developed this notion; in [T$_1$] I proved that if X_M is homeomorphic to the Cantor set, then $X = X_M$. I. Juhász (personal communication) asked whether this generalized to arbitrary cardinals, i.e. if X_M is homeomorphic to a generalized Cantor discontinuum D^λ, where D is the 2-point discrete space, then does $X = X_M$? We shall show that the answer is yes for small λ, but not necessarily for very huge ones.

The following technical result is the key observation for our work on Juhász' problem. It and Corollary 2 are due independently to Lucia Junqueira, conversations with whom have been very helpful. Note that when we write "X_M", we are implicitly assuming that $X \in M$. Also note that $D^\lambda \in M$ implies $\lambda \in M$.

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Theorem 1. *Let λ be a set. (Most of the time, it will be a cardinal.) Suppose $(D^\lambda)_M$ is compact. Then $(D^\lambda)_M$ is homeomorphic to $D^{\lambda \cap M}$.*

Proof. Let $h : (D^\lambda)_M \to D^{\lambda\cap M}$ be defined by $h(f) = f|(\lambda \cap M)$. Claim h is a homeomorphism. Since $(D^\lambda)_M$ is compact and also T_2 (since D^λ is [JT$_1$]), it suffices to show h is continuous, one-one, and has dense image. Let $[p] = \{g \in D^{\lambda\cap M} : g|\mathrm{dom}(p) = p\}$, where p is a finite partial function from $\lambda \cap M$ into D. Then $h^{-1}([p]) = \{f \in D^\lambda \cap M : f|\mathrm{dom}(p) = p\}$. But this is open in $(D^\lambda)_M$. h is one-one, since if $f_1 \neq f_2$ are in $D^\lambda \cap M$, $f_1|(\lambda\cap M) \neq f_2|(\lambda \cap M)$ by elementarity. Finally, given any non-empty basic open $[p]$ in $D^{\lambda\cap M}$, since $\mathrm{dom}(p) \subseteq M$, $p \in M$, so the function f defined by

i) $f|\mathrm{dom}(p) = p$,

ii) $f|\big(\lambda - \mathrm{dom}(p)\big) = 0$,

is in $(D^\lambda)_M$, and $h(f) \in [p]$. □

Corollary 2. *If $\lambda \subseteq M$ and $(D^\lambda)_M$ is compact, then $(D^\lambda)_M = D^\lambda$.*

Proof. In this case, h is the identity function. □

Corollary 3. *Let μ be the least ordinal not included in M. Then if λ is a cardinal less than μ and $(D^\lambda)_M$ is compact, then $D^\lambda = (D^\lambda)_M$.*

The proof is immediate.

Corollary 4. *Let μ be the least ordinal not included in M. If X_M is homeomorphic to D^λ, $\lambda < \mu$, and $D^\lambda \in M$, then $X = X_M$.*

Proof. By [J], since X_M is compact, so is X and there is a continuous map from X onto X_M. Relativizing, there is a continuous map from X_M onto $(D^\lambda)_M$. Hence $(D^\lambda)_M$ is compact, so D^λ and hence $2^\lambda \subseteq M$. Therefore $\lambda^+ \subseteq M$. We now do some easy calculation of cardinal functions. See [H] for definitions and theorems. Using a straightforward argument done in detail in [T_1], we see that X has no right- or left-separated subspaces of size $\geq \lambda^+$, else X_M would. But $w(X_M) = \lambda$. Since X_M and hence [T_1] X is T_3, it follows that $|X \cup \mathcal{T}| \leq 2^\lambda$, so $X \cup \mathcal{T} \subseteq M$, so $X = X_M$. □

Theorem 5. *The first cardinal λ – if any – such that $(D^\lambda)_M$ is compact for some M but $\neq D^\lambda$ must be strongly inaccessible.*

Proof. The first cardinal – if any – for which $(D^\lambda)_M$ is compact but $\neq D^\lambda$ cannot be $\leq 2^\kappa$ for some $\kappa < \lambda$, $\kappa \in M$. By elementarity, we can omit '$\kappa \in M$'. The point is that – since D^κ is a continuous image of D^λ – $(D^\lambda)_M$ compact implies $(D^\kappa)_M$ is compact implies $D^\kappa = (D^\kappa)_M$ implies $2^\kappa \subseteq M$ implies $\lambda \subseteq M$ implies $(D^\lambda)_M = D^\lambda$. The first such cardinal can also not be singular, since $D^\lambda \in M$ implies $\lambda \in M$ implies $cf(\lambda) \in M$ implies $D^{cf(\lambda)} \in M$ (since $D^\lambda \in M$). Then, since λ is least and – as before – $(D^{cf(\lambda)})_M$ is compact, $(D^{cf(\lambda)})_M = D^{cf(\lambda)}$. Therefore $D^{cf(\lambda)}$ and a fortiori $cf(\lambda) \subseteq M$. But then there is a set S of cardinals cofinal in λ included in M. For each $\sigma \in S$, $D^\sigma \in M$ and $(D^\sigma)_M$ is compact, so $D^\sigma = (D^\sigma)_M$ so D^σ and hence $\sigma \subseteq M$. But then $\lambda \subseteq M$ and so $D^\lambda = (D^\lambda)_M$, contradiction. Thus M thinks λ is strongly inaccessible, so it is. □

Corollary 6. *Suppose X_M is homeomorphic to $D^\lambda \in M$ and λ is less than the first strongly inaccessible cardinal. Then $X = X_M$.*

Proof. As for Corollary 4 above. □

Thus if there are no strongly inaccessible cardinals, Juhasz' problem is solved. A less draconian solution is given by the following two results. $0^\#$ is a set of natural numbers, the existence of which has large cardinal strength. See [K]. $V = L$ implies $0^\#$ does not exist.

Corollary 7. *If* $0^{\#}$ *does not exist and* $|M| \geq \lambda$ *and* $(D^\lambda)_M$ *is compact, then* $(D^\lambda)_M = D^\lambda$.

Proof. This follows immediately from

Lemma 8. [KT] *If* $0^{\#}$ *does not exist and* $|M| \geq \lambda$, *then* $M \supseteq \lambda$.

Corollary 9. *If* $0^{\#}$ *does not exist and* X_M *is homeomorphic to* $D^\lambda \in M$, *then* $X = X_M$.

Proof. $|M| \geq |X_M| = 2^\lambda$, so $2^\lambda \subseteq M$, so as in the proof of Corollary 4, $X = X_M$. □

By going to very large cardinals, we *can* find a λ such that $(D^\lambda)_M$ is compact but not equal to D^λ.

Definition. A cardinal λ is η-*extendible* if there is a ζ and an elementary embedding $j : V_{\lambda+\eta} \to V_\zeta$, with critical point λ.

See [K] to find out about such cardinals and about 2-huge ones, which we shall shortly introduce. Here we shall only mention that η-extendible cardinals are weaker in consistency strength than supercompact cardinals.

Observe that for $\eta \geq 1$,

$$\begin{aligned} D^{j(\lambda)} \cap j``V_{\lambda+\eta} &= \{j(S) : j(S) \in D^{j(\lambda)} \quad \text{and} \quad S \in V_{\lambda+\eta}\} \\ &= \{j(S) : S \in D^\lambda\} \\ &= j``D^\lambda\ . \end{aligned}$$

Now if we want $D^{j(\lambda)} \in j``V_{\lambda+\eta}$, we need $\eta \geq 2$, for then $D^\lambda \in V_{\lambda+\eta}$, so $j(D^\lambda) = D^{j(\lambda)} \in j``V_{\lambda+\eta}$. We would be done if our definition of X_M used "V_θ" instead of "$H(\theta)$" since $j``V_{\lambda+\eta}$ is an elementary submodel of V_ζ. To get $H(\theta)$, we use the fact that for inaccessible θ, $V_\theta = H(\theta)$, and work with a larger cardinal.

Definition. λ is 2-*huge* if there is an elementary embedding $j : V \to N$, an inner model, with critical point λ such that ${}^{j(j(\lambda))}N \subseteq N$.

2-hugeness has considerably more consistency strength than supercompactness and assures us that $j``V_{j(\lambda)} \in N$, as is $j``D^\lambda$. $j``V_{j(\lambda)}$ is an elementary submodel of $V_{j(j(\lambda))} = H(j(j(\lambda)))$ (since $j(j(\lambda))$ is inaccessible by elementarity). As before, $D^{j(\lambda)} \in j``V_{j(\lambda)}$ and $D^{j(\lambda)} \cap j``V_{j(\lambda)} = j``D^\lambda$, which is compact T_2. $(D^{j(\lambda)})_{j``V_{j(\lambda)}}$ is also a $T_2$ (since $D^{j(\lambda)}$ is and $T_2$ "goes down" [$\mathrm{JT}_1$]) topology on $D^{j(\lambda)} \cap j``V_{j(\lambda)}$ that is weaker than the subspace topology and hence the two topologies are equal by compactness. Both $j``V_{j(\lambda)}$ and $V_{j(j(\lambda))}$ are in N; the proof that the former is an elementary submodel of the latter can be carried out in N. Thus, N thinks there is an elementary submodel M of $H(j(j(\lambda)))$ such that $(D^{j(\lambda)})_M$ is compact T_2 but $\neq D^{j(\lambda)}$ (since $j(\lambda) > \lambda$). By elementarity, in V there is an elementary submodel M' of $H_{j(\lambda)}$ such that $(D^\lambda)_{M'}$ is compact T_2 but $\neq D^\lambda$. We have proved

Theorem 8. *If λ is 2-huge, then there is an elementary submodel M such that $(D^\lambda)_M$ is compact but $\neq D^\lambda$.*

There are several problems that remain:

What is the consistency strength of the existence of a λ such that $(D^\lambda)_M$ is compact but $\neq D^\lambda$?

Could such a λ be a successor cardinal?

Must the first such λ be "larger" than merely 'strongly inaccessible'?

After this paper was completed, Lucia Junqueira [JT_2] proved that the condition that $|M| \geq \lambda$ can be removed from Corollary 7. It follows that the existence of a compact $(D^\lambda)_M \neq D^\lambda$ has consistency strength at least equal to the existence of $0^\#$. In [JT_2] we discuss in general when X_M compact implies $X_M = X$. In [T_2] we investigate the particular case of when X_M is a dyadic compactum; the results obtained generalize those in this paper. Of course there are simple examples in ZFC of X's which are not equal to X_M, even if the latter is compact. For example, let X be the one-point compactification of an uncountable discrete space and let M be countable.

In [Ku], K. Kunen considerably sharpened the large cardinal bounds of Theorems 5 and 8.

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