Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Summer 2000
Pages 351–360

ON ALMOST HOMEOMORPHIC SPACES

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Abstract

The purpose of this paper is to present an F-space X with countably many isolated points with the property that every almost homeomorphism on X is a homeomorphism. We also make a link between properties of the fundamental group of a space and the (non-)existence of a non-trivial homeomorphism.

Introduction

In [1] Friedler and Kitover defined a continuous surjection $\varphi : K \to K$ to be an almost homeomorphism if there is a finite subset $F \subseteq K$ such that the restriction of φ to $K \backslash F$ is a bijection onto $K \backslash \varphi(F)$. This definition was motivated by the study of the Fredholm spectrum of the composition operators in spaces of continuous functions. (See [1, Appendix] for a detailed explanation.) We define AH to be the class of all spaces on which every almost homeomorphism is a homeomorphism. In [1] it was shown that: every compact F-space with at most finitely many P-points is in AH; $\beta\mathbb{N}\backslash\mathbb{N}$ is not in AH; a compact basically disconnected space with no isolated points might not be in AH; every compact n-dimensional manifold is in AH; every simply connected, compact space is in AH; the product of two locally connected spaces with no isolated points is in AH. In section 1 of this paper we examine almost homeomorphisms on spaces with infinitely many isolated points and observe that a

Mathematics Subject Classification: 54G05.
Key words: Almost homeomorphism, F-space, Fundamental group.

basically disconnected, compact space with infinitely many isolated spaces is not in AH. We present an example of a compact F space with infinitely many isolated points which is in AH. In section 2 we consider the open question of characterizing those subsets of the plane which are in AH. We show that, for an arcwise connected space K, if the fundamental group of K is finite, Abelian, or finitely generated then $K \in AH$. However, we construct two arcwise connected subsets K and K' of the plane which have isomorphic fundamental groups, but $K \in AH$ while $K' \notin AH$.

All spaces are assumed to be Hausdorff. We use the notation and terminology of [1]. In particular, $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{BN}$ and $E(X)$ is the absolute of the space X.

1. Infinitely Many Isolated Points

A space is an *F-space* if the closures of any two disjoint co-zero sets are disjoint and is *basically disconnected* if the closure of any co-zero set is open (so that a basically disconnected space is an F-space.)

We show below that a compact, basically disconnected space with infinitely many isolated points cannot be in AH. In [1] it was shown that a compact, basically disconnected space with at most finitely many isolated points and no P-points must be in AH while a compact, basically disconnected space with at most finitely isolated points might or might not be in AH.

Lemma 1. *Let K be a basically disconnected compact space with infinitely many isolated points. Then $K \notin AH$.*

Proof. Let $k_1, k_2, ...$ be infinitely many pairwise distinct isolated points in K. The set $\{k_n : n \in \mathbb{N}\}$ is a co-zeroset and is C^*-embedded in its closure $K_0 = \mathrm{cl}\{k_n : n \in \mathbb{N}\}$. So K contains K_0 as a clopen copy of βN. Finally because $\beta N \notin AK$ (for example we can consider the mapping φ defined as $\varphi(1) = 1, \varphi(n+1) = n$ for $n > 1$ and its extension to βN) our proposition is proved. $\square$

A compact F-space with finitely many P-points must be in AH [1, Corollary 1.3]. But if an F-space has infinitely many isolated points then, if the space is basically disconnected, it cannot be in AH by 1.1. We give an example below of a compact F-space (which is not bascially disconnected) with infinitely many isolated points which is AH. Unfortunately we need CH for the example. The problem whether the continuum hypothesis is essential for the existence of such an example we did not examine. The following four lemmas will be needed in our construction. The proofs of Lemma 3 below is straightforward and so omitted.

Lemma 2. [2] *If X is extremally disconnected and has non-measurable cardinal, then X contains no non-isolated P-points.*

Lemma 3. *If $A \subset X$ is a closed P-set of the normal, extremally disconnected space X, then $\mathrm{cl}_{\beta X} A$ is a P-set of βX.*

Lemma 4. [3] *If X, Y are compact F-spaces, A is a closed P-set of X, and $f : A \to Y$ is a continuous map, then the adjunction space $X \cup_f Y$ is an F-space.*

Lemma 5. *Assume $A \subseteq B \subseteq X$ and B is a closed P-set of the space X and A is a closed P-set of B. If X is compact and zero-dimensional then A is a P-set of X.*

Proof. Observe first that the trace of a clopen subset of X on B is a P-set in X. Next, if U_n is open in X with $A \subset U_n$, $n \in \mathbb{N}$, then there is a relatively clopen subset $C \subset B$ with $A \subset C \subset \cap(U_n \cap X)$. As C is a P-set it follows that $A \subset C \subset \mathrm{int} \cap \mathrm{U_n}$. $\square$

Lemma 6. *The weight of $E(2^m)$, the absolute of 2^m, is m^{ω_0}.*

Proof. It is clear that $w(E(2^m)) \geq w(2^m) = m$. But $E(2^m))$ is extremally discconnected, so the weight is an ω_0 power, so $w(E(2^m)) \geq m^{\omega_0}$. On the other hand, the space 2^m is c.c.c., so there cannot be more than m^{ω_0} regular open sets.

(Take a base of 2^m and observe that each regular open set U contains countably many pairwise disjoint basic sets the union of which is dense in U.) Therefore, $E(2^m)$ has at most m^{ω_0} clopen sets. $\square$

Example 1. (CH). A compact F-space K with infinitely many isolated points and $K \in AH$.

Find a sequence $\{\Delta_\alpha : \alpha < \omega_1\}$ of non-measurable cardinals with the property:

$$\omega_0 < \Delta_\beta = \Delta_\beta^{\omega_0} < \Delta_\alpha \text{ for all } \beta < \alpha < \omega_1$$

For each $\alpha < \omega_1$ let Y_α be a compact extremally disconnected space with no P-points and with the property that "every non-empty open subset of Y_α has weight Δ_α." Let $E_\alpha \subseteq Y_\alpha$ be a closed nowhere dense P-set of Y_α. Any such pair (Y_α, E_α) will suffice for our construction. (In Remark 1 below we show that such a pair exists.) Note that as a compact subset of Y_α, E_α is C^*-embedded in Y_α.

Let $Y = \bigoplus\{Y_\alpha : \alpha < \omega_1\}$ and let $E = \bigoplus\{E_\alpha : \alpha < \omega_1\}$. Then *(i)* Y is extremally disconnected, normal, has no isolated points, and is of non-measurable cardinality, so that by Lemma 2 , βY contains no P-points. *(ii)* E is closed in Y, so is C^*-embedded. Thus, $\mathrm{cl}_{\beta Y} E = \beta E$. (*iii*) E is a closed nowhere dense $P-$set in Y, so that that by Lemma 3 βE is a closed nowhere dense P-set in βY.

Under the continuum hypothesis the set of all P-points of $\mathbb{N}^*$ is dense and of cardinality ω_1. Write $\{p_\alpha : \alpha < \omega_1\}$ for this set in $\mathbb{N}^*$.

The map $g : E \to \mathbb{N}^*$ defined by $g(E_\alpha) = p_\alpha$ is a continuous map and can be extended continuously to a map $\beta g : \beta E \to \mathbb{N}^*$. As E is C^*-embedded in Y, we see that the map βg is defined on the closed subspace $\mathrm{cl}_{\beta Y} E$.

Consider first the adjunction space

$$T = \beta Y \cup_{\beta g} \mathbb{N}^*.$$

We list some properties of the space T.

T1: By Lemma 4 the space T is an F–space. It can be seen as the space βY in which the P–set βE is replaced by the P–set $\mathbb{N}^*$.

T2: By Lemma 5 all the P-points of $\mathbb{N}^*$ are still P-points of T.

T3: The only P–points of T are the P–points of $\mathbb{N}^*$.

T4: To each $p_\alpha \in \mathbb{N}^*$ an open subspace $Y'_\alpha = Y_\alpha - E_\alpha$ is assigned. Moreover, every open subset of Y'_α has weight Δ_α and $p_\alpha \in \operatorname{cl} Y'_\alpha$.

T4: The P-point p_α is the only P-point of $\mathbb{N}^*$ which is in the closure of an open subset of T with the property that every non-empty open subset has weight Δ_α.

Proof. Let $p \neq p_\alpha$ be a $P-$point in $\mathbb{N}^*$ and consider the set $A = \{p\} \cup \operatorname{cl}\{p_\beta : \beta < \alpha\}$. Note that $p_\alpha \notin A$ so that $A \cap \operatorname{cl} Y'_\alpha = \emptyset$. Now let V be an open set with $A \subseteq V \subseteq \operatorname{cl} V \subseteq T - \operatorname{cl} Y'_\alpha$. Every open set with $p \in \operatorname{cl} U$ will intersect V, so it suffices to show that every open subset $W$of V contains a subspace W' of weight not equal to Δ_α. Let W be an open subset of V. If W intersects $\bigcup\limits_{\gamma>\alpha} (Y_\gamma - E_\gamma)$, then clearly the weight of W is strictly larger than Δ_α by the choice of Y_γ, so take $W' = W$. If not, then W intersects $\bigcup\limits_{\beta<\alpha} (Y_\beta - E_\beta)$. Since $\Delta_\alpha = \Delta_\alpha^{\omega_0}$, Δ_α is not a countable limit ordinal. Thus, the weight of $\bigcup\limits_{\beta<\alpha} (Y_\beta - E_\beta)$ is $\sup\{\Delta_\beta : \beta < \alpha\}$ which is strictly less than Δ_α. So, W contains the open set $W' = W \cap \bigcup\limits_{\beta<\alpha} (Y_\beta - E_\beta)$, which has weight strictly less than Δ_α. $\square$

T5: If $h : T \to T$ is a homeomorphism with $h(\mathbb{N}^*) = \mathbb{N}^*$, then $h(p) = p$ for every $p \in \mathbb{N}^*$.

Proof. T4 gives a pure topological difference between P-points p_α, so $h(p_\alpha) = p_\alpha$ for all α. Since $\{p_\alpha : \alpha < \omega_1\}$ is dense in $\mathbb{N}^*$, the result follows. $\square$

Next consider again the map $\beta g : \beta E \to \mathbb{N}^*$ and the inclusion map $i : \mathbb{N}^* \subset \beta\mathbb{N}$. Let $h = i \circ \beta g : \beta E \to \beta\mathbb{N}$.

The next space we look at is the adjunction space

$$X = \beta Y \cup_h \beta\mathbb{N}.$$

We list some properties of the space X.

X1: X is an F–space.

X2: X is T together with countably many isolated points.

X3: X can be seen as the space βY in which the P–set βE is replaced by the P–set $\beta\mathbb{N}$.

X4: The P-points of ω^* are no longer P–points in $\beta\mathbb{N}$ and so there are no P–points in X. The only P–points of X are the isolated points.

X5: The subset $\mathbb{N}^*$ can be rediscovered from the the topological structure of X as the points which are in the closure of the isolated points.

We claim that X is the required F-space.

Proof of the claim. Assume $f : X \to X$ is an almost homeomorphism which is not a homeomorphism. Then there is a finite subset $F \subseteq X$ such that the restriction of f to $X \backslash F$ is a bijection onto $X \backslash f(F)$ and each point of F is identified with at least one other point of F. If there were two or more points in $F \cap X \backslash \mathbb{N}$ which mapped to the same point, then, since $X \backslash \mathbb{N}$ has no P-points, it would follow from Theorem 1.1 of [1] that X would not be an F-space. Furthermore, if $x \in T$, then $f(x) \in T$ for otherwise $f^{-1}f(x)$ would be a finite, clopen subset of $X \backslash \mathbb{N}$ – which is impossible. It follows that $f(T) \subseteq T$, $f : T \to T$ is injective, and $F \cap \mathbb{N} \neq \emptyset$. But then $f(T) = T$, for otherwise the set $T \backslash f(T)$, which is open in T and hence uncountable, could be covered by the images of the countable set $\mathbb{N}$ (since $f : X \to X$ is surjective.) It follows that the restriction of f to T is a homeomorphism of T onto T.

Since f is an almost homeomorphism, $\operatorname{card} \mathrm{f}^{-1}(\mathrm{t}) > 1$ for only finitely many $t \in X$. But then we have $f(t) \in \mathbb{N}$ for all $t \in N$ except for finitely many points. From this it follows that $f(\mathbb{N}^*) = \mathbb{N}^*$, so that by T5, f must be the identity on $\mathbb{N}^*$.

Since $F \cap N \neq \emptyset$ and $f : X \to X$ is surjective, we can find an infinite subset $A \subset \mathbb{N}$ with $A \cap f(A) = \emptyset$ and $f(A) \subset \mathbb{N}$. But then $f(\operatorname{cl} A) \subset \operatorname{cl} f(A)$ and since $\operatorname{cl}(A) \cap \operatorname{cl} f(A) = \emptyset$, we see that f is not the identity on $\mathbb{N}^*$.

We have our contradiction.

Remark 1. We show here that the pair (Y_α, E_α) exists.

First, consider the Cantor space 2^{Δ_α}. Note that the weight of every non-empty open subset of 2^{Δ_α} is Δ_α (since every open subset contains a clopen copy of the space 2^{Δ_α}) and also note that 2^{Δ_α} contains no $P-$sets since 2^{Δ_α} is c.c.c.. Now consider the compact ordinal space $\omega_1 + 1$. Note that the endpoint ω_1 of $\omega_1 + 1$ is a P-point. In the product space $(\omega_1 + 1) \times 2^{\Delta_\alpha}$, the weight of every non-empty open subset is Δ_α. Finally, define Y_α to be the absolute of $(\omega_1 + 1) \times 2^{\Delta_\alpha}$ and let E_α be the preimage under the natural map $\pi_\alpha : E((\omega_1 + 1) \times 2^{\Delta_\alpha}) \to (\omega_1 + 1) \times 2^{\Delta_\alpha}$ of the set $\{\omega_1\} \times 2^{\Delta_\alpha}$. To verify that the weight of Y_α is Δ_α, observe that the weight of Y_α is the weight of $E(\ 2^{\Delta_\alpha})$, since Y_α is the Stone-Cech compactification of ω_1 disjoint copies of $E(2^{\Delta_\alpha})$. But the weight of the absolute of 2^κ is κ^{ω_0} by Lemma 6.

2. The Fundamental Group and Almost Homeomorphisms

The main open problem concerning almost homeomorphisms is to characterize those subsets of the plane that are in AH. Theorem 10 below shows that the fundamental group is useful in distinguishing subsets of the plane which are in AH, yet Example 2 demonstrates that the fundamental group will not yield a complete characterization.

We need the following lemmas. Lemmas 7 and 9 are well-known and Lemma 8 follows easily.

Lemma 7. *Let $A \subset X$ be a retract of X and let $a_0 \in A$. If $r : X \to A$ is a retraction then the map $r^* : \Pi_1(X, a_0) \to \Pi_1(A, a_0)$ between the fundamental groups is surjective.*

Lemma 8. *Let X be an arcwise connected space. Then for any finite collection of points $x_0, \cdots x_\ell$ there exists a tree-stucture in X (i.e. a graph without loops) containing these points.*

Note that these trees are absolute retracts for the class of compact spaces (even for the class of normal spaces) and that the fundamental group of a tree is trivial.

Lemma 9. *Let G be a graph with k loops. Then the fundamental group $\Pi_1(G)$ of G is the free group on k-generators. In particular, if $k > 1$ then $\Pi_1(G)$ is not abelian.*

Theorem 10. *Let X be a compact arcwise connected space.*

(1) If the fundamental group $\Pi_1(X)$ is finite then $X \in AH$.

(2) If the fundamental group of X is Abelian, then $X \in AH$.

(3) If the fundamental group of X is finitely generated, then $X \in AH$.

We assume first that the more general situation that $f : X \to Y$ is a finite-to-one surjective map, almost everywhere f is injective, except for the points $y_1, \cdots y_k$ in Y. By this we mean that $\mathrm{card}\, \mathrm{f}^{-1}(\mathrm{y}) = 1$, for all $y \in Y$, except for the points y_i. But we do assume that $f^{-1}(y_i)$ is finite. For such a situation we have the following construction.

Construction of the map $f : X \to Y$.

Write $f^{-1}(y_i) = \{x_{i,0}, \cdots, x_{i,\ell_i}\}$. So, Y is the space obtained from X by identifying $\{x_{i,0}, \cdots, x_{i,\ell_i}\}$ to one point, for $i = 1, \cdots k$. As X is arcwise connected we can find in X a tree T

containing the finite set $\bigcup_{i=0..k}\{x_{i,0},\cdots,x_{i,\ell_i}\}$. Let $r : X \to T$ be a retraction. The map f induces a retraction $r_Y : Y \to f(T)$. But $f(T)$ is a graph with at least k loops in it (In fact, $\ell_1 + \cdots \ell_k$ loops). Choose $a_0 \in f(T)$. We conclude that there exists a surjective group homomorphism ${r_Y}^* : \Pi_1(Y, a_0) \to \Pi_1(f(T), a_0)$ between the fundamental groups of Y and $f(T)$. (In particular we observe that the fundamental group of Y is infinite.)

End construction.

Now we can prove the results.

Let us assume now that Y is homeomorphic to X, so that $f : X \to Y$ can be considered to be an almost homeomorphism $f : X \to X$ which is not a homeomorphism.

(1) The first claim follows directly from the previous observation that the fundamental group of X is infinite.

(2) The second claim follows from the observation that the map $f^2 : X \to X$ is also an almost homeomorphism and the above construction applied to $g = f^2$ gives a tree T' containing at least three points which are identified under g and so $g(T')$ is a non–Abelian group. But then $\Pi_1(X)$ cannot be Abelian, as there do not exist surjective homomorphisms from Abelian to non–Abelian groups.

(3) Finally the third claim. Let us assume that $\Pi_1(X)$ has N generators. Take m large enough so that the map $f^m : X \to X$ has the property that $\ell_1 + \cdots \ell_k > N$. But if this is the situation then our construction induces a surjective map from $\Pi_1(X)$- a group with N generators- onto $\Pi_1(graph)$ the free group on $> N$-generators. But this group cannot be generated by less than N generators.

Corollary 11. [1] *If X is a compact, arcwise connected, simply connected space, then $X \in AH$*

The following example is due to Friedler and Kitover.

Example 2. Two subsets, K and K', of the plane with isomorphic fundamental groups, but $K \in AH$ and $K' \notin AH$.

Let K be the following compact subset of the plane R^2 . K consists of:

1. The union of the circles centered at $(\frac{n}{n+1}, 0)$ with radii $r_n = \frac{1}{2(n+1)(n+2)}$, $n = 0, 1....$

2. The union of the circles centered at $(-\frac{n}{n+1} \pm \frac{1}{6(n+1)(n+2)}, 0)$ with radii $\frac{1}{6(n+1)(n+2)}$, $n = 1, 2....$

3. All the segments of the x axis between the named circles.

4. Points $(1, 0)$ and $(-1, 0)$.

Let $K^{'}$ be the union of K and the vertical segment between the points $(-1/2, -1)$ and $(-1/2, 1)$. The compact sets K and $K^{'}$ are path connected and their fundamental groups are clearly isomorphic. But $K \notin AH$ (identifying the points $(0, -1/4)$ and $(0, 1/4)$ in K we obtain a space homeomorphic to K) and it is easy to see that $K^{'} \in AH$.

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