Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Summer 2000

Pages 435–455

ON LINEAR COFINAL COMPLETENESS

Bruce S. Burdick*

Abstract

We investigate linear versions of some known completeness properties. We show that linear cofinal completeness implies supercompleteness, and that in metric uniform spaces (or, more generally, in uniform spaces with a linearly ordered basis) linear cofinal completeness is equivalent to cofinal completeness. Examples are given to show that completeness, supercompleteness, and cofinal completeness are all distinct from their linear versions.

0. Introduction

In 1958, Corson [Co] considered spaces in which weakly Cauchy filters clustered. Howes [H1] called this property "cofinal completeness" in 1971. (Császár [Cs] called this "ultracompleteness"

* The author wishes to thank Norm Howes for numerous conversations and a great deal of correspondence on this topic in the late 80's and early 90's. In particular, Howes persistently directed the author's attention to the linear versions of cofinal completeness and supercompleteness, and this persistence made this paper possible. After reading an earlier version of this paper Howes pointed out that there was an unnecessary hypothesis in our Lemma 1.1 and thus paved the way for the more general Theorem 1.1. Some of these results have been announced without proof in Howes's paper [H4] and his book [H5].

We also wish to thank the referee for considerably strengthening this paper by contributing new results to Section 2.

Mathematics Subject Classification: 54A20, 54B20, 54D20, 54D45, 54E15, 54E35, 54E50.

Key words: Locally fine, complete, supercomplete, cofinally complete, uniformly paracompact, locally compact, metrizable, cofinally Čech complete.

in 1975.) Meanwhile, Isbell [I1] in 1962 characterized spaces $(X, \mathcal{U})$ for which the hyperspace $(2^X, 2^{\mathcal{U}})$ is complete and called such spaces "supercomplete." In the last paragraph of that paper he pointed out that he had proved along the way that cofinal completeness implies supercompleteness and he mentions that he and Corson knew in 1958 of a space that is supercomplete but not cofinally complete.

There is in topology a long-standing but very low-profile tradition of studying the extent to which topological properties can be characterized using well-ordered nets. Aleksandrov and Uryson, for instance, showed that compactness can be characterized in terms of well-ordered chains of closed sets (see the discussion on pages 17–21 of [AU]), and from this it follows that a space is compact iff each well-ordered net in the space clusters. The notions of closure and continuity can also be characterized in terms of the clustering of linearly ordered nets (see [H2]) and several researchers have studied spaces for which the convergence of linearly ordered nets suffices to determine closure (see [Bo] and [Ny] for results and further references). In 1980 Howes [H2] looked at a linear version of cofinal completeness, and found that some of the consequences given for cofinal completeness in [H1] go through for this linear cofinal completeness. This program was continued in [H4].

While it is interesting to see how far these linearly ordered nets will go in characterizing topological and uniform properties, our Examples 3.1 and 3.2 below show that completeness, supercompleteness, and cofinal completeness are all distinct from their linear versions. However there is an encouraging note in our Theorem 1.1, where we show that linear cofinal completeness implies supercompleteness, and there is our Theorem 2.1, which states that cofinal completeness and linear cofinal completeness are equivalent for pseudometrizable uniform spaces.

1. Properties Related to Completeness

Definition 1.1. In a uniform space $(X,\mathcal{U})$ a filter $\mathcal{F}$ is *weakly Cauchy* if for every $U \in \mathcal{U}$ there is a U-small set $S \subseteq X$ which has non-empty intersection with every member of $\mathcal{F}$. A net $\xi : D \to X$ is *cofinally Cauchy* if for every $U \in \mathcal{U}$ there is a cofinal set $C \subseteq D$ such that $\xi[C]$ is U-small. A space is *cofinally complete* if every cofinally Cauchy net clusters, or, equivalently, if every weakly Cauchy filter clusters. A filter is *stable* if for every $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that for all $F' \in \mathcal{F}$ we have $F \subseteq U[F']$. A net $\xi : D \to X$ is *almost Cauchy* if for any $U \in \mathcal{U}$ there is a $d \in D$ and a set $\mathcal{C}$ of cofinal subsets of D such that for each $C \in \mathcal{C}$, $\xi[C]$ is U-small, and for each $d' \in D$, if $d' \geq d$ then $d' \in \cup\mathcal{C}$. A space is *supercomplete* if each almost Cauchy net clusters, or, equivalently, if every stable filter clusters[1]. We will say that a space is *linearly cofinally complete (linearly supercomplete, linearly complete)* if for any ordinal α any cofinally Cauchy (almost Cauchy, Cauchy) net $\xi : \alpha \to X$ clusters[2].

Definition 1.2. A collection $\mathcal{S}$ of subsets of a uniform space $(X,\mathcal{U})$ is *uniformly locally finite* if there is a $U \in \mathcal{U}$ such that for any $x \in X$ we have $U[x]$ intersects only finitely many members of $\mathcal{S}$. We will say (see Rice [Ri]) a space is *uniformly paracompact* if every open cover has a uniformly locally finite refinement.

Proposition 1.1. *For a uniform space $(X,\mathcal{U})$ the following are equivalent:*

(1) $(X,\mathcal{U})$ *is cofinally complete.*

[1] Originally, Isbell called a space supercomplete if it had a complete hyperspace [I1] [I2]. That condition is equivalent to the definition given here. This author stated this in [Bu] before discovering that Isbell mentioned this in the last paragraph of [I1]. We note that Császár gave this as an open problem in 1975 [Cs].

[2] Howes was probably the first person to consider the linear cofinal completeness property in the paper [H2]

(2) *Any open cover $\mathcal{O}$ of X which is closed under finite unions is uniform.*

(3) *X is paracompact and all locally finite collections in X are uniformly locally finite.*

(4) *$(X,\mathcal{U})$ is uniformly paracompact.*

Corollary 1.1. *A space is cofinally complete if (a) it is paracompact and fine, or (b) it is uniformly locally compact.*

Both of the results in Corollary 1.1 are implicit in Corson's paper [Co]. Proposition 1.1 is a combination of several results in [Fr] (the corollary to Theorem 1.4, and Proposition 1.8) although it may be older. It is interesting that Rice [Ri] in 1977 proved the equivalence of properties (4) and (2) while in 1978 Fletcher and Lindgren [FL] proved (for quasi-uniform spaces) the equivalence of properties (1) and (2). It apparently was Smith [Sm] in 1978 who, in the course of reviewing the paper by Rice, first noticed the connection between these two papers.

Propostion 1.1 has a proof that leads in a natural way from property (1) through properties (2), (3), and (4), and then back to (1). Therefore one is tempted to ask if linear cofinal completeness has a similar characterization with a similar proof. This leads to our next proposition, but a few definitions are needed first.

Definition 1.3. For an ordinal γ an indexed collection $\{S_\alpha\}_{\alpha<\gamma}$ of subsets of a space $(X,\mathcal{U})$ is *locally bounded* if for each point $x \in X$ there is a neighborhood O of x and some $\beta < \gamma$ such that for all $\alpha < \gamma$ we have $O \cap S_\alpha \neq \phi$ implies $\alpha \leq \beta$. The collection is *uniformly locally bounded* if there is a $U \in \mathcal{U}$ such that for each $x \in X$ there is some $\beta < \gamma$ such that for all $\alpha < \gamma$ we have $U[x] \cap S_\alpha \neq \phi$ implies $\alpha \leq \beta$. A *shrinking* of an indexed cover $\{O_\alpha\}_{\alpha<\gamma}$ is a cover $\{\hat{O}_\alpha\}_{\alpha<\gamma}$ such that for each $\alpha < \gamma$, $\hat{O}_\alpha \subseteq O_\alpha$. We use cA for the closure of a set A.

Proposition 1.2. *For a uniform space* $(X,\mathcal{U})$ *the following are equivalent:*

(1) $(X,\mathcal{U})$ *is linearly cofinally complete.*

(2) *Any open cover* $\mathcal{O}$ *of* X *which is well-ordered by inclusion is uniform.*

(3) *Any locally bounded indexed collection in* X *is uniformly locally bounded.*

(4) *Any indexed open cover* $\{O_\alpha\}_{\alpha<\gamma}$ *has a uniformly locally bounded indexed open shrinking* $\{\hat{O}_\alpha\}_{\alpha<\gamma}$.

Proof. (1) implies (2): We refer the reader to Howes [H2, proof of Part 1 of Theorem 1] for a proof of the following: If $(X,\mathcal{U})$ is linearly cofinally complete then any weakly Cauchy filter with a base that is well-ordered by reverse inclusion has a cluster point. Let $\mathcal{C}$ be an open cover of X which is well-ordered by inclusion. Suppose $\mathcal{C}$ is not uniform. Then, in particular, $O \neq X$ for each $O \in \mathcal{C}$. Let $\mathcal{F}$ be the filter generated by the complements of members of $\mathcal{C}$. $\mathcal{F}$ has no cluster points so by the statement above $\mathcal{F}$ is not weakly Cauchy. So there is a $U \in \mathcal{U}$ such that no U-small set intersects every member of $\mathcal{F}$. Thus every U-small set is contained in some member of $\mathcal{C}$. So $\mathcal{C}$ is uniform, a contradiction.

(2) implies (3): Let $\{S_\alpha\}_{\alpha<\gamma}$ be a locally bounded indexed collection of subsets of X. For each $\alpha < \gamma$ let O_α be the complement of the closure of $\cup_{\beta>\alpha} S_\beta$. $\{O_\alpha\}_{\alpha<\gamma}$ is an open cover of X well-ordered by inclusion, so it is uniform by (2). Therefore $\{S_\alpha\}_{\alpha<\gamma}$ is uniformly locally bounded.

(3) implies (4): We first show that property (3) implies paracompactness. We see that property (3) implies property (1) since if $\xi : \gamma \to X$ is a net with no cluster points then $\{\xi(\alpha)\}_{\alpha<\gamma}$ is a locally bounded indexed family. Since linear cofinal completeness implies paracompactness by [H2, Theorem 1], we see that property implies paracompactness.

Let $\{O_\alpha\}_{\alpha<\gamma}$ be an open cover. Then by paracompactness there is a locally finite indexed open shrinking $\{\hat{O}_\alpha\}_{\alpha<\gamma}$. Since locally finite implies locally bounded, we see by property (3) that $\{\hat{O}_\alpha\}_{\alpha<\gamma}$ is uniformly locally bounded.

(4) implies (1): Suppose $\xi : \gamma \to X$ is a net with no cluster points. For each $\alpha < \gamma$ let $O_\alpha = X - c(\{\xi(\beta) | \ \beta > \alpha\})$. Then $\{O_\alpha\}_{\alpha<\gamma}$ is an indexed open cover. Let $\{\hat{O}_\alpha\}_{\alpha<\gamma}$ be a uniformly locally bounded indexed open shrinking. Then there is a $U \in \mathcal{U}$ such that for any $x \in X$, $U[x]$ does not contain the ξ-image of a cofinal subset of γ. □

A version of this Proposition 1.2 was announced without proof as Theorem 7 in [H4].

Definition 1.4. A cover $\mathcal{C}$ of X is called *uniformly locally uniform* if there is a uniform cover $\mathcal{C}'$ such that on each $S \in \mathcal{C}'$, the trace of $\mathcal{C}$ on S is uniform. A space $(X,\mathcal{U})$ is called *locally fine* if every uniformly locally uniform cover is uniform. Given a space $(X,\mathcal{U})$ the uniformity $\lambda\mathcal{U}$ is the coarsest one finer than $\mathcal{U}$ such that $(X, \lambda\mathcal{U})$ is locally fine. λ is constructed by transfinite recursion (see [GI] or [I2]).

The next theorem of Isbell's [I1] is our chief reason for interest in the functor λ.

Isbell's Theorem. *A space $(X,\mathcal{U})$ is supercomplete if and only if it is paracompact and $\lambda\mathcal{U}$ is fine.*

This author originally proved the next Lemma with an unnecessary hypothesis included and is grateful to Norm Howes for pointing this out and making possible the more general statement here. The evolution of this result is told in more detail in [H4], and the last two results of Section 1 here are announced there without proof.

Lemma 1.1. [*with* N. Howes] *For locally fine spaces $(X,\mathcal{U})$ the following are equivalent:*

(1) *$(X,\mathcal{U})$ is paracompact and fine.*

(2) *$(X,\mathcal{U})$ is cofinally complete.*

(3) $(X,\mathcal{U})$ *is linearly cofinally complete.*

(4) $(X,\mathcal{U})$ *is supercomplete.*

Proof. We have already remarked in Corollary 1.1 that property (1) implies property (2). Property (2) clearly implies properties (3) and (4). Isbell's Theorem gives us that property (4) implies property (1) for locally fine spaces. To finish the proof it suffices to show that for locally fine spaces property (3) implies property (2). So suppose that $(X,\mathcal{U})$ is linearly cofinally complete. We will show by contradiction that $(X,\mathcal{U})$ is cofinally complete.

If $(X,\mathcal{U})$ is not cofinally complete then by Proposition 1.1 there is an open cover which is closed under finite unions and which is not uniform. Then there is such a cover, $\{O_\alpha\}_{\alpha<\gamma}$, of least cardinality, where γ is an initial ordinal. γ must be infinite since otherwise one of the O_α's would be equal to X. By Proposition 1.2, there is a uniformly locally bounded indexed open shrinking $\{\hat{O}_\alpha\}_{\alpha<\gamma}$ of $\{O_\alpha\}_{\alpha<\gamma}$. So there is a $U \in \mathcal{U}$ such that for each $x \in X$ there is a $\beta_x < \gamma$ so that $U[x]$ doesn't intersect any $\hat{O}_\alpha$ with $\beta_x \le \alpha < \gamma$.

For each $x \in X$ let $\mathcal{C}_x$ be the open cover of X formed by taking all finite unions of the members of $\{\hat{O}_\alpha\}_{\alpha<\beta_x} \cup \{\cup_{\alpha\ge\beta_x}\hat{O}_\alpha\}$. Then $\mathcal{C}_x$ is uniform, since its cardinality is smaller than γ. So the collection of finite unions of members of $\{\hat{O}_\alpha\}_{\alpha<\beta_x}$ is a uniform cover of $U[x]$, and hence $\{O_\alpha\}_{\alpha<\gamma}$ is a uniform cover of $U[x]$. We have shown that $\{O_\alpha\}_{\alpha<\gamma}$ is uniformly locally uniform, and so by local fineness it must be uniform, a contradiction. □

The property of linear supercompleteness cannot be included in Lemma 1.1 because of Example 3.2 below.

Theorem 1.1. *For all spaces* $(X,\mathcal{U})$*, linear cofinal completeness implies supercompleteness.*

Proof. Suppose $(X,\mathcal{U})$ is linearly cofinally complete. Then $(X,\lambda\mathcal{U})$ is, too. By Lemma 1.1, $(X,\lambda\mathcal{U})$ is paracompact and fine. By Isbell's Theorem $(X,\mathcal{U})$ is supercomplete. □

This Theorem 1.1 then makes Lemma 1.1 an immediate consequence of Isbell's Theorem and Corollary 1.1.

2. A Theorem on Metrizable Uniform Spaces

The proof of Hahn's Theorem[3], that complete metric spaces are supercomplete, can actually be used to establish the following.

Extended Hahn's Theorem. *Let $(X,\mathcal{U})$ be a pseudometrizable uniform space. Then the following are equivalent:*

(1) $(X,\mathcal{U})$ *is supercomplete.*

(2) $(X,\mathcal{U})$ *is linearly supercomplete.*

(3) $(X,\mathcal{U})$ *is complete.*

(4) $(X,\mathcal{U})$ *is linearly complete.*

It is also well known that complete metric spaces need not be cofinally complete (see Example 3.3). We will show below, using the next two lemmas, that in pseudometrizable uniform spaces linear cofinal completeness is equivalent to cofinal completeness.

The next lemma is a modification of Theorem 5 in [Ri] and has almost the same proof.

Lemma 2.1. *In a linearly cofinally complete pseudometric uniform space $(X,\mathcal{U})$ the set of points which do not have a compact neighborhood is compact.*

Proof. Let F be the collection of points which have no compact neighborhood. F is closed and so it is complete, since by Theorem 1.1 we have that $(X,\mathcal{U})$ is complete. Suppose that F is not compact. Then it must not be totally bounded, so we can choose an infinite uniformly discrete sequence $\{x_n\}$ of points in F. Let $\{F_n\}$ be a uniformly discrete sequence of closed sets, with the diameters of the F_n's converging to 0, and such that for each n, F_n is a neighborhood of x_n. For each n, choose a

[3] Kuratowski [Ku] gives the reference [Ha].

countable family $\{O^n_m\}$ of open sets which covers F_n but has no finite subcover, and such that $O^n_m \cap F_k = \phi$ for each n, m, and $k \neq n$. Now consider the following sequence of open sets: $X - \cup F_n$, O^1_1, O^1_2, O^2_1, O^1_3, O^2_2, O^3_1, O^1_4, $\cdots$, and for each n, let G_n be the union of the first n sets in this sequence. $\{G_n\}$ is an open cover of X which is well-ordered by inclusion, but it can't be uniform, contradicting Proposition 1.2. □

Lemma 2.2. *In a linearly cofinally complete pseudometric uniform space* $(X,\mathcal{U})$ *local compactness is equivalent to uniform local compactness.*

Proof. Let the pseudometric d generate the uniformity $\mathcal{U}$ and suppose that $(X,\mathcal{U})$ is not uniformly locally compact. For each $n \in \omega$ choose an x_n such that the closed ball $\overline{B}_d(x_n, 2^{-n})$ is not compact. We show that the sequence $\{x_n\}$ is cofinally Cauchy by contradiction.

Suppose $\{x_n\}$ is not cofinally Cauchy. Then for some $\epsilon > 0$ we have that for any n the open ball $B_d(x_n, \epsilon)$ contains only finitely many of the x_n's. So we can construct a subsequence x_{n_k} for which the family $\{\overline{B}_d(x_{n_k}, 2^{-n_k})\}$ is uniformly discrete. We then proceed as in the last proof to choose for each k a countable family $\{O^k_m\}$ which covers $\overline{B}_d(x_{n_k}, 2^{-n_k})$ but has no finite subcover, and this will lead, as above, to a contradiction to the linear cofinal completeness property.

So by linear cofinal completeness the sequence $\{x_n\}$ has a cluster point x. Then $(X,\mathcal{U})$ is not locally compact at x, for if the closed ball $\overline{B}_d(x, \epsilon)$ is compact then, since the open ball $B_d(x, \epsilon/2)$ contains cofinally many $\{x_n\}$, there will be some x_n with $\overline{B}_d(x_n, 2^{-n}) \subseteq \overline{B}_d(x, \epsilon)$, a contradiction. □

Theorem 2.1. *In a pseudometric uniform space* $(X,\mathcal{U})$ *linear cofinal completeness is equivalent to cofinal completeness.*

Proof. For the pseudometrizable linearly cofinally complete uniform space $(X,\mathcal{U})$ let $\mathcal{C}$ be an open cover which is closed under finite unions. By Lemma 2.1 there is a compact set $K \subseteq X$ such that every point not in K has a compact neighborhood. There is some $O^* \in \mathcal{C}$ such that $K \subseteq O^*$. Take an open $U^* \in \mathcal{U}$ such that $U^* \circ U^*[K] \subseteq O^*$.

The set $A = X - U^*[K]$ is a locally compact subspace of X, so it is uniformly locally compact by Lemma 2.2. So there is some $U \in \mathcal{U}$ such that for every $x \in A$ there is an $O \in \mathcal{C}$ with $U[x] \cap A \subseteq O$, and this implies that $U[x] \subseteq O \cup O^*$. Then $U \cap U^*$ witnesses that $\mathcal{C}$ is a uniform cover, and by Proposition 1.1 we have shown that $(X, \mathcal{U})$ is cofinally complete. □

This Theorem 2.1 then implies that Lemmas 2.1 and 2.2 follow from known properties of cofinally complete spaces.

The following definition is from [Ro].

Definition 2.1. A uniformizable space is *cofinally Čech complete* if there is a countable collection $\{\mathcal{C}_n\}_{n \in \omega}$ of open covers of X such that if $\mathcal{F}$ is a filter on X where for each $n \in \omega$ there is an $O \in \mathcal{C}_n$ which meets all the members of $\mathcal{F}$, then $\mathcal{F}$ clusters in X.

Definition 2.2. Analogously, let's define a uniformizable space to be *linearly cofinally Čech complete* if there is a countable collection $\{\mathcal{C}_n\}_{n \in \omega}$ of open covers of X such that for any filter $\mathcal{F}$ on X which has a filterbase linearly ordered by inclusion, and such that for each $n \in \omega$ there is an $O \in \mathcal{C}_n$ which meets all the members of $\mathcal{F}$, we have that $\mathcal{F}$ clusters in X.

Romaguera [Ro] has given characterizations of metrizable spaces which admit a cofinally complete metric. The following combines Theorem 1 and Theorem 2 in [Ro] with some new characterizations. We note that property (5) in the Proposition was shown in [HI, Theorem 3.1] to be equivalent in general to the statement that $\beta X - X$ is locally compact.

Proposition 2.1. *For a pseudometrizable space* $(X, \mathcal{T})$ *the following are equivalent:*

(1) $(X, \mathcal{T})$ *admits a cofinally complete pseudometric.*

(2) $(X, \mathcal{T})$ *admits a linearly cofinally complete pseudometric.*

(3) $(X, \mathcal{T})$ *is cofinally Čech complete.*

(4) $(X, \mathcal{T})$ *is linearly cofinally Čech complete.*

(5) *The set of points in* $(X, \mathcal{T})$ *which admit no compact neighborhood is a compact set.*

Proof. The equivalence of (1), (3), and (5) is established in [Ro]. (1) and (2) are equivalent by Theorem 2.1. (3) clearly implies (4), and a proof that (4) implies (2) could be constructed from the proof of Theorem 1 in [Ro]. □

The referee has suggested that we include some generalizations to uniformities that have linearly ordered bases. When we say below that a space has a well-ordered base $\{V_\alpha | \ \alpha < \kappa\}$, we intend that whenever $\alpha < \beta$, $V_\beta \subseteq V_\alpha$.

Definition 2.3. A space is *κ-compact* if every open cover of cardinality κ has a subcover of smaller cardinality. It is *$[\kappa, \infty)$-compact* if every open cover has a subcover of cardinality smaller than κ, and it is *$[\kappa, \infty)^r$-compact* if for every regular cardinal $\alpha \geq \kappa$ every open cover of cardinality α has a subcover of smaller cardinality. (For the latter two terms we have been influenced by [HV]. We will need to use the last term in Section 3.) We modify the properties of local compactness and uniform local compactness to *local $[\kappa, \infty)$-compactness* and *uniform local $[\kappa, \infty)$-compactness* in the obvious way. A uniform space $(X, \mathcal{U})$ is κ-bounded if for any $U \in \mathcal{U}$ there is a set $S \subseteq X$ of cardinality less than κ with $X = U[S]$.

We should remark that we are not assuming any separation properties besides uniformizablity. However, just as the closures of compact sets are compact in uniform spaces, in a uniform space with a well-ordered base $\{V_\alpha | \ \alpha < \kappa\}$ with κ regular, the closure of a $[\kappa, \infty)$-compact set is $[\kappa, \infty)$-compact. Therefore, in these spaces, local compactness may be stated in terms of neighborhoods having $[\kappa, \infty)$-compact supersets or having $[\kappa, \infty)$-compact closure.

We will make use of the fact that for a regular cardinal κ, spaces with a uniform basis of cardinality κ are $[\kappa,\infty)$-compact if and only if they are κ-compact. This is a straightforward generalization of the usual proof that countable compactness is equivalent to compactness in pseudometric spaces.

We also need the following result of N. Howes (a consequence of Theorem 2 in [H4]): For a regular cardinal κ, a space which is κ-bounded and linearly cofinally complete is $[\kappa,\infty)$-compact.

Lemma 2.3. *In a linearly cofinally complete uniform space* $(X,\mathcal{U})$ *with a well-ordered base* $\{V_\alpha|\ \alpha<\kappa\}$ *with* κ *regular, the set of points which do not have a* $[\kappa,\infty)$*-compact neighborhood is* $[\kappa,\infty)$*-compact.*

Proof. Let F be the collection of points which have no $[\kappa,\infty)$-compact neighborhood. F is closed and so it is linearly cofinally complete. Suppose that F is not $[\kappa,\infty)$-compact. Then by Howes's result mentioned above, it must not be κ-bounded, so we can choose an infinite uniformly discrete sequence $\{x_\alpha|\ \alpha<\kappa\}$ of points in F. Let $\{F_\alpha\}$ be a uniformly discrete sequence of closed sets, with each F_α being V_α-small, and such that each F_α is a neighborhood of x_α. For each α, since F_α is not κ-compact, choose a family $\{O^\alpha_\beta|\ \beta<\kappa\}$ of open sets which covers F_α but has no subcover of smaller cardinality, and such that $O^\alpha_\beta\cap F_\gamma=\phi$ for each β, α, and $\gamma\neq\alpha$. Now for each α let

$$G_\alpha=(X-\cup F_\alpha)\cup\left(\cup_{\beta,\gamma<\alpha}O^\gamma_\beta\right).$$

$\{G_\alpha\}$ is an open cover of X which is well-ordered by inclusion, but it can't be uniform, contradicting Proposition 1.2. □

Lemma 2.4. *In a linearly cofinally complete uniform space* $(X,\mathcal{U})$ *with a well-ordered base* $\{V_\alpha|\ \alpha<\kappa\}$ *with* κ *regular, local* $[\kappa,\infty)$*-compactness is equivalent to uniform local* $[\kappa,\infty)$*-compactness.*

Proof. This is a generalization of our Lemma 2.2 in the spirit of the proof of Lemma 2.3. □

Lemma 2.5. *In a linearly cofinally complete uniform space* $(X,\mathcal{U})$ *with a well-ordered base* $\{V_\alpha | \ \alpha < \kappa\}$ *with* κ *regular, if* $\mathcal{C}$ *is an open cover of* X*, then there exists* $\alpha < \kappa$ *such that every* $V_\alpha[x]$*,* $x \in X$ *is either contained in some member of* $\mathcal{C}$ *or has* $[\kappa, \infty)$*-compact closure.*

Proof. Without loss of generality we may assume that the V_α's are open relations. Let K be the set of points which do not have a $[\kappa, \infty)$-compact neighborhood. By Lemma 2.3 we may find an β such that for any $x \in K$, $V_\beta \circ V_\beta[x]$ is contained in some member of $\mathcal{C}$. Let $A = X - V_\beta[K]$. Then by Lemma 2.4 find γ such that for any $x \in A$ we have $V_\gamma[x] \cap A$ has $[\kappa, \infty)$-compact closure. Then if we take α such that $V_\alpha \circ V_\alpha \subseteq V_\beta$ and $V_\alpha \subseteq V_\gamma$ then α will satisfy the conclusion of the Lemma. □

The last three results of this section, and their proofs, are lifted almost verbatim from the referee's report. We have only changed some notation to make it consistent with the rest of our paper.

Lemma 2.6. *Let* $(X,\mathcal{U})$ *be a uniform space with a well-ordered base of equivalence relations,* $\{V_\alpha | \ \alpha < \kappa\}$*. The following are equivalent for an open cover* $\mathcal{C}$ *of* X*.*

(1) $\mathcal{C}$ *does not have a uniformly locally finite refinement.*

(2) *For each* $\alpha < \kappa$ *there exists* x *such that every partition of* $V_\alpha[x]$ *into clopen sets refining* $\mathcal{C}$ *is infinite.*

Proof. If (2) fails, let $\mathcal{P}$ be a partition of X into clopen sets refining the partition $\{V_\alpha[x] | \ x \in X\}$, such that each $V_\alpha[x]$ is partitioned into finitely many sets, each contained in some member of $\mathcal{C}$. Then $\mathcal{P}$ witnesses the failure of (1).

If (1) fails, let $\mathcal{W}$ be a uniformly locally finite refinement of $\mathcal{C}$ and let α be such that for every $x \in X$, $V_\alpha[x]$ meets only finitely many members of $\mathcal{W}$. Using ultraparacompactness, we can partition each $V_\alpha[x]$ into clopen sets, each contained in some member of $\mathcal{W}$. Given x, let $\{W_{1,x}, \ldots, W_{n,x}\}$ list the members of $\mathcal{W}$ that meet $V_\alpha[x]$, and let $C_{i,x}$ be the union of all members of the partition of $V_\alpha[x]$ that are contained in $W_{i,x}$ but not in $W_{j,x}$ if $j < i$. The existence of these $C_{i,x}$ contradicts (2). □

Lemma 2.7. *Let* $(X,\mathcal{U})$ *be a uniform space with a well-ordered base of equivalence relations,* $\{V_\alpha |\ \alpha < \kappa\}$ *with* κ *regular. Then* X *is linearly cofinally complete if and only if* X *is cofinally complete.*

Proof. Let $\mathcal{C}$ be an open cover of X and, by Lemma 2.5, let α_0 be such that, for each $x \in X$, $V_{\alpha_0}[x]$ is either contained in some member of $\mathcal{C}$ or is $[\kappa, \infty)$-compact. Suppose $\mathcal{C}$ does not have a uniformly locally finite refinement. Then there exists $x_0 \in X$ such that $V_{\alpha_0}[x_0]$ is not contained in any member of $\mathcal{C}$ and, moreover, x_0 is like x in (2) of Lemma 2.6 with α_0 playing the role of α. Using $[\kappa, \infty)$-compactness of $V_{\alpha_0}[x_0]$, pick $\alpha_1 < \kappa$ such that the trace of $\mathcal{C}$ on $V_{\alpha_0}[x_0]$ is refined by the partition $\{V_{\alpha_1}[y] |\ y \in V_{\alpha_0}[x_0]\}$ of $V_{\alpha_0}[x_0]$.

Suppose α_η and x_η have been defined for all $\eta < \xi < \kappa$, and if ξ is a successor ordinal, $\xi = \eta + 1$, we also assume $\alpha_{\eta+1}$ has been defined. Assume by induction that the sets $V_{\alpha_0}[x_\eta]$ are distinct for distinct η, and that each $\{V_{\alpha_{\eta+1}}[y] |\ y \in V_{\alpha_0}[x_\eta]\}$ refines the trace of $\mathcal{C}$ on $V_{\alpha_0}[x_\eta]$. If ξ is a limit ordinal, let α_ξ be the supremum of $\{\alpha_\eta |\ \eta < \xi\}$, and let x_ξ be chosen by (2) of Lemma 2.6 applied to $\alpha = \alpha_\xi$. Then $V_{\alpha_0}[x_\xi] \cap V_{\alpha_0}[x_\eta] = \phi$ whenever $\eta < \xi$, and we pick $\alpha_{\xi+1} (> \alpha_\xi)$ so that $\{V_{\alpha_{\xi+1}}[y] |\ y \in V_{\alpha_0}[x_\xi]\}$ refines the trace of $\mathcal{C}$ on V_{α_0}.

With x_ξ and α_ξ defined for all $\xi < \kappa$, let $\{P_{n,\xi} |\ n \in \omega\}$ be a partition of $V_{\alpha_\xi}[x_\xi]$ into infinitely many clopen sets. Let $W_0 = X - \bigcup\{V_{\alpha_\xi}[x_\xi] |\ \xi < \kappa\}$. Then W_0 is clopen, because all the x_ξ's are in distinct members of $\{V_{\alpha_0}[x] |\ x \in X\}$. For each $n \in \omega$, let

$$W_{n+1} = W_n \cup \bigcup\{P_{n,\xi} |\ \xi < \kappa\}.$$

Then $\{W_n|\ n \in \omega\}$ is an ascending non-uniform open cover of X, contradicting linear cofinal compleness of X. □

Theorem 2.2. *Let $(X,\mathcal{U})$ be a uniform space with a linearly ordered base. Then it is linearly cofinally complete if and only if it is cofinally complete.*

Proof. The case where the cardinality of the base has countable cofinality is exactly the pseudometrizable case, covered by Theorem 2.1. If the cardinality of the base has uncountable cofinality then it is an elementary fact that it has a well-ordered base of equivalence relations, and Lemma 2.7 applies. □

3. Examples and Questions

Example 3.1. *A linearly cofinally complete space that is not cofinally complete.* Let κ be a singular cardinal and let X be a discrete space of cardinality κ. Consider the uniformity $\mathcal{U}$ generated by the basis of all equivalence relations on X which have fewer than κ equivalence classes. Fried [Fr] has considered this example and pointed out that it is not uniformly paracompact. A direct proof that it is not cofinally complete is that the cofinite filter is weakly Cauchy but doesn't cluster.

$(X,\mathcal{U}_\kappa)$ is linearly cofinally complete. For suppose $\{O_\alpha\}_{\alpha<\gamma}$ is an open cover of X such that $O_\alpha \subseteq O_\beta$ whenever $\alpha < \beta < \gamma$. Then if $\operatorname{cf}\gamma > \kappa$ we must have $O_\alpha = X$ for some $\alpha < \gamma$, and therefore the cover is uniform. If, on the other hand, $\operatorname{cf}\gamma < \kappa$, let $f : \operatorname{cf}\gamma \to \gamma$ be an increasing cofinal map. Then $\{O_{f(\alpha)}\}_{\alpha<\operatorname{cf}\gamma}$ is a uniform cover which refines $\{O_\alpha\}_{\alpha<\gamma}$. By Proposition 1.2 we have shown the desired result.

Example 3.2. *A locally fine linearly supercomplete space which is not complete.* Let X be an infinite set and let $\mathcal{F}$ be a non-principal ultrafilter on X. Define a uniformity $\mathcal{U}$ on X by saying that $U \in \mathcal{U}$ iff there is an $F \in \mathcal{F}$ such that $\Delta_X \cup (F \times F) \subseteq U$. $\mathcal{U}$ is the finest uniformity on X which makes $\mathcal{F}$ Cauchy. $(X,\mathcal{U})$

is clearly not complete. Therefore it is not linearly cofinally complete, by Theorem 1.1. We claim that it is linearly supercomplete.

Suppose that α is an ordinal and $f : \alpha \to X$ is an almost Cauchy net with no cluster points. For every $\gamma \in \alpha$ the cardinality of the image $\{f(\beta) | \ \beta \geq \gamma\}$ cannot be less than $\operatorname{cf} \alpha$. So we can define by recursion a cofinal map $s : \operatorname{cf} \alpha \to \alpha$ such that $f \circ s$ is one-to-one. Partition $\operatorname{cf} \alpha$ into two cofinal subsets A and B. At most one of the sets $f \circ s[A]$ and $f \circ s[B]$ can be a member of $\mathcal{F}$. So we may choose an $F \in \mathcal{F}$ which is disjoint from one of these sets, say $f \circ s[B]$. The relation $U = \Delta_X \cup F \times F \in \mathcal{U}$ is such that no points of $f \circ s[B]$ are related to any other points in X. But there must be some $\gamma \in \alpha$ such that $\{\beta | \ \beta \geq \gamma\}$ can be partitioned into cofinal subsets each of which has a U-small f-image. Since there must be a $\delta \in B$ such that $s(\delta) \geq \gamma$ we have a contradiction.

Note that for any filter $\mathcal{F}$, a space constructed as above is locally fine, and so the linear supercompleteness property cannot be included in Lemma 1.1.

Example 3.3. *A supercomplete space which is not linearly cofinally complete.* Several metrizable examples of this are in the literature [Cs], [H3]. As suggested by Corollary 2 in [Ro], our Lemma 2.1 shows that any complete metric space which is homogeneous but not locally compact is an example. Both Császár's and Howes's examples are of this type. The referee has pointed out that another important example is ω^ω with the product metric, i.e., the irrationals with the appropriate complete metric.

Example 3.4. *A complete space which is not linearly supercomplete.* Isbell [I2] shows that if X is a discrete space of cardinality $2^{\aleph_0}$ and $\mathcal{U}$ is the finest $\aleph_1$-bounded uniformity on X, then $(X, \mathcal{U})$ is a complete space which is not supercomplete. In fact it is easy to show that $(X, \mathcal{U})$ is not linearly supercomplete.

These four examples show that the following six properties:

(1) Cofinal completeness,

(2) Linear cofinal completeness,

(3) Supercompleteness,

(4) Linear supercompleteness,

(5) Completeness, and

(6) Linear completeness

are all distinct.

However some questions remain unanswered. Among these are the following, which pertain to linear supercompleteness.

(1) Is there a space which is complete and linearly supercomplete but not supercomplete?

(2) Is there a space which is linearly supercomplete but not paracompact? Could it also be complete, and thus answer (1)? (Supercomplete spaces are paracompact; see [I1] or [I2].)

The referee has brought to our attention the paper [AMP]. The authors of that paper have given several examples of spaces with well-ordered uniform bases which are complete but not supercomplete, thus showing that Hahn's Theorem does not extend to uniform spaces with well-ordered bases.

In Section 5 of that paper the authors consider the set ${}^{\kappa}2$ of functions from κ to $\{0,1\}$, together with the κ-product uniformity $\mathcal{U}$, whose basis elements are generated from the usual subbasis for the product by allowing intersections of fewer than κ many subbasis elements. They show in Theorem 13 that for a regular uncountable cardinal κ, supercompleteness of $({}^{\kappa}2,\mathcal{U})$ is equivalent to $[\kappa,\infty)$-compactness of $({}^{\kappa}2,\mathcal{U})$, which in turn is equivalent to κ being a weakly compact cardinal by a previously known result. We add here a few more equivalent statements to that Theorem.

Proposition 3.1. *For a regular uncountable cardinal κ, the following are equivalent:*

(1) $({}^{\kappa}2, \mathcal{U})$ *is* $[\kappa, \infty)$*-compact*

(2) $({}^{\kappa}2, \mathcal{U})$ *is fine.*

(3) $({}^{\kappa}2, \mathcal{U})$ *is cofinally complete.*

(4) $({}^{\kappa}2, \mathcal{U})$ *is linearly cofinally complete.*

(5) $({}^{\kappa}2, \mathcal{U})$ *is supercomplete.*

(6) $({}^{\kappa}2, \mathcal{U})$ *is linearly supercomplete.*

(7) κ *is weakly compact.*

Proof. The space $({}^{\kappa}2, \mathcal{U})$ is always paracompact—every open cover has a discrete refinement. So properties (1) through (6) imply each other in that order, using the fact that $({}^{\kappa}2, \mathcal{U})$ has a well-ordered base $\{V_\alpha |\ \alpha < \kappa\}$ to get from (1) to (2), using Corollary 1.1 to get from (2) to (3), and using Theorem 1.1 to get from (4) to (5). As mentioned above, we have in [AMP] the equivalence of (1), (5), and (7). So it remains to show that property (6) implies any of the others.

First we show (6) implies that κ is strongly inaccessible. If not then by Fact 10 in [AMP], ${}^{\kappa}2$ with the κ-product uniformity is uniformly isomorphic to ${}^{\kappa}\kappa$ with the κ-product uniformity. The proof of Theorem 1 in [AMP] provides a nested sequence of closed sets G_α in ${}^{\kappa}\kappa$, indexed by κ, which has empty intersection. Their proof that the sequence is a Cauchy net in the hyperspace may be modified to show the G_α's are a base for a stable filter. This is impossible given linear supercompleteness.

Now if κ is strongly inaccessible then, again by the discussion preceeding Proposition 12 in [AMP], we see that $({}^{\kappa}2, \mathcal{U})$ is κ-bounded. Since for any infinite cardinal κ metacompactness plus κ-compactness implies $[\kappa, \infty)$-compactness, the following Proposition is more than what we need to finish the proof that (6) implies (1). $\square$

Proposition 3.2. *If $(X,\mathcal{U})$ is linearly supercomplete and κ-bounded then it is $[\kappa,\infty)^r$-compact.*

Proof. It is required to show that if $(X,\mathcal{U})$ is linearly supercomplete and κ-bounded, then for any regular cardinal α with $\alpha \geq \kappa$, $(X,\mathcal{U})$ is α-compact. It is easily checked that (for α regular) α-compactness is the same as the property that any net $\xi : \alpha \to X$ clusters.

Suppose ξ fails to be almost Cauchy. Then there is a $U \in \mathcal{U}$ and a cofinal set $S \subseteq \alpha$ such that for any $x \in S$ the set $\xi^{-1}[U[\xi(x)]]$ is not cofinal in α. By κ-bounded there is a cover $\mathcal{C}$ of X consisting of fewer than κ many U-small sets. The restriction $\xi|_S$ must be cofinally in one of the sets in $\mathcal{C}$ and this is a contradiction. So ξ is almost Cauchy, and by linear supercompleteness, ξ has a cluster point. □

Corollary 3.1. *Let κ be a regular cardinal. If $(X,\mathcal{U})$ is metacompact, linearly supercomplete, and κ-bounded then it is $[\kappa,\infty)$-compact.*

This Corollary is a strengthening of Theorem 7 in [AMP], but that Theorem has an unnecessary hypothesis (since supercompleteness implies paracompactness one doesn't have to say "weakly paracompact" [=metacompact] in the Theorem). If it turns out that linear supercompleteness implies paracompactness (a "no" to Question 2 above), or even metacompactness, then this Corollary here would become a strengthening of a revised Theorem 7 there.

References

[AU] Paul Alexandroff and Paul Urysohn, *Mémoire sur les espaces topologiques compactes*, Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam **14** (1929), 1–96.

[AMP] Giuliano Artico, Umberto Marconi, and Jan Pelant, *On supercomplete ω_μ-metric spaces*, Bulletin of the Polish Academy of Sciences, Mathematics, **44**, No. 3 (1996), 299–310.

[Bo] James R. Boone, *A note on linearly ordered net spaces*, Pacific Journal of Mathematics **98** (1982), 25–35.

[Bu] Bruce S. Burdick, *A note on completeness of hyperspaces*, in "General Topology and Applications, Fifth Northeast Conference," S. J. Andima, etal., eds., Lecture Notes in Pure and Applied Mathematics **134**, Marcel Dekker, New York, 1991, pp. 19–24.

[Cs] Á. Császár, *Strongly complete, supercomplete and ultracomplete spaces*, in "Mathematical Structures—Computational Mathematics—Mathematical Modelling, Papers dedicated to Professor L. Iliev's 60th Anniversary," Sofia, 1975, pp. 195–202.

[Co] H. Corson, *The determination of paracompactness by uniformities*, American Journal of Mathematics **80** (1958), 185–190.

[FL] P. Fletcher and W. F. Lindgren, *C-complete quasi-uniform spaces*, Archiv der Mathematik **30** (1978), 175–180.

[Fr] Jan Fried, *On paracompactness in uniform spaces*, Commentationes Mathematicae Universitatis Carolinae **26** (1985), 373–385.

[GI] Seymour Ginsberg and J. R. Isbell, *Some operators on uniform spaces*, Transactions of the American Mathematical Society **93** (1959), 145–168.

[Ha] H. Hahn, "Reele Funktionen," Akademische Verlagsgesellschaft, Leipzig, 1932.

[HI] Melvin Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Mathematics Journal **25** (1958), 82–105.

[HV] R. E. Hodel and J. E. Vaughan, *A note on $[a, b]$-compactness*, General Topology and its Applications **4** (1974), 179–189.

[H1] N. Howes, *On completeness*, Pacific Journal of Mathematics **38** (1971), 431–440.

[H2] —, *A note on transfinite sequences*, Fundamenta Mathematicae **106** (1980), 213–226.

[H3] —, *Paracompactifications, preparacompactness, and some problems of K. Morita and H. Tamano*, Questions and Answers in General Topology **10** (1992), 191–204.

[H4] —, *Relatively fine spaces*, Questions and Answers in General Topology **12** (1994), 71–83.

[H5] —, "Modern Analysis and Topology," Springer-Verlag, New York, 1995.

[I1] J. R. Isbell, *Supercomplete spaces*, Pacific Journal of Mathematics **12** (1962), 287–290.

[I2] —, "Uniform Spaces," American Mathematical Society, Providence, 1964.

[Ku] C. Kuratowski, "Topologie I, Espaces Métrisables, Espaces Complets," Monografje Matematyczne, Warsaw, 1933.

[Ny] P. J. Nyikos, *Convergence in topology*, in "Recent Progress in General Topology," M. Hušek and J. van Mill, eds., North-Holland, Amsterdam, 1992, pp. 537-570.

[Ri] Michael D. Rice, *A note on uniform paracompactness*, Proceedings of the American Mathematical Society **62** (1977), 359–362.

[Ro] Salvador Romaguera, *On cofinally complete metric spaces*, Questions and Answers in General Topology **16** (1998), 165–170.

[Sm] James C. Smith, *Review of "A note on uniform paracompactness" by Michael D. Rice*, # 9036, Mathematical Reviews **55** (1978).

Deptartment of Mathematics, Roger Williams University, Bristol, Rhode Island 02809

E-mail address: `bburdick@rwu.edu`