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UNIVERSALS IN OUR TIME

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Abstract

The paper considers universals, a topic introduced by Lebesgue and part of the classical development of descriptive set theory, in the context of modern set-theoretic topology and, in particular, by surveying recent work of the Oxford group. Interest centres on which spaces have universals 'parametrised' by the Cantor set and how this varies when the universal is at a different level of the Borel hierarchy or the space under consideration has a G_δ-diagonal or is compact. Cardinal invariants and consistency considerations play an important role.

1. Introduction

One problem in developing topological theory and in its appreciation by the mathematical community is its third-order character and the resulting importance of counter-examples and set-theoretic considerations, so anathema to many working mathematicians. Topologists recognized this long ago and many attempts have been made to make the subject more tractable. How may one describe the open sets of a space in a 'nice way'? How could one best describe, for example, the complement of a Julia, or Mandelbrot, set in the plane? For separable metric spaces, which is where the discussion commences here, one can

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give a 'listing' of open sets by means of sequences of elements from a countable basis. But what would a 'nice listing' look like?

Interpreting classical theory in our own times is often a fruitful way of developing a subject - and such is the strand of the current work of the Oxford general topology group surveyed here. This strand concerns 'universals', sets used by the ancients of Descriptive Set Theory to attack the listing problem. The work goes back to H. Lebesgue (1905) and, amongst the more influential authors, one can name N. Lusin, O. Nikodym and W. Sierpinski (see [8, 9, 10, 11, 12]). I select one aspect of their work as background to the current discussion.

Suppose $\mathcal{M}$ is a family of subsets of Euclidean spaces (the family may have sets in many, or all, $\mathbb{R}^n$). Let $P(\mathcal{M})$ denote the collection of all projections of these sets into Euclidean spaces of one lower dimension and let $C(\mathcal{M})$ be the collection of all complements of these sets in the spaces in which they lie ($A \in \mathcal{M}, A \subseteq \mathbb{R}^m \iff \mathbb{R}^m \backslash A \in C(\mathcal{M})$).

Now let $\mathcal{F}$ denote the family of all closed sets in all $\mathbb{R}^n$. It turns out that $P(\mathcal{F})$ consists of the F_σ's, $CP(\mathcal{F})$ the G_δ's, $PCP(\mathcal{F})$, the analytic sets and $PCP(\mathcal{F}) \cap CPCP(\mathcal{F})$ the Borel subsets of Euclidean spaces. Indeed, the *projective hierarchy* $\{P_n : 1 \leqslant n < \omega\}$ is defined in this way by the induction $C_n = C(P_n)$ and $P_{n+1} = P(C_n)$, for $n \geqslant 1$, where $P_1 = P(\mathcal{F})$. The projective hierarchy extends the Borel hierarchy $\{\Sigma_\alpha, \Pi_\alpha (1 \leqslant \alpha < \omega_1)\}$ see section 3 below. Before reproducing one typical result, we give our basic definition in a general context.

Definition 1.1. Suppose that X and Y are topological spaces. A subset U of $X \times Y$ is an *open* (resp. *closed*, resp. $\Sigma_\alpha-$, resp. $\Pi_\alpha-$, resp. $P_\alpha-$) *universal for* X *parametrised by* Y if U is open (resp. closed, resp. belongs to $\Sigma_\alpha-$, resp. Π_α, resp. P_α) and if for each open (resp. closed, resp. member of Σ_α, resp. Π_α, resp. P_α) V in X, there is y in Y such that $V = \{x : (x, y) \in U\} \equiv U^y$. We may equivalently speak of 'X having an open universal parametrised by Y' etc.

N.B. Thus, the topology on X consists precisely of all the sets U^y as Y varies, but it could well happen that $U^y = U^z$ for distinct elements y, z of the parametrising space Y.

An example of a classical theorem, and one which motivates some of our current work, is the following.

Theorem 1.2. (see [12]) *Given any positive integer n, there is in $\mathbb{R}^2$ a P_n-universal for $\mathbb{R}$ parametrised by itself.*

N.B. Indeed, Theorem 1.2 can be (seemingly) strengthened by replacing $\mathbb{R}^2$ with $\mathbb{R}^{m+1}$ (m a positive integer) and $\mathbb{R}$ (the space X in the definition) by a hyperplane in $\mathbb{R}^m$ parallel to a co-ordinate axis.

All spaces are assumed to be both regular and Hausdorff. For concepts not defined here, the reader is referred to R. Engelking [2].

2. Open Universals

We start by making some elementary observations, the first of which is folklore, but generates much of what we say in this section.

Proposition 2.1. (see [7]) *Every separable metric space X has an open universal set U parametrised by a Cantor set.*

Proof. If $\mathcal{B}$ is a countable basis, $f \in 2^{\mathcal{B}}$ and $U^f = \bigcup\{B \in \mathcal{B} : f(B) = 0\}$, put $U = \bigcup\{U^f \times \{f\} : f \in 2^{\mathcal{B}}\}$. □

Lemma 2.2. [5] *Suppose that U is an open universal for X parametrised by Y. Then*

(a) $c(U)$, the complement of U, is a closed universal for X parametrised by Y (and conversely)

(b) *if* $X' \subseteq X$, $Y' \supseteq Y$, *there exists an open universal for* X' *parametrised by* Y'.

Lemma 2.2 (b), the 'subspace-superspace lemma', has some interesting consequences: if Y has property $\mathcal{P}$ entails X has Q, then it implies that X has Q hereditarily. Further, in the case when Y is Tychonoff, one adds nothing by insisting that Y be compact - just embed Y in any Hausdorff compactification.

Natural question 2.3. *Which spaces have an open universal parametrised by* 2^ω*?*

N.B. The question remains the same if one replaces the parametrising space by a second countable space (indeed, by any cosmic space).

The range of spaces available for such Cantor parametrisations may be judged from the following result. The imposition of a G_δ-diagonal and compactness to strengthen results obtained for general spaces is typical.

Theorem 2.4. [5] *Suppose that* X *has an open universal parametrised by* Y. *Then*

(a) $w(X) \leqslant nw(Y), hd(X) \leqslant hL(Y), hc(X) \leqslant hc(Y), hL(X) \leqslant hd(Y)$

(b) *if* X *has a* G_δ*-diagonal*

$$hd(X^\omega) \leqslant hL(Y), hc(X^\omega) \leqslant hc(Y), hL(X^\omega) \leqslant hd(Y)$$

(c) *(consistent and independent) if* X *is compact and zero-dimensional and* Y *has the countable chain condition hereditarily, then* X *has a countable basis (and is hence metrisable).*

N.B. The duality between hereditary density and hereditary Lindelöf degree in (a) is noteworthy. In (b), X^ω may be replaced by X^n for each positive integer n, as is well-known. It is not known if 'zero-dimensional' can be omitted in (c) (see [5]).

The flavour of proofs involving universals can be tasted in the following partial justification of 2.4(a).

Proposition 2.5. *Under the hypotheses of 2.4, if Y is hereditarily separable then X is hereditarily Lindelöf.*

Proof. Suppose that X has the open universal U but is not hereditarily Lindelöf and that $\{V_\alpha : \alpha < \omega_1\}$ is a strictly increasing family of open subsets of X with no countable subcover. Pick x_α in each $V_\alpha \backslash \bigcup\{V_\beta : \beta < \alpha\}$ and y_α such that $V_\alpha = U^{y_\alpha}$ (notation of Definition 1.1). Then, as U is open, there are T_α open in X and W_α open in Y such that

$$(x_\alpha, y_\alpha) \in T_\alpha \times W_\alpha \subseteq U \quad \textit{but } (x_\beta, y_\beta) \notin T_\alpha \times W_\alpha \qquad (\beta < \alpha)$$

The proof is completed by showing that $\{y_\alpha : \alpha < \omega_1\}$ is not separable. If D were a countable dense subset, then there is $\beta < \omega_1$, such that $D \subseteq \{y_\alpha : \alpha \leqslant \beta\}$. But then $W_\beta \cap D = \varnothing$. □

3. Borel Universals

We use the following (abbreviated) notation for the Borel Hierarchy $\{\Sigma_\alpha, \Pi_\alpha (1 \leqslant \alpha < \omega_1)\}$ in a space $X : \Sigma_1$ for the family of open subsets of X and, inductively, $\Pi_\alpha (\alpha \geqslant 1)$ for the family of complements of elements of Σ_α in X and $\Sigma_\alpha (\alpha \geqslant 2)$ for the family of *countable* unions of sub-families from $\Pi_{\alpha-1}$ for successor α and the *countable* union of subfamilies from all $\Pi_\beta, \beta < \alpha$, for limit ordinal α. Thus, Σ_2 consists of all the F_σ's, Π_2 the G_δ's, etc. As in 2.2(a), a space has a Σ_α-universal if and only if it has a Π_α-universal.

Theorem 3.1. [6] *Suppose that the topological space* (X, τ) *has weight* κ *and* $0 < \alpha < \omega_1$. *Then* X *has a* Σ_α*-universal parametrised by* $D(|\tau|)^\omega$ *and one parametrised by* 2^κ. *(*$D(\lambda)$ *is the discrete space of cardinality* λ.*)*

That universals may be used to show that the Borel hierarchy in a Polish space stretches all the way up to ω_1 is well known (see [7]). (Sketch proof: if U is a Σ_α-universal for a Polish space X parametrised by 2^ω, then A (defined by $y \in A$ *if* $(y, y) \notin U$) belongs to $\Pi_\alpha = \Sigma_\alpha$, as the diagonal is homeomorphic to X. As U is universal, there exists $y_\circ \in 2^\omega$ with $A = U^{y_\circ}$. But then a simple diagonal argument gives a contradiction (see [7], 22.4)).

Natural question 3.2. *Are there non-metric spaces with* Σ_α*- (or* Π_α*-) universals parametrised by* 2^ω*?*

For $\alpha = 1$ and general X, and for α finite and X compact, the answer is 'no'; for, in such cases, $w(X) \leqslant nw(Y)$, extending 2.4(a) (see [6]). For general X and $\alpha \geqslant 2$, the answer is 'yes'. Even for $\alpha = 2$, there are a wide range of examples. Gartside and Lo give, under CH, an L-space (hereditarily Lindelöf, but not separable) and, under $\mathfrak{b} = \aleph_1$, even a strong S-space (all finite powers hereditarily separable, but not Lindelöf) with a G_δ-universal parametrised by the Cantor set. They also provide (in [6]) an example in ZFC of a non-metrisable, but hereditarily separable and hereditarily Lindelöf space with G_δ-universal set parametrised by 2^ω, namely, a λ-set in $\mathbb{R}$ when considered as a subset of the Sorgenfrey line. (A λ-*set* is an uncountable subset of the reals in which every *countable* subset is a G_δ.) The crucial fact here is that any subspace of the Sorgenfrey line is of 'Kunen-line-type', that is, closed sets differ from Euclidean closed sets by at most a countable set.

For $\omega \leqslant \alpha < \omega_1$ and X compact, the question provides a natural link with modern set-theoretic topology (as do the L- and S-space examples above). Only partial answers are known; only consistent results can be expected and much of what has been shown depends on adding the assumption $2^{\aleph_\circ} < 2^{\aleph_1}$. For fuller discussions of the issues involved, see [1, 6].

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