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SPACES IN WHICH POINT FINITE OPEN COVERS HAVE FINITE SUBCOVERS

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Abstract

In this note we first give a slight generalization of Iseki and Kasahara theorem. We also prove the existence of a Hausdorff space X in which 1) point-ω_α open covers have a subcover with cardinality less that ω_α, 2) X has a closed-discrete subset with cardinality ω_α and finally 3) X has a point-ω_α open cover with cardinality $\geq \omega_\alpha$. We finally prove that every Hausdorff space X is a closed nowhere dense subspace of a Hausdorff space Z_X in which point-finite (resp. point-countable) open covers have finite (resp. countable) subcovers. We also pose a related conjecture.

Introduction.

G.Aquaro has proved in 1965 that point-countable and in particularly point-finite open covers of a countably compact space have finite subcoverings. Actually a countably compact space X without any separation axiom satisfies the following covering property: For any open covering $\mathcal{U}$ of X there exists a finite set F of X such that $X = st(F,\mathcal{U}) = \bigcup\{st(x,\mathcal{U}) : x \in F\}$. The last mentioned covering property determines a new and strictly larger class of starcompact spaces which is defined by Fleischman [Fl]. One may define in here in this context a starcompact non countably compact T_1 space X_m for each fixed ordinal number $m \in \omega$

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satisfying $1 \leq m$. In fact let $A = \{r_n : n \in \omega\}$ be any countably infinite set of reals and let $X_m = [0, \omega_m) \cup A$ where $[0, \omega_m)$ be equipped with the usual order topology and let the basic nbhds of each r_n in X_m be defined as $G_n(\alpha) = \{r_n\} \cup [\alpha, \omega_m)$ where $\alpha < \omega_m$. Note that X_m is T_1 but not Hausdorff and A is closed and discrete in X_m. Furthermore X_m is starcompact. In fact if $\mathcal{U}$ is any open cover of X_m and if the well defined members $U_n \in \mathcal{U}$ be determined so that $r_n \in U_n$ and $G_n(\alpha_n) \subseteq U_n$ hold for each $n \in \omega$, then, by the aid of an element $\beta \in X_m$ with $\sup_n \alpha_n < \beta < \omega_m$ we have $[\beta, \omega_m) \subseteq \bigcap_{n \in \omega} U_n$ and $A \cup [\beta, \omega_m) \subseteq \bigcup_{n \in \omega} U_n \subseteq \mathrm{st}(\beta, \mathcal{U})$. Thus the starcompactness of X_m follows easily since $[0, \beta]$ is compact in X_m. It is evident that any regular uncountable ordinal number κ can also be taken instead of ω_m in above. On the other hand it is known that a Hausdorff space is starcompact iff it is countably compact, see [E], Problem 3.12.22d.

Another related well known result has been proved by Iseki and Kasahara in 1957: A regular Hausdorff space X is countably compact if and only if every point-finite open cover of X has a finite subcover. Z.Frolik proved on the other hand in 1960 that this equivalency does not necessarily hold without regularity condition. He defined a non countably compact Hausdorff space in which point-finite open covers have finite subcovers. Actually, point finite open covers of this space are necessarily finite. In here one must notice that his non regular counterexample space is not even $\aleph_0$-collectionwise Hausdorff, see [Fr] or [E] page 241. As is well known, a topological space X is called $\aleph_0$-collectionwise Hausdorff if points of any countable closed-discrete subset is separated by suitable pairwise disjoint nbhds of these points and it is also well known that regular Hausdorff spaces are $\aleph_0$-collectionwise Hausdorff. On the other hand there exist $\aleph_0$-collectionwise Hausdorff (even collectionwise Hausdorff) non-regular Hausdorff spaces. In fact any countably compact Hausdorff space is collectionwise Hausdorff since closed-discrete subsets are necessarily finite in such spaces, but,

examples of non-regular countably compact Hausdorff spaces are already known. Take for instance $X = \{p\} \cup (\omega_1 \times [0,1])$ where $p \notin \omega_1 \times [0,1]$ and define the basic nbhd's of the point p as $G_\alpha(p) = \{p\} \cup ([\alpha+1, \omega_1) \times [0,1))$ for each $\alpha < \omega_1$ and let $\omega_1 \times [0,1]$ be equipped with the usual product topology. This countably compact Hausdorff space is not regular since p and $\omega_1 \times \{1\}$ are not separated in X.
The aim of this paper is written in the Abstract.

Results.

The following result slightly generalizes Iseki and Kasahara's Theorem after the remarks in above.

Proposition 1. *An $\aleph_0$-collectionwise Hausdorff space is countably compact if and only if every point-finite open cover has a finite subcover. $\aleph_0$-collectionwise Hausdorff condition is essential.*

Proof. Suppose that $K = \{x_n : n \in \omega\}$ is a closed-discrete subset in an $\aleph_0$-collectionwise Hausdorff space and let $\mathcal{U} = \{U_n : n \in \omega\}$ be the pairwise disjoint open family separating the points of K, i.e. let $x_n \in U_n$ for each $n \in \omega$. Then point-finite open cover $\mathcal{U} \cup \{X - K\}$ has no finite subcovering. It is not difficult to observe that the Hausdorff space Y which will be described roughly in Corollary 1 or Frolik's counterexample space proves that $\aleph_0$-collectionwise Hausdorff condition is essential. □

Now we are going to construct (by a method which is completely different from Frolik's) a non countably compact Hausdorff space X in which point-finite open covers have a finite subcover and yet X has an infinite point-finite open cover. Afterwards we somehow generalize this construction and thus we are able to prove the last two important propositions of this note. Thus we learn a way of defining non countably compact Hausdorff spaces in which point-finite (resp. point-countable) open covers have finite (resp. countable) subcovers.

Proposition 2. *There is a Hausdorff space X with a set of isolated points I so that $X - I$ is countable infinite and closed-discrete and yet any open family point-finite on I and covering $X - I$ has a finite subfamily which covers $X - I$.*

Proof. Let us define first the set I of all functions $f \in ([\omega]^{<\omega})^\omega$ satisfying $n \notin f(m)$ or $m \notin f(n)$ for each $(n, m) \in \omega \times \omega$. Thus in particularly $n \notin f(n)$ holds for each $n \in \omega$ and $f \in I$. Now set $X = \omega \cup I$ and let us define a nbhd of $n \in \omega$ by $F \in [\omega]^{<\omega}$ satisfying $n \notin F$ as the set $W(n, F) = \{n\} \cup \{f \in I : F \subseteq f(n)\}$. We declare each point of I as an isolated point, so $X - I$ is in fact closed-discrete in X. X is a Hausdorff space since $W(n, \{m\}) \cap W(m, \{n\}) = \emptyset$ for each $n \neq m$. Besides $f \notin W(n, F_f)$ for any non-empty and finite $F_f \subseteq \omega$ satisfying $F_f \not\subseteq f(n)$ and $n \notin F_f$. Now suppose that $\mathcal{U}$ is an open covering for $X - I$. If none of its finite subfamilies covers $X - I$ then we can define by induction an infinite subfamily $\{U_{n_i} : i \in \omega\}$ of $\mathcal{U}$ such that $U_{n_i} \neq U_{n_k}$ for $i \neq k$ and

$$W(N_{n_i}, F_{n_i}) \subseteq U_{n_i} \ , \ N_{n_i} \in \omega - \bigcup_{k<i} F_{n_k} \qquad (i \in \omega).$$

Suppose that the open sets U_{n_k} for each $k < i$ have already been defined. Then by the above supposition we can determine an $U_{n_i} \in \mathcal{U}$ and $N_{n_i} \in \omega$, $F_{n_i} \in [\omega]^{<\omega}$ so that

$$N_{n_i} \in U_{n_i} - (\bigcup_{k<i} U_{n_k} \cup \bigcup_{k<i} F_{n_k})$$

and $W(N_{n_i}, F_{n_i}) \subseteq U_{n_i}$. Thus the induction process is over. Now if $\mathrm{card}(\omega - \{N_{n_i} : i \in \omega\}) = \aleph_0$ let us write $\omega - \{N_{n_i} : i \in \omega\} = \{N_{m_i} : i \in \omega\}$ and define g by $g(N_{n_i}) = F_{n_i}$ and $g(N_{m_i}) = \{N_{m_{i+1}}, N_{m_{i+2}}, \ldots, N_{m_{i+i}}\}$ for each $i \in \omega$. Then we evidently have $g \in I$ and furthermore

$$g \in W(N_{n_i}, F_{n_i}) \subseteq U_{n_i} \qquad (\forall i \in \omega).$$

If $\omega - \{N_{n_i} : i \in \omega\}$ is non empty and finite set

$$\{N_{m_1}, N_{m_2}, \ldots, N_{m_k}\},$$

then one can easily define a similar $g \in I$ this time as $g(N_{m_i}) = \{N_{m_{i+1}}, N_{m_{i+2}}, \ldots, N_{m_k}\}$ for each $i < k$ and $g(N_{m_k}) = \emptyset$ and besides $g(N_{n_i}) = F_{n_i}$ for each $i \in \omega$. Thus X and I satisfy all conditions stated in proposition. It is also easy to observe by defining such an element $g \in I$ that X is not $\aleph_0$-collectionwise Hausdorff, since for any infinite subset $A = \{n_i : i \in \omega\} \subseteq \omega \subseteq X$ one can not define the pairwise disjoint nbhds of the elements of A in X. □

Proposition 3. *Let X be a Hausdorff space and I be subset of X so that any open family covering $X - I$ and point-finite on I has a finite subfamily covering $X - I$. Then there is a Hausdorff space Y which embeds $X - I$ as a closed nowhere dense subspace and in which every point-finite open cover has a finite subcover.*

Proof. Without loss of generality we can assume that I is a set of isolated points; otherwise we can define a finer topology on X leaving the nbhds of any $x \in X - I$ as the same and isolating each $x \in I$ and this new space X evidently satisfy all conditions of the statement. (Actually such a Hausdorff space does exist by Proposition 2). Let us take now a regular uncountable cardinal κ satisfying card $I \leq \kappa$. Such a κ does exist since $\aleph_{\alpha+1}$ is regular and uncountable cardinal number for each ordinal α . Define first $f : \mathrm{succ} \cap \kappa \to I$ where succ is the class of all successor ordinals, such that we have $\kappa = \mathrm{card}\ (f^{-1}(x))$ for each $x \in I$ and then define $Y = (X - I) \cup \kappa$. If $x \in X - I$ and G_x is a nbhd of x in X and $\alpha \in \kappa$, let $W(G_x, \alpha) = (G_x - I) \cup (f^{-1}(G_x \cap I) - \alpha)$ be declared as a basic nbhd of x in Y and let κ be open and have the well known order topology. Notice that Y is a Hausdorff space and $f^{-1}(G_x \cap I) - \alpha \subseteq \kappa$ is open in Y but $X - I$ may possibly be a nondiscrete set in Y. Take any point-finite open cover $\mathcal{U}$ of Y. Write $\mathcal{U}' = \{U \in \mathcal{U} : U \cap (X - I) \neq \emptyset\}$. Then there are $G_\gamma(U)$ and $\alpha_\gamma(U)$ (where γ belongs to some index set $\Lambda(U)$) for each $U \in \mathcal{U}'$ such that

$$\bigcup\{W(G_\gamma(U), \alpha_\gamma(U)) \ : \ \gamma \in \Lambda(U)\} \subseteq U,$$

$$U \cap (X - I) \;=\; \bigcup\{G_\gamma(U) - I \;:\; \gamma \in \Lambda(U)\}. \qquad (1)$$

It is not difficult to observe that

$$\Big\{\bigcup\{G_\gamma(U) \;:\; \gamma \in \Lambda(U)\} \;:\; U \in \mathcal{U}'\Big\}$$

is an open cover of $X - I$. We claim that it is point-finite on I. If not then there would be a $x \in I$ and distinct $U_n \in \mathcal{U}'$ so that $x \in G_{\gamma_n}(U_n)$ for each $n \in \omega$; thus by using $\alpha > \sup_n \alpha_{\gamma_n}(U_n)$ satisfying $\alpha \in f^{-1}(x)$ we would easily have

$$\alpha \in f^{-1}(G_{\gamma_n}(U_n) \cap I) - \alpha_{\gamma_n}(U_n) \subseteq W\big(G_{\gamma_n}(U_n), \alpha_{\gamma_n}(U_n)\big) \subseteq U_n$$

for each $n \in \omega$. Thus by assumption the open family $\big\{\bigcup\{G_\gamma(U) : \gamma \in \Lambda(U)\} : U \in \mathcal{U}'\big\}$ has a finite subfamily covering $X - I$ and so $\mathcal{U}$ has a finite subfamily covering the same set after (1). Since κ is countably compact subspace of Y we conclude that $\mathcal{U}$ has a finite subcover. □

Corollary 1. *There is a non countably compact Hausdorff space X in which point-finite open covers have a finite subcover and yet there exists an infinite point-finite open cover in X.*

Proof. In the last two propositions we already proved that there exists a Hausdorff space Y in which every point-finite open cover has a finite subcover. We also observed that Y would be a non countably compact space whenever we guarantee the existence of an infinite closed-discrete subset in Y. In fact, if we define the Hausdorff space Y of Proposition 3 by the aid of the Hausdorff space $Z^* = Z \cup I$ which is equipped with the topology described in Proposition 2 whereas $Z = \{z_n : n \in \omega\}$ and

$$I = \{f \in ([Z]^{<\omega})^Z \;:\; \forall (n,m) \in \omega \times \omega, z_n \notin f(z_m) \vee z_m \notin f(z_n)\}$$

then $Z^* - I \subseteq Y = (Z^* - I) \cup \kappa$ will be the required infinite closed-discrete subset of Y as one can easily observed. Now let X be the free union of Y with $\omega_0 + 1$ where the latter one is equipped with the usual order topology which is evidently compact and possesses and infinite point-finite open cover. It is straightforward to see that X have all the required properties. □

After the necessary revisions one can easily obtain the following generalization:

Proposition 4. *There exists a Hausdorff space X and a subset I of X so that $X - I$ is closed-discrete and any open family covering $X - I$ and point-ω_α on I has a subfamily with cardinality $< \omega_\alpha$ covering $X - I$. Furthermore there is a Hausdorff space Y which embeds $X - I$ as a closed nowhere dense subspace and in which every point-ω_α open cover has a subcover with cardinality $< \omega_\alpha$.*

Corollary 2. *There is a Hausdorff space X in which point-ω_α open covers have a subcover with cardinality $< \omega_\alpha$ and yet X has a closed-discrete subset with cardinality ω_α. Furthermore X has a point- ω_α open cover with cardinality ω_α.*

Now we prove a somewhat general form of Proposition 2:

Proposition 5. *For any Hausdorff space X there is a Hausdorff space $Y_X = X \cup I$ in which X is embedded as a closed nowhere dense subset and I a set of isolated points so that any open family of Y_X covering X and point-finite (resp. point-countable) on I has a finite (resp. countable) subfamily covering X.*

Proof. Let $\mathcal{B}$ be a base for X. Define now the set I of all functions $f \in ([\mathcal{B}]^{<\omega})^{\mathcal{B}}$satisfying $B_1 \notin f(B_2)$ or $B_2 \notin f(B_1)$ for any $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$. Let $Y_X = X \cup I$ and $W(B, \mathcal{F}) = B \cup \{f \in I : \mathcal{F} \subseteq f(B)\}$ for any $B \in \mathcal{B}$ and $\mathcal{F} \in [\mathcal{B}]^{<\omega}$ satisfying $B \notin \mathcal{F}$. Define a topology on Y_X where each element of I is an isolated point and an open nbhd of $x \in X$ in Y_X contains an intersection $\bigcap_{k<n} W(B_k, \mathcal{F}_k)$ where $x \in \bigcap_{k<n} B_k$. Thus $X = Y_X - I$ is closed

and nowhere dense and Y_X is a Hausdorf space. If $\mathcal{U}$ is an open family covering X and having no finite covering subfamilies then we can easily define $U_{n_i} \in \mathcal{U}$ for each $i \in \omega$ and $B_{n_i(k)} \in \mathcal{B}$, $\mathcal{F}_{n_i(k)} \in [\mathcal{B}]^{<\omega}$ where $k < N_i$, $N_i \in \omega$ such that there exists point $x_{n_i} \in X$ for each $i \in \omega$ and furthermore the conditions

$$x_{n_i} \in U_{n_i} - (\bigcup_{j<i} \bigcup_{k<N_j} B_{n_j(k)})$$

$$x_{n_i} \in \bigcap_{k<N_i} B_{n_i(k)} \subseteq \bigcap_{k<N_i} W(B_{n_i(k)}, \mathcal{F}_{n_i(k)}) \subseteq U_{n_i}$$

hold. Notice that $B_{n_i(k)}$ and $B_{n_j(k')}$ are different for any $k < N_i$ and $k' < N_j$ whenever $i \neq j$. One can now easily define an element $g \in I$ such that $g \in \bigcap_{k<N_i} W(B_{n_i(k)}, \mathcal{F}_{n_i(k)})$ for each $i \in \omega$. Thus Y_X have all the properties of the statement. For proving the statement written in parenthesis one should work this time with the set I of all functions $f \in ([\mathcal{B}]^{\leq\omega})^{\mathcal{B}}$ satisfying the old condition. The topology on $Y_X = X - I$ is defined just as in above and this space have all the required properties. □

Proposition 6. *Any Hausdorff space X can be embedded as a closed nowhere dense subset of a Hausdorff space Y_X in which point-finite (resp. point-countable) open covers have finite (resp. countable) subcovers.*

Proof. Let X be any Hausdorff space and let $Y_X = X \cup I$ be the Hausdorff space, after Proposition 5, in which every open family in Y_X covering X and point-finite (resp. point-countable) on I has a finite (resp. countable) subfamily covering X. If we define now an anologous topology on the set $Z_X = (Y_X - I) \cup \kappa$ just as in Proposition 3, i.e. if we define first the function $f : \text{succ} \cap \kappa \to I$ and then the basic nbhds of any point $x \in Y_X - I$ as

$$W(G_x, \alpha) = (G_x - I) \cup f^{-1}((G_x \cap I) - \alpha)$$

just exactly in that proposition, then, the whole proof works now where the regular uncountable cardinal number κ satisfy this time $\text{card}(Y_X - X) + \aleph_1 < \kappa$. Then, as one can easily see, the resulting space Z_X have all the required properties since κ is countably compact in Z_X and therefore every point-countable open cover of κ admits a finite subcover.

The reader will easily observe that the resulting space Z_X of Proposition 6 is non countably compact iff X is so. Thus, by utilizing this method, one can obtain non countably compact Hausdorff spaces in which point-finite (resp.point-countable) open covers have finite (resp. countable) subcovers. □

We close the paper with the following conjecture. As is well known a topological space is called $[\kappa_1, \kappa_2]$-compact, if and only if, every open cover with cardinality κ_2 has a subcover with cardinality $< \kappa_1$, see [S].

Conjecture. *Every $[\kappa_1, \kappa_2]$-compact Hausdorff space X can be embedded as a closed nowhere dense subset of a Hausdorff space Y_X in which point $< \kappa_2$ open covers have a subcover with cardinality $< \kappa_1$.*

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