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GENERALIZED FIRST CLASS SELECTORS FOR USCO MAPS IN $C_{p(K)}$

R. W. Hansell

Abstract

Recall that a compact Hausdorff space K has property $\mathcal{N}^*$ if, whenever X is a Baire space and $f : X \to C_p(K)$ is continuous, then f is norm continuous at each point of a dense subset of X. It is shown that, if X is an hereditarily Baire space and K is any compact Hausdorff space with property $\mathcal{N}^*$, then any upper semi-continuous map $F : X \to C_p(K)$ with nonempty compact-values (both properties relative to the topology of pointwise convergence), then F has a selector f that is a PC map relative to the sup-norm topology—that is, for any nonempty closed set $A \subset X$, the restriction map $f|A$ has a point of norm continuity. A variant of this result is also presented where the domain space X is somewhat stronger and K is a general compact Hausdorff space.

1. Introduction

In recent years considerable attention has been given to the following problem: Let X be a metric space and Y a Banach space and suppose $F : X \to Y$ is weakly upper semi-continuous taking nonempty set-values. When does F have a "nice" selector relative to the norm topology on Y?

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That is, does there exist a "nice" map $f : X \to (Y, \text{norm})$ such that $f(x) \in F(x)$ for each $x \in X$? Elementary examples show that continuous selectors fail to exist even in the best of cases (see, e.g., [H1]). By imposing conditions on the Banach space Y and/or the values of F, selectors of norm Borel class 2, then norm Borel class 1 and finally norm Baire class 1 were obtained (see [JR1], [JR2], [HJT], [H1], [S], [Sr]). Assuming only nonempty set-values Srivatsa [Sr] showed that norm Baire class 1 selectors (that is, a norm pointwise limit of a sequence of norm continuous maps) always exist when X is metrizable. The question was then raised as to whether extensions of this result were possible for a nonmetric domain X, and in particular when X is a compact Hausdorff space [Sr, Remark 2.7]. However, to obtain norm Baire class 1 selectors when X is a nonmetrizable compact Hausdorff space would impose strong restrictions on the Banach space Y. For example, the identity map from a weakly compact subset $X \subset Y$ would be of norm Baire class 1 only if X is norm σ-compact.

A more interesting problem results when the desired selector is required only to be a "generalized" first class map. A study of such maps was made in [H2] and in more detail in [K]. Since most of the natural generalized first class maps coincide for hereditarily Baire spaces, and hence for compact Hausdorff spaces (see [K, Theorem 2.3] and [H2, Theorems 2.2 and 2.8]), we seek primarily selectors of the following type. A map $f : X \to Y$ between topological spaces X and Y is said to be a *PC map* [H2] if for each nonempty closed set $A \subset X$, the restriction map $f|A$ has a point of continuity (such maps are also called *barely continuous* [NM], and are said to have *PCP* or the *point of continutiy property* [K]).

Our principal result gives the following theorem on first class selectors for upper semi-continuous compact-value maps defined on an arbitrary compact Hausdorff space.

Theorem 1. *Let X and K be compact Hausdorff spaces and let $F : X \to C_p(K)$ be upper semi-continuous and have nonempty compact values relative to the topology of pointwise convergence. Then F has a selector that is a PC map relative to the norm topology on $C(K)$.*

Here $C(K)$ is the space of all real-valued functions on the space K with the usual supremum norm, and $C_p(K)$ denotes the same space with the topology of pointwise convergence. An important tool used in the proof of Theorem 1 is the existence of minimal usco maps. By an *usco map* $F : X \to Y$ we mean a set-valued map with nonempty compact values $F(x) \subset Y$, and which is *upper semi-continuous* (abbreviated *usc*) in the sense that, whenever V is open in Y and contains $F(x)$, then there is an open neighborhood U of x in X such that $F(x^*) \subset V$ for all $x^* \in U$. An usco map is said to be *minimal* if it does not properly contain the graph of another usco map. It is a consequence of the axiom of choice that the graph of any usco map between Hausdorff spaces contains a minimal usco map (see, e.g., [HJT, §2]).

Theorem 1 is in fact true for a much wider class of domain spaces. To describe this class we need to recall some terminology. A topological space X is said to be a *Namioka space* if, whenever K is a compact Hausdorf space and $f : X \to C_p(K)$ is a continuous map, f is norm continuous at each point of a dense (and necessarily G_δ) subset of X (compare [L, p. 230]). We say X is a *special Namioka space* [HJT, p. 206] (or, X is a "Namioka space for minimal usco maps" [L, p. 230]) if, whenever K is a compact Hausdorf space and $F : X \to C_p(K)$ is a minimal usco map, then F is point-valued and norm usc at each point of some dense (again, necessarily G_δ) subset of X. As noted in [L, p. 230] "all Namioka spaces that we are able to recognize in topological terms" are special Namioka. In particular, this is true of any class of Namioka spaces preserved under minimal usco maps [HJT, Lemma 2].

A related concept is that of a compact Hausdorff space K having property $\mathcal{N}^*$. The space K is said to have *property* $\mathcal{N}^*$ if, whenever X is a Baire space and $f : X \to C_p(K)$ is a continuous map, f is norm continuous at each point of a dense subset of X (see [NP] for recent results related to this property). Labuda [L, Theorem 1] has shown that if K has property $\mathcal{N}^*$, then, whenever $F : X \to C_p(K)$ is a minimal usco map, F is point-valued and norm usc at each point of some dense G_δ subset of X.

A space X is said to be *hereditarily Baire* if every nonempty closed subspace of X is a Baire space (that is, has the Baire category property). Since every nonempty open subset of a Baire space is a Baire space, it follows that every nonempty FG subset of a hereditarily Baire space is a Baire space, where by a FG set we mean a set expressible as the intersection of a closed and an open set. Note that if the compact Hausdorff space K has property $\mathcal{N}^*$ and X is hereditarily Baire, then for every nonempty FG subset T of X and every minimal usco map $F : T \to C_p(K)$, F is point-valued and norm usc at each point of a dense subset of T. On the other hand, if X is (for example) a regular strongly countably complete space [N], then every nonempty FG subset is special Namioka (this can be deduced from the fact that every $(\sigma - \beta)$-unfavourable space is special Namioka [HJT, Lemma 3]).

We can now state in full generality our principal selection theorem.

Theorem 2. *Let K be compact Hausdorff space and let X be such that, whenever T is a nonempty FG subset of X and $F : T \to C_p(K)$ is a minimal usco map, then F is point-valued and norm usc at some point of T. Then each usco map from X to $C_p(K)$ has a selector that is a PC map relative to the norm topology on $C(K)$.*

2. Preliminaries

All topological spaces considered are assumed to be Hausdorff spaces. A partition $\mathcal{E}$ of a topological space X is said to be *scattered* if each nonempty subcollection $\mathcal{H}$ of $\mathcal{E}$ has a nonempty member that is relatively open in $\bigcup \mathcal{H}$. If E_1 has this property for $\mathcal{E}$ and E_2 has this property for $\mathcal{E} \setminus \{E_1\}$ and so on, then it is easy to see that a partition $\mathcal{E}$ is scattered if and only if it can be well-ordered so that

$$U_E \equiv \bigcup_{E^* \leq E} E^*$$

is open in X for each $E \in \mathcal{E}$. These *associated open sets* U_E satisfy

$$E = U_E \setminus \bigcup_{E^* < E} U_{E^*},$$

and so each $E \in \mathcal{E}$ is an FG set in X. Note also that for any nonempty $A \subset X$, if E is the least member of $\mathcal{E}$ such that $A \cap E \neq \emptyset$, then

$$A \cap E = A \cap U_E$$

and so $A \cap E$ is relatively open in A.

The following lemma is an easy consequence of the definitions (see, also [NP, Proposition 2.1] where a scattered partition is also called a disjoint exhaustive cover).

Lemma 1. *If $\mathcal{E}$ is a scattered partition for X and, for each $E \in \mathcal{E}$, $\mathcal{H}_E$ is a scattered partition of the subspace E, then $\bigcup_{E \in \mathcal{E}} \mathcal{H}_E$ is a scattered partition for X.*

The following two lemmas isolate the basic properties of PC maps used in the construction of the selectors in Theorems 1 and 2.

Lemma 2. *Let $f : X \to Y$ be a map and $\mathcal{E}$ a scattered partition of X such that the restriction $f|E$ is a PC map for each $E \in \mathcal{E}$. The f is a PC map. In particular, if f is constant on each $E \in \mathcal{E}$, then f is a PC map relative to any topology on Y.*

Proof. Let A be any nonempty closed subset of X, and let E be the least member of $\mathcal{E}$ such that $A \cap E \neq \emptyset$. Then $A \cap E$ is open relative to A, so any point of continuity for $f|A \cap E$ (which exists since $f|E$ is a PC map) will also be a point of continuity for $f|A$. □

Lemma 3. *Suppose X is a hereditarily Baire space, Y a metric space, and let $f_n : X \to Y$ $(n = 1, 2, \dots)$ be a sequence of PC maps converging uniformly to f. Then f is a PC map.*

Proof. Let A be any nonempty closed subset of X. By a result of Michael and Namioka [MN, Remark 3.3], $f_n|A$ is continuous at each point of a dense G_δ subset G_n of A for each n. Since A is a Baire space, there exists some $x \in \bigcap_{n=1}^{\infty} G_n$. Since $f_n|A$ converges uniformly to $f|A$, the latter is also continuous at x.□

Our last lemma isolates the property of minimal usco maps utilized in the proof of Theorem 2. If F and G are two set-valued maps defined on the space X we write $F \subset G$ to mean, for all $x \in X$, $F(x) \subset G(x)$ and $G(x) \neq \emptyset \Rightarrow F(x) \neq \emptyset$. Also, for any $E \subset X$ we use $F(E)$ to denote the set $\bigcup\{F(x) : x \in E\}$.

Lemma 4. *Suppose K is compact Hausdorff and X is a topological space such that, whenever T is a nonempty closed subspace of X and $F_T : T \to C_p(K)$ is a minimal usco map, then F_T is point-valued and norm usc at some point of T. If $F : X \to C_p(K)$ is an usco map and $\epsilon > 0$, then there exist a scattered partition $\mathcal{E}$ of X and usco maps $F_E : E \to C_p(K)$ for each $E \in \mathcal{E}$ such that $F_E \subset F|E$ and norm-diam$[F_E(E)] < \epsilon$.*

Proof. We may assume that F is a minimal usco map. By assumption F is single-valued and norm usc at some point $x \in X$. Hence there is an open neighborhood U_0 of x such that norm-diam$[F(U_0)] < \epsilon$. Put $E_0 = U_0$. If $X \setminus U_0 \neq \emptyset$, then $F|(X \setminus U_0)$ has a minimal usco refinement $F_1 \subset F|(X \setminus U_0)$. Since $X \setminus U_0$ is closed in X, by assumption there is a nonempty open set U_1 in X such that $E_1 \equiv U_1 \cap (X \setminus U_0) \neq \emptyset$ and norm-diam$[F_1(E_1))] < \epsilon$. Note that $E_0 \cup E_1 = U_0 \cup U_1$, so $\{E_0, E_1\}$ is a scattered partition of its union. If $X \setminus (U_0 \cup U_1)$ is nonempty, we may apply the same argument to $F|(X \setminus (U_0 \cup U_1))$ to obtain a minimal usco refinement F_2 and a nonempty open set U_2 in X such that, $E_2 \equiv U_2 \cap (X \setminus (U_0 \cup U_1)) \neq \emptyset$ and norm-diam$[F_2(E_2)] < \epsilon$. It is clear that this method will generate the desired scattered partition of X. □

3. Proof of Theorem 2

Note that the assumptions in Theorem 2 show that Lemma 4 may be applied to any nonempty FG subset of X. With $\epsilon = 1$ we apply Lemma 4 to X to obtain a scattered partition $(E_{\alpha_1})_{\alpha_1 \in A}$ of X and usco maps $F_{\alpha_1} : E_{\alpha_1} \to C_p(K)$ with $F_{\alpha_1} \subset F|E_{\alpha_1}$ and norm-diam$[F_{\alpha_1}(E_{\alpha_1})] < 1$ for each $\alpha_1 \in A$. Since each E_{α_1} is an FG set in X, we may apply Lemma 4 with $\epsilon = 1/2$ to F_{α_1} and obtain a scattered partition $(E_{\alpha_1\alpha_2})_{\alpha_2 \in A(\alpha_1)}$ of E_{α_1} and usco maps $F_{\alpha_1\alpha_2} : E_{\alpha_1\alpha_2} \to C_p(K)$ such that $F_{\alpha_1\alpha_2} \subset F_{\alpha_1}|E_{\alpha_1\alpha_2}$ and norm-diam$[F_{\alpha_1\alpha_2}(E_{\alpha_1\alpha_2})] < 1/2$ for each $\alpha_2 \in A(\alpha_1)$. By Lemma 1,

$$\{E_{\alpha_1\alpha_2} : \alpha_1 \in A \text{ and } \alpha_2 \in A(\alpha_1)\}$$

is a scattered partition of X. Since $E_{\alpha_1\alpha_2}$ is an FG set in X we can apply the above argument with $\epsilon = 1/3$ to each of these subspaces. In this way we obtain for each $n = 1, 2, \dots$ a scattered partition $(E_{\alpha_1\cdots\alpha_n})$ of X, with $\alpha_i \in A(\alpha_1 \cdots \alpha_{i-1})$ for $i = 2, \dots, n$, and usco maps $F_{\alpha_1\cdots\alpha_n} : E_{\alpha_1\cdots\alpha_n} \to C_p(K)$ such that (with the conventions $F_{\alpha_0} = F$, $E_{\alpha_0} = X$ and $A_{\alpha_0} = A$)

(1) $F_{\alpha_1\cdots\alpha_n} \subset F_{\alpha_1\cdots\alpha_{n-1}}|E_{\alpha_1\cdots\alpha_n}$,

(2) $E_{\alpha_1\cdots\alpha_{n-1}} = \bigcup_{\alpha_n \in A(\alpha_1\cdots\alpha_{n-1})} E_{\alpha_1\cdots\alpha_n\alpha_n}$,

(3) norm-diam$[F_{\alpha_1\cdots\alpha_n}(E_{\alpha_1\cdots\alpha_n})] < 1/n$,

(4) $\{E_{\alpha_1\cdots\alpha_n} : \alpha_i \in A(\alpha_1 \cdots \alpha_{i-1}),\ i = 1, \dots, n\}$ is a scattered partition of X for each $n = 1, 2, \dots$.

Choose $y_{\alpha_1\cdots\alpha_n} \in F_{\alpha_1\cdots\alpha_n}(E_{\alpha_1\cdots\alpha_n})$ arbitrarily and define maps $f_n : X \to C_p(K)$ by requiring that f_n take the constant value $y_{\alpha_1\cdots\alpha_n}$ on $E_{\alpha_1\cdots\alpha_n}$ for each member of the partition $(E_{\alpha_1\cdots\alpha_n})$. Then each f_n is a norm PC map by (4) and Lemma 2. Moreover, for any $x \in E_{\alpha_1\cdots\alpha_n}$ and $n, m \geq N$, both $f_n(x)$ and $f_m(x)$ belong to $F_{\alpha_1\cdots\alpha_n}(E_{\alpha_1\cdots\alpha_n})$ by (2), and so $\|f_n(x) - f_m(x)\| < 1/N$ by (3). Thus the sequence $\{f_n\}$ is norm-uniformly Cauchy, and so converges uniformly to some $f : X \to C(K)$. It follows from Lemma 3 that f is a norm PC map. It remains only to show that f is a selector for F.

Let $x \in X$, and choose $\alpha_1, \alpha_2, \ldots$ such that $x \in \bigcap_{n=1}^{\infty} E_{\alpha_1 \cdots \alpha_n}$. Then (1) implies that

$$F(x) \supset F_{\alpha_1}(x) \supset \cdots \supset F_{\alpha_1 \cdots \alpha_n}(x) \supset \cdots,$$

and thus it follows from (3) that $F(x)$ has norm-distance less than $1/n$ from any point in $F_{\alpha_1 \cdots \alpha_n}(E_{\alpha_1 \cdots \alpha_n})$. In particular, $F(x)$ has norm-distance less than $1/n$ to $f_n(x)$. Hence $F(x)$ has norm-distance zero to $f(x)$, and so $f(x) \in F(x)$ as required.

4. Borel Class 1 Selectors

It is natural to ask when the selector in Theorem 2 is Borel measurable and of class 1. Recall that for metric spaces X and Y, a map $f : X \to Y$ is said to be of Borel class 1 if the inverse image of an open set in Y is an F_σ set in X. In more general spaces in which open sets are not necessarily F_σ sets it is more natural to define $f : X \to Y$ to be of *Borel class 1* if the inverse image of every open set in Y is a $(FG)_\sigma$ set in X (that is, a countable union of FG sets). In [H2] a study was made of the relationship between PC maps and Borel class 1 maps (called $(FG)_\sigma$ measurable maps in [H2]). In particular, the following was established [H2, Theorem 2.7]:

(*) Let $f : X \to Y$ be a PC map with Y a metric space. If X has the property that every open collection has a σ-relatively discrete refinement, then f is of Borel class 1.

Recall that a collection of subsets $\mathcal{E}$ of a space X is relatively discrete if the members of $\mathcal{E}$ are pairwise disjoint and each member is open relative to $\bigcup \mathcal{E}$. Spaces X having the covering property in (*) are said to be *hereditarily σ-relatively metacompact* in [DJP]. The weak topology of any Banach space having an equivalent Kadec norm $\|\cdot\|$ (i.e., the norm and weak topologies coincide on the unit sphere for $\|\cdot\|$) has this covering property [H3], as does any compact subset of $C_p(K)$ for any compact Hausdorff K [Y].

It follows that whenever the domain space X in Theorem 2 has this covering property the selector will also be of Borel class 1.

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Mathematics Department, University of Connecticut, Storrs, Connecticut, 06269

E-mail address: `hansell@math.uconn.edu`