# Topology Proceedings

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| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

**TOPOLOGY PROCEEDINGS**
Volume 25, Summer 2000

Pages 539–551

# A GENERALIZATION OF THE CONSTRUCTION OF CONTAINING SPACES

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**Abstract**

In this paper we give a generalization of the construction of containing spaces given in [3].

## 1. Introduction

All spaces considered in the paper are assumed to be $T_0$-spaces of weight less than or equal to a given infinite cardinal denoted by $\tau$. The notions and notation introduced in the paper [3] are assumed to be known.

In [3], for a given collection $\mathbf{S}$ of $T_0$-spaces of weight less than or equal to $\tau$, a containing $T_0$-space denoted by $T(\mathbf{M}, R)$ was constructed, where $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and $R \equiv \{\sim^s \colon s \in \mathcal{F}\}$ is an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. This containing space is uniquely determined by $\mathbf{S}$, $\mathbf{M}$, and R. We note that if for a class $\mathbb{P}$ of spaces for any collection $\mathbf{S}$ of elements of $\mathbb{P}$ by a suitable choosing of the co-mark $\mathbf{M}$ and the family R the containing space $T(\mathbf{M}, R)$ belongs to $\mathbb{P}$, then in $\mathbb{P}$ there exists a universal element. If for "almost all" co-marks $\mathbf{M}$ and families R the containing spaces $T(\mathbf{M}, R)$ belong to $\mathbb{P}$, then $\mathbb{P}$ is said to be saturated.

We recall that $\mathcal{F}$ is the set of all finite subsets of $\tau$ (a cardinal is identified with the set of all ordinals of cardinality less than this cardinal) and the notion of the $\mathbf{M}$-admissibility means that:

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*Mathematics Subject Classification*: 54B99, 54C25.
*Key words*: Containing space, Universal space, Saturated class, $P_\nu$-space.

(1) The family R is admissible, that is, the following conditions are satisfied: (a) $\sim^s = \mathbf{S} \times \mathbf{S}$ if $s = \emptyset \in \mathcal{F}$, (b) $\sim^s \subset \sim^t$ if $t \subset s \in \mathcal{F}$, and (c) the number of $\sim^s$-equivalence classes is finite for every $s \in \mathcal{F}$, and
(2) The family R is a final refinement of the standard family $\mathrm{R}_{\mathbf{M}} \equiv \{\sim^s_{\mathbf{M}}: s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$.

In the present paper we generalize the construction of containing spaces given in [3]. This generalization concerns the above mentioned (indexed) families R and $\mathrm{R}_{\mathbf{M}}$ of equivalence relations on $\mathbf{S}$. First, instead of the indexing set $\mathcal{F}$ we consider here the set $\mathcal{P}_\nu(\tau)$ consisting of all subsets of $\tau$ of cardinality less than a fixed infinite cardinal $\nu$. Furthermore, the elements of the $\mathbf{M}$-standard family is defined here by $\nu$-algebras (see below the definition) instead of algebras in [3]. Finally, the above property (c) of the definition of an admissible family is replaced by the following: the number of $\sim^s$-equivalence classes is less than a fixed infinite cardinal $\mu$. For containing spaces to be $\mathrm{T}_0$-spaces of weight $\leq \tau$ the cardinals $\nu$ and $\mu$ must satisfy some additional conditions.

The given generalization is carried out in parallels to the construction of [3]. The notation, definitions, lemmas, and propositions considered here have a similar correspondence in [3]. Many of them use the same notation and have the same formulation as the corresponding ones though they are related here to the "infinite case". The proofs of almost all lemmas and propositions are ommited. These proofs are similar to the proofs of the corresponding lemmas and propositions of [3].

As it is mensioned in [3] the construction of containing spaces gives a satisfactory answer to problem 7 of A. Arhangel'skiĭ and V. Fedorchuk posed in §3 of [1]. The present generalization can be considered as a complement to this answer. Moreover, this generalization give us the opportunity to consider saturated classes in a more general sense. As an example of a "new" saturated class we consider the class of all $P_\nu$-spaces (see below the definition).

**Agreement.** In the paper we denote by $\nu$ and $\mu$ two fixed infinite cardinals satisfying the following conditions:
(1) The cardinals $\nu$ and $\tau$ are not $\pi$-accessible for any cardinal $\pi$ less than $\nu$ (that is, the sum of less than $\nu$ many cardinals, which are less than $\nu$ or $\tau$, is less than $\nu$ or $\tau$, respectively),
(2) $|\mathcal{P}_\nu(\tau)| \leq \tau$ (therefore, $\nu \leq \tau$), and
(3) $|\mathcal{P}(\mathcal{P}(\pi))| < \mu \leq \tau^+$ for every cardinal $\pi$ less than $\nu$.
It is easy to see that the above conditions are satisfied if $\nu = \omega$ and $\mu$ is an arbitrary cardinal such that $\omega \leq \mu \leq \tau^+$. We note that if $\nu = \mu = \omega$, then the given here construction of containing spaces coincides with that given in [3].

## 2. Marked Spaces

**Definitions.** By a *$\nu$-algebra of subsets of a set* $X$, or briefly by a *$\nu$-algebra* of $X$, we mean a set of subsets of $X$, which is closed under the complements and under the sums of less than $\nu$ many elements of this set. A $\nu$-algebra of $X$ coincides with an algebra of $X$ if $\nu = \omega$.

We have: (a) the empty set and the set $X$ are elements of any $\nu$-algebra of $X$, (b) the set of all subsets of $X$ is a $\nu$-algebra of $X$, and (c) the intersection of any number of $\nu$-algebras of $X$ is also a $\nu$-algebra of $X$. Therefore, any given set of subsets of $X$ is contained in a $\nu$-algebra of $X$. The intersection of all such $\nu$-algebras is called the *minimal $\nu$-algebra* containing the given set of subsets of $X$.

Let $A$ be a $\nu$-algebra of a set $X$ and $i$ be a mapping of $A$ into the set $\mathcal{P}(Y)$ of all subsets of a set $Y$. The mapping $i$ is said to be a *homomorphism* if the following conditions are satisfied: (a) $i(X \setminus U) = Y \setminus i(U)$ for every $U \in A$ and (b) $i(\cup\{U_\lambda : \lambda \in \Lambda\}) = \cup\{i(U_\lambda) : \lambda \in \Lambda\}$ for every set $\Lambda$ of cardinality $< \nu$ and for every elements $U_\lambda$ of $A$. It is easy to verify that if $i$ is a homomorphism, then the set $i(A)$ is a $\nu$-algebra of the set $Y$.

An one-to-one homomorphism is called an *isomorphism*. It is easy to see that if $i$ is an isomorphism of $A$ onto $i(A) \subset \mathcal{P}(Y)$, then the inverse mapping of $i(A)$ onto $A \subset \mathcal{P}(X)$ is also an isomorphism. It is also clear that the composition of two homomorphisms is a homomorphism and, therefore, the composition of two isomorphisms is an isomorphism.

**Definition.** Let $X$ be a marked space with a mark $\{U_\delta^X : \delta \in \tau\}$ and let $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. The minimal $\nu$-algebra of $X$ containing the sets $U_\delta^X$, $\delta \in s$, is called the *s-algebra* of the marked space $X$. This $\nu$-algebra is denoted by $A_s^X$. If $s$ is finite, then $\nu$-algebra $A_s^X$ coincides with the algebra $A_s^X$ of [3].

**Notation.** Let $X$ be a marked space with a mark $\{U_\delta^X : \delta \in \tau\}$ and let $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. For every $\delta \in \tau$ we put

$$X_{(\delta,0)} = U_\delta^X \text{ and } X_{(\delta,1)} = X \setminus U_\delta^X.$$

For every $f \in 2^s$ we put

$$X_{(s,f)} = \cap\{X_{(\delta,f(\delta))} : \delta \in s\}.$$

We denote by $2_X^s$ the set of all $f \in 2^s$ such that $X_{(s,f)} \neq \emptyset$.

For every $\delta \in s$ we put

$$u(X, s, \delta) = \{f \in 2_X^s : f(\delta) = 0\}.$$

For every element $u$ of $\mathcal{P}(2_X^s)$ we put

$$X_{(s,u)} = \cup\{X_{(s,f))} : f \in u\}$$

if $u \neq \emptyset$ and $X_{(s,u)} = \emptyset$ if $u = \emptyset$.

Finally, we denote by $A(2_X^s)$ the set of all elements $u$ of $\mathcal{P}(2_X^s)$ such that $X_{(s,u)} \in A_s^X$ and by $i_X^s$ the mapping of $A(2_X^s)$ into $A_s^X$ such that $i_X^s(u) = X_{(s,u)}$ for every $u \in A(2_X^s)$. In particular, $\emptyset \in A(2_X^s)$ and $i_X^s(\emptyset) = \emptyset \in A_s^X$. Also, for every $f \in 2_X^s$, $\{f\} \in A(2_X^s)$ and $i_X^s(\{f\}) = X_{(s,f)} \in A_s^X$.

**Lemma 2.1.** *Let $X$ be a marked space and $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. Then, the set $A(2_X^s)$ is a $\nu$-algebra of $2_X^s$, $u(X, s, \delta) \in A(2_X^s)$ for every $\delta \in s$, and the mapping $i_X^s$ is an isomorphism of $A(2_X^s)$ onto $A_s^X$ such that*

$$i_X^s(u(X, s, \delta)) = X_{(\delta,0)}$$

*for every $\delta \in s$.*

**Corollary 2.2.** *Let $X$ be a marked space and $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. Then, the set $A(2^s_X)$ is the minimal $\nu$-algebra of $2^s_X$ containing the sets $u(X, s, \delta)$, $\delta \in s$.*

**Definition.** For every $s \in \mathcal{P}_\nu(\tau)$ on the class of all marked spaces an equivalence relation, denoted by $\sim^s_m$, is defined as follows: two marked spaces $X$ and $Y$ is said to be $\sim^s_m$-*equivalent* if either $s = \emptyset$ or $s \neq \emptyset$ and there exists an isomorphism $i$ of $A^X_s$ onto $A^Y_s$, called *natural*, such that $i(X_{(\delta,0)}) = Y_{(\delta,0)}$ for every $\delta \in s$.

It is easy to see that if $\emptyset \neq t \subset s \in \mathcal{P}_\nu(\tau)$, $X$ and $Y$ are two $\sim^s_m$-equivalent marked spaces, and $i$ is the corresponding natural isomorphism of $A^X_s$ onto $A^Y_s$, then $X \sim^t_m Y$ and the restriction to $A^X_t$ of $i$ is the natural isomorphism of $A^X_t$ onto $A^Y_t$, that is, the equivalence relation $\sim^s_m$ is contained in the equivalence relation $\sim^t_m$.

**Lemma 2.3.** *Let $X$ and $Y$ be marked spaces and $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. The following condition are equivalent:*
(1) *$X$ and $Y$ are $\sim^s_m$-equivalent.*
(2) *$2^s_X = 2^s_Y$.*
(3) *$u(X, s, \delta) = u(Y, s, \delta)$ for every $\delta \in s$.*
(4) *$A(2^s_X) = A(2^s_Y)$.*

**Lemma 2.4.** *Let $s$ be an element of $\mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$. Then, the cardinality of the set $C(\sim^s_m)$ of all $\sim^s_m$-equivalence classes is equal to $|\mathcal{P}(2^s)|$.*

*Proof.* By Lemma 2.3, $|C(\sim^s_m)| \leq |\mathcal{P}(2^s)|$. We prove that $|C(\sim^s_m)| = |\mathcal{P}(2^s)|$. For this, by the same lemma it sufficies to prove that for every subset $A \subset 2^s$ there exists a space $X$ and an indexed base $B^X$ for $X$ such that the subset $2^s_X$ of $2^s$ corresponding to the marked space $X$ (with the mark $B^X$) is equal to $A$.

Let $A$ be a subset of $2^s$. We put $X = A$. For every $\delta \in s$ we put

$$u(A, s, \delta) = \{f \in A : f(\delta) = 0\}.$$

On the set $X$ we consider a topology for which the set

$$\{u(A, s, \delta) : \delta \in s\} \cup \{X \setminus u(A, s, \delta) : \delta \in s\}$$

is a subbase for open sets. Since $|s| \leq \tau$ the weight of the constructed space $X$ is $\leq \tau$.

Let $f$ and $g$ be two distinct elements of $X$. Then, there exists an element $\delta \in s$ such that $f(\delta) \neq g(\delta)$. Therefore, either $f(\delta) = 0$ and $g(\delta) = 1$ or $f(\delta) = 1$ and $g(\delta) = 0$. Then, either $f \in u(A, s, \delta)$ and $g \notin u(A, s, \delta)$ or $f \notin u(A, s, \delta)$ and $g \in u(A, s, \delta)$. This means that $X$ is a $\mathrm{T}_0$-space.

Let $B^X \equiv \{U_\delta^X : \delta \in \tau\}$ be a mark of $X$ such that $U_\delta^X = u(A, s, \delta)$ for every $\delta \in s$. By definition, an element $f \in 2^s$ belongs to $2_X^s$ if and only if

$$\cap\{X_{(\delta, f(\delta))} : \delta \in s\} \neq \emptyset.$$

It is easy to verify that the intersection $\cap\{X_{(\delta, f(\delta))} : \delta \in s\}$ is not empty if and only if $f \in A$. Thus, $f \in 2_X^s$ if and only if $f \in A$, which means that $2_X^s = A$. □

**Definition.** Let $X$ be a space and $s \in \mathcal{P}_\nu(\tau) \setminus \emptyset$. For every $x \in X$ there exists a unique element $f$ of $2^s$ such that $x \in X_{(s,f)}$. We define a mapping $d_s^X$ of $X$ into $2^s$ setting $d_s^X(x) = f$. It is supposed that $d_s^X(X) = \emptyset$ if $X = \emptyset$. Obviously, $d_s^X(X) = 2_X^s$.

## 3. Construction of Containing Spaces

**Definition.** Let $\mathbf{S}$ be a collection of spaces and $\mathbf{M}$ a co-mark of $\mathbf{S}$. For every $s \in \mathcal{P}_\nu(\tau)$ we define on $\mathbf{S}$ an equivalence relation denoted by $\sim_{\mathbf{M}}^s$ as follows: two elements $X$ and $Y$ of $\mathbf{S}$ are $\sim_{\mathbf{M}}^s$-equivalent if and only if either $s = \emptyset$ or $s \neq \emptyset$ and the marked spaces $X$ and $Y$ with the marks $\mathbf{M}(X)$ and $\mathbf{M}(Y)$, respectively, are $\sim_m^s$-equivalent. The family

$$\mathrm{R}_{\mathbf{M}}^\nu \equiv \{\sim_{\mathbf{M}}^s : s \in \mathcal{P}_\nu(\tau)\}$$

of equivalence relations on $\mathbf{S}$ is called $(\mathbf{M}, \nu)$-*standard.*

**Definitions.** Let **S** be a collection of spaces. A family

$$\mathrm{R} \equiv \{\sim^s : s \in \mathcal{P}_\nu(\tau)\}$$

of equivalence relations on **S** is said to be *$\mu$-admissible* if the following conditions are satisfied: (a) $\sim^s = \mathbf{S} \times \mathbf{S}$ if $s = \emptyset$, (b) if $s \subset t \in \mathcal{P}_\nu(\tau)$, then the equivalence relation $\sim^t$ is contained in the equivalence relation $\sim^s$, and (c) for every $s \in \mathcal{P}_\nu(\tau)$ the number of $\sim^s$-equivalence classes is less than $\mu$. The set $\cup\{\mathrm{C}(\sim^s) : s \in \mathcal{P}_\nu(\tau)\}$ is denoted by $\mathrm{C}(\mathrm{R})$.

By Lemmas 2.3 and 2.4 and the condition (3) of the Agreement the $(\mathbf{M}, \nu)$-standard family of equivalence relations on **S** is $\mu$-admissible.

Let $\mathrm{R}_0 \equiv \{\sim_0^s : s \in \mathcal{P}_\nu(\tau)\}$ and $\mathrm{R}_1 \equiv \{\sim_1^s : s \in \mathcal{P}_\nu(\tau)\}$ be two $\mu$-admissible families of equivalence relations on **S**. It is said that $\mathrm{R}_1$ is a *final refinement* of $\mathrm{R}_0$ if for every $s \in \mathcal{P}_\nu(\tau)$ there exists an element $t$ of $\mathcal{P}_\nu(\tau)$ such that $\sim_1^t \subset \sim_0^s$.

A $\mu$-admissible family R of equivalence relations on **S** is said to be *$(\mathbf{M}, \nu, \mu)$-admissible* if R is a final refinement of the $(\mathbf{M}, \nu)$-standard family $\mathrm{R}_{\mathbf{M}}^{\nu}$.

In what follows it is assumed that an arbitrary non-empty collection of spaces (of weight $\leq \tau$) denoted by **S** is given.

It is also assumed that an arbitrary co-mark of **S** is given. This co-mark is denoted by **M**. Moreover, we assume that for every $X \in \mathbf{S}$,

$$\mathbf{M}(X) = \{U_\delta^X : \delta \in \tau\}.$$

Finally, it is assumed that an $(\mathbf{M}, \nu, \mu)$-admissible family of equivalence relations on **S** is given. This family is denoted by

$$\mathrm{R} \equiv \{\sim^s : s \in \mathcal{P}_\nu(\tau)\}.$$

**Definition.** Suppose that **S** contains a non-empty element and consider the set of all pairs $(x, X)$, where $X \in \mathbf{S}$ and $x \in X$. On this set an equivalence relation denoted by $\sim_{\mathbf{M}}^{\mathrm{R}}$ is defined as follows: two pairs $(x, X)$ and $(y, Y)$ are *$\sim_{\mathbf{M}}^{\mathrm{R}}$-equivalent* if

$$X \sim^s Y \text{ and } d_s^X(x) = d_s^Y(y)$$

for every $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$.

**Notation.** The set of all $\sim^{\mathrm{R}}_{\mathbf{M}}$-equivalence classes is denoted by $\mathrm{T}(\mathbf{M},\mathrm{R}) \equiv \mathrm{T}$. It is supposed that $\mathrm{T}(\mathbf{M},\mathrm{R}) = \emptyset$ if all elements of $\mathbf{S}$ are empty.

For every element $\mathbf{H}$ of $\mathrm{C}(\mathrm{R})$ the set of all $\mathbf{a} \in \mathrm{T}$ for which there exists an element $(x, X)$ of $\mathbf{a}$ such that $X \in \mathbf{H}$ is denoted by $\mathrm{T}(\mathbf{M},\mathrm{R},\mathbf{H}) \equiv \mathrm{T}(\mathbf{H})$. It is easy to see that if $\mathbf{a} \in \mathrm{T}(\mathbf{H})$, then for every $(x, X) \in \mathbf{a}$, $X \in \mathbf{H}$.

**Notation.** Let $s \neq \emptyset$ and $t$ be elements of $\mathcal{P}_\nu(\tau)$ such that $\sim^t \subset \sim^s_{\mathbf{M}}$ and let $\mathbf{H}$ be an element of $\mathrm{C}(\sim^t)$. Suppose that an element $X$ of $\mathbf{H}$ is chosen. We denote by $2^s_{\mathbf{H}}$ and $A(2^s_{\mathbf{H}})$ the sets $2^s_X$ and $A(2^s_X)$, respectively, and for every $\delta \in s$ by $u(\mathbf{H}, s, \delta)$ we denote the set $u(X, s, \delta)$. Since $X \sim^s_{\mathbf{M}} Y$ for every $X, Y \in \mathbf{H}$, the sets $2^s_{\mathbf{H}}$, $A(2^s_{\mathbf{H}})$, and $u(\mathbf{H}, s, \delta)$ are independent of the element of $\mathbf{H}$ we choose to define them. (See Lemma 2.3).

For every $u \in A(2^s_{\mathbf{H}})$ we denote by $\mathrm{T}_{(s,u)}(\mathbf{H})$ the set of all elements $\mathbf{a}$ of $\mathrm{T}$ for which there exists an element $(x, X)$ of $\mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in X_{(s,u)}$.

Finally we put

$$A^{\mathbf{H}}_s = \{\mathrm{T}_{(s,u)}(\mathbf{H}) : u \in A(2^s_{\mathbf{H}})\}.$$

Since $\mathrm{T}_{(s,u)}(\mathbf{H})$ is a subset of $\mathrm{T}(\mathbf{H})$, $A^{\mathbf{H}}_s$ can be considered as a set of subsets of $\mathrm{T}(\mathbf{H})$.

**Lemma 3.1.** *Let $s, t \in \mathcal{P}_\nu(\tau)$, $s \neq \emptyset$, $\sim^t \subset \sim^s_{\mathbf{M}}$, $\mathbf{H} \in \mathrm{C}(\sim^t)$, and $u \in A(2^s_{\mathbf{H}})$. Then, the set $\mathrm{T}_{(s,u)}(\mathbf{H})$ coincides with the set of all $\mathbf{a} \in \mathrm{T}$ such that for every $(x, X) \in \mathbf{a}$ we have $X \in \mathbf{H}$ and $x \in X_{(s,u)}$.*

**Lemma 3.2.** *Let $s, t \in \mathcal{P}_\nu(\tau)$, $s \neq \emptyset$, $\sim^t \subset \sim^s_{\mathbf{M}}$, and $\mathbf{H} \in \mathrm{C}(\sim^t)$. Let also $u, v_\lambda \in A(2^s_{\mathbf{H}})$, where $\lambda \in \Lambda$ and $|\Lambda| < \nu$. Then, setting $w = 2^s_{\mathbf{H}} \setminus u$ and $v = \cup\{v_\lambda : \lambda \in \Lambda\}$, we have*

$$\mathrm{T}(\mathbf{H}) \setminus \mathrm{T}_{(s,u)}(\mathbf{H}) = \mathrm{T}_{(s,w)}(\mathbf{H}) \text{ and}$$
$$\mathrm{T}_{(s,v)}(\mathbf{H}) = \cup\{\mathrm{T}_{(s,v_\lambda)}(\mathbf{H}) : \lambda \in \Lambda\}.$$

**Corollary 3.3.** *If $s, t \in \mathcal{P}_\nu(\tau)$, $s \neq \emptyset$, $\sim^t \subset \sim^s_{\mathbf{M}}$, and $\mathbf{H} \in \mathrm{C}(\sim^t)$, then the set $A^{\mathbf{H}}_s$ is a $\nu$-algebra of the set $\mathrm{T}(\mathbf{H})$.*

**Lemma 3.4.** *Let $s, t \in \mathcal{P}_\nu(\tau)$, $s \neq \emptyset$, $\sim^t \subset \sim^s_{\mathbf{M}}$ and $\mathbf{H} \in \mathrm{C}(\sim^t)$. Then, the mapping $i^s_{\mathbf{H}}$ of the set $A(2^s_{\mathbf{H}})$ onto the set $A^{\mathbf{H}}_s$, for which $i^s_{\mathbf{H}}(u) = \mathrm{T}_{(s,u)}(\mathbf{H})$ for every $u \in A(2^s_{\mathbf{H}})$, is an isomorphism.*

**Corollary 3.5.** *Let $s, t \in \mathcal{P}_\nu(\tau)$, $s \neq \emptyset$, $\sim^t \subset \sim^s_{\mathbf{M}}$, $\mathbf{H} \in \mathrm{C}(\sim^t)$, and $X \in \mathbf{H}$. Then, the mapping $i \equiv i^s_{\mathbf{H}} \circ (i^s_X)^{-1}$ is an isomorphism of $A^X_s$ onto $A^{\mathbf{H}}_s$ such that for every $\delta \in s$, $i(X_{(\delta,0)}) = \mathrm{T}_{(s,u)}(\mathbf{H})$, where $u = u(\mathbf{H}, s, \delta)$.*

**Notation.** For every $\delta \in \tau$ and $\mathbf{H} \in \mathrm{C}(\mathrm{R})$ we denote by $U^{\mathrm{T}}_\delta(\mathbf{H})$ the set of all $\mathbf{a} \in \mathrm{T}$ for which there exists an element $(x, X)$ of $\mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in U^X_\delta$. The set of all such subsets of T is denoted by $B^{\mathrm{T}}$.

Let $\kappa$ be a subset of $\tau$. The set of all elements $U^{\mathrm{T}}_\delta(\mathbf{H})$ of $B^{\mathrm{T}}$ for which $\delta \in \kappa$ is denoted by $B^{\mathrm{T}}_\kappa$.

**Lemma 3.6.** *Suppose that $\kappa$ is a subset of $\tau$ such that for every $X \in \mathbf{S}$ the set*

$$\{U^X_\delta : \delta \in \kappa\}$$

*is a base for $X$. Then, the set $B^{\mathrm{T}}_\kappa$ is a base for a topology on the set $\mathrm{T}(\mathbf{M}, \mathrm{R})$. Moreover, the set $B^{\mathrm{T}}$ is a base for the same topology.*

**Proposition 3.7.** *The space T is a $\mathrm{T}_0$-space of the weight $\leq \tau$.*

**Notation.** Let $X \in \mathbf{S}$. For every $x \in X$ there exists a unique element $\mathbf{a}$ of T, which contains the pair $(x, X)$. We denote by $e^X_{\mathrm{T}}$ the mapping of $X$ into T such that

$$e^X_{\mathrm{T}}(x) = \mathbf{a}.$$

**Proposition 3.8.** *For every $X \in \mathbf{S}$ the mapping $e^X_{\mathrm{T}}$ is an embedding of the space $X$ into the space* T.

Thus, the set T equiped with the topology, for which the set $B^{\mathrm{T}}$ is a base for open subsets, is a *containing space for the collection* **S** *corresponding to the co-mark* **M** *and the family* R.

## 4. The$(\nu, \mu)$-saturated Classes

**Definition.** A class $I\!P$ of spaces is said to be $(\nu, \mu)$-*saturated* if for every collection $\mathbf{S}$ of spaces belonging to $I\!P$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $(\mathbf{M}, \nu, \mu)$-admissible family $\mathrm{R}^+$ of equivalence relations on $\mathbf{S}$ such that for every $\mu$-admissible family $\mathrm{R}$ of equivalence relations on $\mathbf{S}$, which is a final refinement of $\mathrm{R}^+$, the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ belongs to $I\!P$.

We note that the notion of $(\omega, \omega)$-saturated class coincides with the notion of a saturated class given in [3].

**Proposition 4.1.** *In any non-empty $(\nu, \mu)$-saturated class of spaces there exists a universal element.*

**Proposition 4.2.** *The intersection of not more than $\tau$ many $(\nu, \mu)$-saturated classes is a $(\nu, \mu)$-saturated class.*

**Proposition 4.3.** *The class of all regular spaces and the class of all completely regular spaces are $(\nu, \mu)$-saturated classes.*

## 5. The Class of $P_\nu$-spaces

**Definition.** A space $X$ is said to be a $P_\nu$-*space* if any intersection of less than $\nu$ many open subsets of $X$ is open. (See, for example, [2]). It is easy to see that a space $X$ is a $P_\nu$-space if and only if for some (every) base $B$ for open subsets of $X$ any intersection of less than $\nu$ many elements of $B$ is open.

**Lemma 5.1.** *There exists an one-to-one mapping $\chi$ of $\mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$ into $\tau$ having the following property: for every space $X$ any base $B^X$ for $X$ of cardinality $\leq \tau$ has an indication $N^X \equiv \{V^X_\varepsilon : \varepsilon \in \tau\}$ such that $V^X_{\chi(s)} = \cap\{V^X_\varepsilon : \varepsilon \in s\}$ for every $s \in \mathcal{P}_\nu(\tau) \setminus \{\emptyset\}$.*

*Proof.* For every ordinal $\delta$ we denote by $\mathcal{P}_\nu(\delta \times \tau)$ the set of all subsets of the set $\delta \times \tau$ of cardinality $< \nu$ and by $\mathcal{P}^\delta_\nu(\delta \times \tau)$ the set consisting of all elements $s$ of $\mathcal{P}_\nu(\delta \times \tau)$ such that for every $\delta_0 \in \delta$ there exists an element $(\delta_1, \eta) \in s$ for which $\delta_0 \leq \delta_1$. Since $\tau$ is not $\nu$-accessible (see the Agreement) for every element $s$ of $\mathcal{P}_\nu(\tau \times \tau)$ there exists a unique element $\delta$ of $\tau$ such that $s \in \mathcal{P}^\delta_\nu(\delta \times \tau)$. Thus, we have

$$\mathcal{P}_\nu(\tau \times \tau) = \cup\{\mathcal{P}^\delta_\nu(\delta \times \tau) : \delta \in \tau\} \text{ and}$$
$$\mathcal{P}^{\delta_0}_\nu(\delta_0 \times \tau) \cap \mathcal{P}^{\delta_1}_\nu(\delta_1 \times \tau) = \emptyset$$

for every $\delta_0, \delta_1 \in \tau$, $\delta_0 \neq \delta_1$. (The second relation follows by the definition of the including sets).

By the assumptions of the Agreement it follows that $|\mathcal{P}^\delta_\nu(\delta \times \tau)| = \tau$, $\delta \neq 0$. Therefore, for every $\delta \in \tau \setminus \{0\}$ there exists an one-to-one mapping $\chi_\delta$ of $\mathcal{P}^\delta_\nu(\delta \times \tau)$ onto the set $\{\delta\} \times \tau$.

For the definition of $\chi$ we consider an one-to-one mapping $\psi$ of $\tau \times \tau$ onto $\tau$. Let $s$ be a non-empty element of $\mathcal{P}_\nu(\tau)$. There exists an element $\delta \neq 0$ of $\tau$ such that $\psi^{-1}(s) \in \mathcal{P}^\delta_\nu(\delta \times \tau)$. We set $\chi(s) = \psi(\chi_\delta(\psi^{-1}(s)))$. It is easy to verify that $\chi$ is one-to-one.

Now, let $B^X$ be a base for a space $X$ of cardinality $\leq \tau$. Let $\{N^X_{(0,\eta)} : \eta \in \tau\}$ be an indication of $B^X$. For every $\delta \in \tau \setminus \{0\}$ we define by induction on $\delta$ an indexed family $\{V^X_{(\delta,\eta)} : \eta \in \tau\}$ of elements of $B^X$ setting

$$V^X_{(\delta,\eta)} = \cap\{V^X_{(\delta',\eta')} : (\delta', \eta') \in \chi^{-1}_\delta(\delta, \eta)\}.$$

The indication $N^X$ of $B^X$ is defined setting $V^X_\varepsilon = V^X_{\psi^{-1}(\varepsilon)}$. It is easy to verify that this indication and the mapping $\chi$ satisfy the condition of the lemma. □

**Proposition 5.2.** *The class of all $P_\nu$-spaces is $(\nu, \mu)$-saturated.*

*Proof.* Let $\mathbf{S}$ be a collection of $P_\nu$-spaces. For every $X \in \mathbf{S}$ we consider the indexed base $N^X \equiv \{V^X_\varepsilon : \varepsilon \in s\}$ constructed in Lemma 4.1. Denote by $\mathbf{M}^+$ the co-mark of $\mathbf{S}$ such that $\mathbf{M}^+(X) = N^X$, that is, $\mathbf{M}^+ = \{N^X : X \in \mathbf{S}\}$. Let $\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$ be an arbitrary co-extension of $\mathbf{M}^+$. Denote by $\theta$ an indicial mapping of this co-extension.

Let $\mathrm{R}^+$ be an $(\mathbf{M},\nu,\mu)$-admissible family of equivalence relations on $\mathbf{S}$. Denote by $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{P}_\nu(\tau)\}$ an arbitrary $(\mathbf{M},\nu)$-admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of $\mathrm{R}^+$.

To prove the proposition it suffices to prove that the containing space $\mathrm{T} \equiv \mathrm{T}(\mathbf{M},\mathrm{R})$ is a $P_\nu$-space. For this, we consider the base $B^{\mathrm{T}}_{\theta(\tau)}$ of the containing space T and prove that any intersection of less than $\nu$ many elements of this base is open. Indeed, let $\{U^{\mathrm{T}}_{\delta_\gamma}(\mathbf{H}_\gamma) : \gamma \in \Gamma\}$ be a set of elements of $B^{\mathrm{T}}_{\theta(\tau)}$, where $|\Gamma| < \nu$. Set $s = \{\delta_\gamma : \gamma \in \Gamma\}$. For every $\gamma \in \Gamma$ denote by $t_\gamma$ an element of $\mathcal{P}_\nu(\tau)$ such that $\mathbf{H}_\gamma$ is a $\sim^{t_\gamma}$-equivalence class and set $t = \cup\{t_\gamma : \gamma \in \Gamma\}$. Obviously, $s$ and $t$ are elements of $\mathcal{P}_\nu(\tau)$.

It is easy to prove that the set $\mathbf{L} \equiv \cap\{\mathbf{H}_\gamma : \gamma \in \Gamma\}$ is the union of some $\sim^t$-equivalence classes. Then, we have

$(*) \cap\{U^{\mathrm{T}}_{\delta_\gamma}(\mathbf{H}_\gamma) : \gamma \in \Gamma\} = \cup\{U^{\mathrm{T}}_{\chi(s)}(\mathbf{H}) : \mathbf{H} \in \mathrm{C}(\sim^t) \text{ and } \mathbf{H} \subset \mathbf{L}\}.$

Indeed, if $\mathbf{a}$ is an element of the left side of $(*)$ and $(x,X) \in \mathbf{a}$, then $x \in U^X_{\delta_\gamma}$ for every $\gamma \in \Gamma$ and $X \in \mathbf{L}$. Therefore,

$$x \in \cap\{U^X_{\delta_\gamma} : \gamma \in \Gamma\} = U^X_{\chi(\delta)},$$

which means that $\mathbf{a} \in U^{\mathrm{T}}_\delta(\mathbf{H})$, where $\mathbf{H}$ is the $\sim^t$-equivalence class containing $X$, that is, $\mathbf{a}$ belongs to the right side of $(*)$. The converse inclusion is proved similarly. Thus, any intersection of less than $\nu$ many elements of $B^{\mathrm{T}}_{\theta(\tau)}$ is open. This completes the proof of the proposition. □

**Corollary 5.3.** *The class of all regular $P_\nu$-spaces is $(\nu,\mu)$-saturated.*

## 6. Some Problems

A necessary condition for a class $I\!\!P$ of spaces to be saturated is the existence of universal elements in this class. Therefore, to prove that a class of spaces is not saturated it suffices to prove that in this class there are no universal elements. However, it is "difficult" to prove that a class is not saturated if in this class there exists a universal element. In this connection we pose the following problems.

(1) Suppose that $(\omega,\omega) \neq (\nu,\mu)$. Is there a saturated class, which is not $(\nu,\mu)$-saturated?

(2) Suppose that $(\omega,\omega) \neq (\nu,\mu)$. Is there a $(\nu,\mu)$-saturated class, which is not saturated? In particular, for $\omega \neq \nu$, is the class of all (regular) $P_\nu$-spaces saturated?

The more general problem is the following.

(3) Suppose that $(\nu_0,\mu_0)$ and $(\nu_1,\mu_1)$ are distinct pairs satisfying the conditions (1)$-$(3) of the Agreement. Is there a class of spaces, which is $(\nu_0,\mu_0)$-saturated but not $(\nu_1,\mu_1)$-saturated?

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