Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Summer 2000

Pages 575–587

ON WEAK REFLECTIONS IN SOME SUPERCLASSES OF COMPACT SPACES I

Martin Maria Kovár*

Abstract

We characterize and study the weak reflections of general topological spaces in classes that contain the class of compact spaces and that are contained in the class of θ-regular spaces in terms of extensions and remainders. In particular, we show that there is no such a weakly reflective subclass of topological spaces.

Notation and Terminology

The main source of the definitions of most used standard topological notions and notions related to the construction of the Wallman compactification is A. Császár's book [Cs]. By a space we always mean a topological space. All spaces are assumed with no separation axioms in general. Especially, compactness, paracompactness and their modifications are understood without T_2 or any other separation axiom. Let Ξ be a collection of sets. We denote by Ξ^F the family of all finite unions of members of Ξ. A family Ξ is called *directed* if each element of Ξ^F is contained in an element of Ξ. Let X be a space. A filter base Φ in X has a *θ-cluster point* $x \in X$ if every closed

* This research is supported by the grant no. 201/00/1466 of the Grant Agency ofthe Czech Republic and by the research intention of the Ministry of Education of the Czech Republic CEZ: J22/98:262200012.

Mathematics Subject Classification: 54C20, 54D35, 54D40.

Key words: Weak reflection, extensions, remainders, the Wallman compactification, θ^n-operator and other θ-notions, θ-regularity.

neighborhood of x and every element of Φ have a nonempty intersection. We will say that $H \subseteq X$ is a *θ^n-neighborhood* of $x \in X$ for $n \in \mathbb{N}$ if there exist open $U_1, U_2, \dots, U_n$ such that $x \in U_1 \subseteq \operatorname{cl} U_1 \subseteq U_2 \subseteq \operatorname{cl} U_2 \subseteq \cdots \subseteq U_n \subseteq \operatorname{cl} U_n \subseteq H$. A filter base Φ has a *θ^n-cluster point* $x \in X$ for some $n \in \mathbb{N}$ if every θ^n-neighborhood of x meets every element of Φ. We leave to the reader to show that if $f : X \to Y$ is a continuous mapping of topological spaces and Φ has a θ^n-cluster point $x \in X$ then $f(x) \in Y$ is a θ^n-cluster point of the image filter base $f(\Phi)$. For further interesting properties of the θ^n-operator the reader is referred to [DG_1] and more generally to the other papers of Dikranjan, Giuli and Tholen. Recall that a space X is said to be *θ-regular* [Ja] if every filter base in X with a θ-cluster point has a cluster point. For the detailed characterization of θ-regular spaces the reader is referred to Janković's paper [Ja] or to some of the author's more recent papers (see the references). Here we mention only a few basic facts regarding θ-regular spaces. For example, compact, paracompact or regular spaces are θ-regular. There are more alternative definitions of local compactness. In this paper we say that a space is *(strongly) locally compact* if its every point has a compact (closed) neighborhood. Strongly locally compact spaces are θ-regular. A product of any family of θ-regular spaces as well as a closed subspace of a θ-regular space is θ-regular. In Hausdorff spaces, θ-regularity coincides with regularity. A space X is said to be *almost compact* if every open filter base in X has a cluster point [Cs].

A filter in a space X is said to be *ultra-closed* (*ultra-open*, respectively) if it is maximal among all filters in X having a base consisting of closed (open, respectively) sets [Cs]. By the Wallman compactification of X we mean the set $\omega X = X \cup \{y \mid y$ is a non-convergent ultra-closed filter in $X\}$. The sets $\mathcal{S}(U) = U \cup \{y \mid y \in \omega X \smallsetminus X, U \in y\}$, where U is open in X, constitute an open base of ωX (see [Cs]). We say that a subset $A \subseteq X$ is *discrete* (in a space X) if for every $x \in X$ there is a neighborhood U of x that contains at most one element of A. In particular, since the singletons of the Wallman remainder are closed in ωX, a subset $A \subseteq \omega X \setminus X$ is discrete in ωX if and only if A is a closed discrete subspace of ωX. If X is a θ-regular space, then every two points $x \in X$, $y \in \omega X \setminus X$ have disjoint neighborhoods in ωX. Conversely, if one splits a compact, or more generally, a θ-regular space K into two disjoint subspaces X, Y (that is, $X \cup Y = K$, $X \cap Y = \varnothing$) such that every two points $x \in X$, $y \in Y$ have disjoint neighborhoods in K, then both the subspaces X, Y are θ-regular (see [Ko_1] and [Ko_3]).

By Top we mean the class of all topological spaces, by Comp we denote its subclass of all compact spaces. The subclass of Top consisting of all θ-regular spaces we denote by $\Theta\mathsf{Reg}$. Obviously, $\mathsf{Comp} \subsetneq \Theta\mathsf{Reg}$. Let $\mathsf{K} \subseteq \mathsf{Top}$ be some class of topological spaces. We say that a space X has a *weak reflection* in K if there exists a space $kX \in \mathsf{K}$ and a continuous mapping $k : X \to kX$ such that for every $Y \in \mathsf{K}$ and any continuous mapping $f : X \to Y$ there exists a continuous mapping $g : kX \to Y$ with $g \circ k = f$. If $\mathsf{Comp} \subseteq \mathsf{K}$ then $k : X \to kX$ is an embedding and we can, for the simplicity, suppose that $X \subseteq kX$. A subclass $\mathsf{K} \subseteq \mathsf{Top}$ is said to be *weakly reflective* [AR] if it is closed under retracts and every space X has a weak reflection in K. One can easily check that weakly reflective subclasses of Top are closed under the products.

1. Introduction

The problem standing at the beginning perhaps is due to Z. Frolík who – about 35 years ago – in some oral communication mentioned a simple and very natural question but which fascinated some topologists for years: *Is there a compactification γX of a space X such that every continuous mapping from X into any compact space Y can be continuously extended to γX?* In other words: *Is the class of compact spaces weakly reflective in the class of topological spaces?* It is extremely difficult to mention all people who worked in the related topics. It was, for example – in the alphabetic order – J. Adámek, A. Dow, H.

Herrlich, M. Hušek, J. Rosický, S. Salbany, S. Todorćevič and S. Watson but perhaps also some other mathematicians. Now, the question is stated explicitly in [AR] and [He_1]. S. Todorćevič posed a modified problem [DW]: *Does there exist a space U such that every space X has a compactification γX embeddable into a power of U such that every continuous mapping from X into a compact T_1 space has a continuous extension onto γX?* In 1991 M. Hušek answered both problems in the negative. In fact, Hušek's main results were the following [Hu]:

Theorem 1.1. (Hušek) *Let X be a space. Then:*

(1) *If the Wallman remainder of X is finite, then the Wallman compactification of X is the weak reflection of X in compact spaces.*

(2) *If X contains an infinite family $\{X_n\}$ of closed noncompact subsets such that $X_n \cap X_m$ is compact for $n \neq m$ then X has no weak reflection in compact spaces.*

(3) *A normal T_1 space has a weak reflection in compact spaces iff its Čech-Stone remainder is finite.*

Regarding the weak reflections in compact spaces now all seemed to be done. In 1994 I tried to construct few simple examples of spaces with or without weak reflections in **Comp**. But all my examples satisfied either the condition (1) or the condition (2). In other words, I was not able to find a space with the infinite Wallman remainder which did not contain a family $\{X_n\}$. M. Hušek in [Hu] (in a joint discussion with colleagues from Univ. of Kansas) derived that if the Wallman remainder of X contains an infinite discrete subspace, then X contains a family $\{X_n\}$ and hence has no weak reflection in compact spaces. In [Ko_2] I proved a similar result regarding the case that the Wallman remainder of X contains an infinite subspace with the co-finite topology. But the Wallman remainder of any space always is T_1 and it is not very difficult to show that every infinite

T_1 space must contain an infinite subspace, which is either discrete or co-finite. Hence in [Ko$_2$] I was able to improve Hušek's results (1) – (3) to the following theorem (but a decent portion of the work were already done by Hušek in his results (1) – (3)):

Theorem 1.2. *A space has a weak reflection in compact spaces iff its Wallman remainder is finite.*

2. Between Compactness and θ-Regularity

The spaces having a weak reflection in compact spaces are fully characterized by Theorem 1.2. Nevertheless, new unsolved problems will arise when one replace compact spaces by another classes. M. Hušek proved that the class of compact spaces is not weakly reflective in **Top** but it is natural to study the weak reflections "below" as well as "above" compactness. In this work we will study the weak reflections of general topological spaces in the subclasses **K** of **Top** satisfying **Comp** $\subseteq$ **K** $\subseteq$ Θ**Reg**. We will start with a characterization of some spaces that have a weak reflection in θ-regular spaces. At first, we need a "θ-regularization" of any space with some features analogous to the Wallman compactification. We introduce it in the following proposition.

Proposition 2.1. *Let X be a space, $\vartheta X = X \cup \{y|\, y$ is a non-convergent ultra-closed filter in X with a θ^n-cluster point, $n \in \mathbb{N}\}$ be a subspace of the Wallman compactification of X. Then ϑX is θ-regular. In particular, for a θ-regular space it follows $\vartheta X = X$.*

Proof. Let $y \in \vartheta X$ and $z \in \omega X \setminus \vartheta X$. Then there exists some θ^n-cluster point $x \in X$ of of y. Let H be any θ^{n+1}-neighborhood of x. Then there exist U_1, U_2, $\ldots$, U_{n+1} open in X such that $x \in U_1$, $\operatorname{cl} U_i \subseteq U_{i+1}$ for $i = 1, 2, \ldots, n$ and $\operatorname{cl} U_{n+1} \subseteq H$. Hence, there is some $F \in y$ such that $F \subseteq \operatorname{cl} U_n \subseteq U_{n+1}$ since $\operatorname{cl} U_n$ is a θ^n-neighborhood of x and y is ultra-closed. On the other hand, x is not a θ^{n+1}-cluster point of z. Therefore there exists

$G \in z$ such that $G \cap H = \varnothing$ which implies that $G \subseteq X \setminus \operatorname{cl} U_{n+1}$. Now, it follows that $\mathcal{S}(U_{n+1})$ and $\mathcal{S}(X \setminus \operatorname{cl} U_{n+1})$ are the disjoint neighborhoods of y and z in ωX. Hence, it follows that ϑX is θ-regular.

Suppose that X is θ-regular. We show that $\vartheta X \setminus X = \varnothing$. Let $y \in \omega X$ be a non-convergent ultra-closed filter in X. Then $\Omega = \{X \setminus G | \, G \in y\}$ is an open directed cover of X. Since X is θ-regular, then for any open directed cover Φ of X there is an open directed cover Ψ of X such that the family $\operatorname{cl} \Psi$ (of the closures of Ψ) refines Φ. Now it can be easily seen that for any $x \in X$ and $n \in \mathbb{N}$ there is a θ^n-neighborhood H of X such that $H \subseteq X \setminus G$ for some $G \in y$. It follows that $H \cap G = \varnothing$, which implies that y has no θ^n-cluster point in X and hence $y \notin \vartheta X$. It follows that $\vartheta X \setminus X = \varnothing$. □

Now we can formulate the theorem.

Theorem 2.2. Let X be a space such that the remainder $\vartheta X \setminus X$ is discrete in ϑX. Then ϑX is a weak reflection of X in θ-regular spaces.

Proof. Let $f : X \to Y$ be a continuous mapping, Y be a θ-regular space. Let $y \in \vartheta X \setminus X$. We show that the filter base $f(y)$ has a limit point in Y. Let $x \in X$ be a θ^n-cluster point of y for some $n \in \mathbb{N}$ in X. Then $z = f(x)$ is a θ^n-cluster point of $f(y)$ in Y. Since Y is θ-regular it follows that $f(y)$ has a cluster point $\tilde{y} \in Y$. Let U be an open neighborhood of $\tilde{y}$. Then $f^{-1}(U)$ is open in X. Since y is ultra-closed, it follows that either $X \setminus f^{-1}(U) \in y$ or $f^{-1}(U) \in y$. But $U \cap f(X \setminus f^{-1}(U)) = \varnothing$ which implies that $f^{-1}(U) \in y$. It follows that $U \in f(y)$ and hence $\tilde{y}$ is a limit point of $f(y)$.

Now, we put $\tilde{f}(y) = \tilde{y}$ for every $y \in \vartheta X \setminus X$ and $\tilde{f}(x) = f(x)$ for every $x \in X$. Suppose that the remainder $\vartheta X \setminus X$ is discrete in ϑX. We only show that then the extension mapping $\tilde{f} : \vartheta X \to Y$ of f is continuous. Obviously, X is open in ϑX so f is continuous at every point $x \in X$. Let $y \in \vartheta X \setminus X$. There exists some open $U \subseteq X$ such that $\mathcal{S}(U) \cap (\vartheta X \setminus X) = \{y\}$. Let

$Q \subseteq Y$ be any open set in Y containing $\tilde{f}(y) = \tilde{y}$. Obviously, there is some $F \in y$ such that $f(F) \subseteq Q$. Then $F \subseteq f^{-1}(Q)$ and so $f^{-1}(Q) \in y$. We put $L = U \cap f^{-1}(Q)$. Since $U \in y$ it follows that $L \in y$ which gives $y \in \mathcal{S}(L)$. Let $t \in \mathcal{S}(L)$. If $t \in L$ then $t \in f^{-1}(Q) \subseteq X$ which implies $\tilde{f}(t) = f(t) \in Q$. If $t \notin L$ then $t \in \mathcal{S}(L) \cap (\vartheta X \smallsetminus X) \subseteq \mathcal{S}(U) \cap (\vartheta X \smallsetminus X) = \{y\}$, so $t = y$. Then $\tilde{f}(t) = \tilde{f}(y) = \tilde{y} \in Q$. It follows that $\tilde{f}(\mathcal{S}(L)) \subseteq Q$ which means that $\tilde{f}$ is continuous at y. Hence $\tilde{f} : \vartheta X \to Y$ is a mapping continuously extending X, so ϑX is a weak reflection of X in θ-regular spaces. □

Further interesting properties of ϑX and its relations to ωX will be studied in a separate paper. Note that it is still unknown whether there exists a space X having a weak reflection in θ-regular spaces such that the remainder $\vartheta X \smallsetminus X$ is not discrete in ϑX. The construction that used the author and M. Hušek to get the result in Theorem 1.2 does not work completely for θ-regular spaces (but it works for strongly locally compact spaces as we show in the forthcoming paper).

Example 2.3. Let X be any regular space with the infinite Wallman remainder (one can take $X = \mathbb{R}$ with the Euclidean topology, for example). Then X is a weak reflection of itself in θ-regular spaces, but X has no weak reflection in compact spaces.

The next proposition is a modification of Hušek's result (2) in Theorem 1.1 mentioned in the introductory section. It gives a characterization of some spaces having no weak reflection in K. But first we need the following two lemma.

Lemma 2.4. *Let X be a dense subspace of a space Y and let every filter base which is open in X has a cluster point in Y. Then Y is almost compact.*

Proof. Obviously, if Ω is an open filter base in Y, the family $\Omega' = \{U \cap X | U \in \Omega\}$ is an open filter base in X having a cluster point by the assumption. But this point is also a cluster point of Ω. □

In fact, the proof of the second lemma is a reformulation of the essential part of the proof of Hušek's result that we mentioned above. We repeat it briefly only because of completeness.

Lemma 2.5. *Let X contain an infinite family $\{X_n\}$ of closed noncompact subsets such that $X_n \cap X_m$ is compact for $n \neq m$ and let κ be any infinite cardinal number. Then there exist a compact space $Z_\kappa(X)$ such that X is dense in $Z_\kappa(X)$ and every compact subspace $Y \subseteq Z_\kappa(X)$ containing X has the cardinality at least κ.*

Sketch of Proof. For any infinite cardinal κ we define $Z_\kappa = X \cup (\kappa \times \omega)$. Let $\{N_n\}_{n=1}^{\infty}$ be a partition of ω with $|N_n| = \omega$ for every $n \in \omega$. The topology on Z_κ will be defined transfinitely on κ such that

$$\alpha < \kappa \Rightarrow X \cup (\alpha \times \omega) \text{ is open in } Z_\kappa :$$

(1) A neighborhood base of $(0,n)$ is composed of the sets $\{(0,n)\} \cup (X \setminus (C \cup \bigcup_{i \in K} X_i))$ for finite $K \subseteq \omega$ and for closed compact sets C in X.

(2) A neighborhood base of $(\beta+1,n)$ is composed of the sets $\{(\beta+1,n)\} \cup \bigcup \{V_x | \, x \in (\{\beta\} \times N_n) \setminus F\}$ for finite sets F, and for neighborhoods V_x of x.

(3) A neighborhood base of (α,n), for α limit, is composed of the sets $\{(\alpha,n)\} \cup \bigcup \left\{V_{(\beta,n)} | \, \gamma < \beta < \alpha\right\}$ for $\gamma < \alpha$, and for neighborhoods V_x of x.

Claim: *Let $S(\delta,k)$ be either of the following three subsets of Z_κ for some $k \in \omega$, $1 \leq \delta < \kappa$: $\{(0,k)\} \cup X_k$, $\{(\delta,k)\} \cup \{\delta-1\} \times N_k$ for isolated δ, $\{(\beta,k) | \, \beta \leq \delta\}$ for limit δ. Then S is closed and compact in Z_κ.* For the proof of Claim, see [Hu].

Now, let $Z_\kappa(X)$ be the Alexandroff one-point compactification of Z_κ and Y be any compact subspace of $Z_\kappa(X)$ containing X. We will prove that $Z_\kappa \subseteq Y$. If $Z_\kappa \nsubseteq Y$, then there exists the least $\delta < \kappa$ such that for some $k \in \omega$ we have $(\delta,k) \notin Y$.

Then $Y \cap S(\delta, k)$ is closed in Y and hence compact. But, this is a contradiction because $Y \cap S(\delta, k)$ can be only one of the following sets: X_k, $\{\delta - 1\} \times N_k$ for isolated δ, $\{(\beta, k) | \beta < \delta\}$ for limit δ. □

Proposition 2.6. *Let* $\mathsf{Comp} \subseteq \mathsf{K} \subseteq \Theta\mathsf{Reg}$ *and let* X *have a weak reflection* kX *in* K. *Then there is no infinite family of closed non-compact subsets* $X_1, X_2, \ldots$ *of* X *such that* $X_p \cap X_q$ *is compact for* $p \neq q$ *and every open filter in* X *containing* $X_1 \cup X_2 \cup \ldots$ *has a cluster point in* kX.

Proof. Let kX be the weak reflection of X in K. Without loss of generality, we may assume that $X \subseteq kX$. Suppose conversely, that there exists such a family $\{X_n\}_{n=1}^{\infty}$ stated in the theorem. Denote $L = \bigcup_{i=1}^{\infty} X_i$ and $M = \mathrm{cl}_{kX} L$. By the assumption, every open filter in L has a cluster point in M. Hence, by Lemma 2.4, M is almost compact. Since M is θ-regular (as a closed subspace of a θ-regular space kX), it follows that M is compact. Let κ be a cardinal number such that $\kappa > |M|$. Let $Z_\kappa(X)$ be the space whose existence is ensured by Lemma 2.5. Since $Z_\kappa(X)$ is compact, it follows that $Z_\kappa(X) \in \mathsf{K}$ and then there exists a continuous extension $f : kX \to Z_\kappa(X)$ of the identity mapping on X. Let $Y = f(M)$. Then $X \subseteq Y \subseteq Z_\kappa(X)$ and Y is compact. It follows from Lemma 2.5 that $|Y| \geq \kappa > |M|$, which is impossible since Y is an image of M. □

Regarding the weak reflections, a small modification only of the topology can completely change the situation (cf. with Example 2.3).

Example 2.7. Let $X = \mathbb{R}$ and ε be the Euclidean topology on X. Denote $M = \{1/n | n = 1, 2, \ldots\}$. The family $\tau_0 = \varepsilon \cup \{U \setminus M | U \in \varepsilon\}$ is a base of a Hausdorff but non-regular topology τ on X. In this topology the space X has no weak reflection in θ-regular spaces.

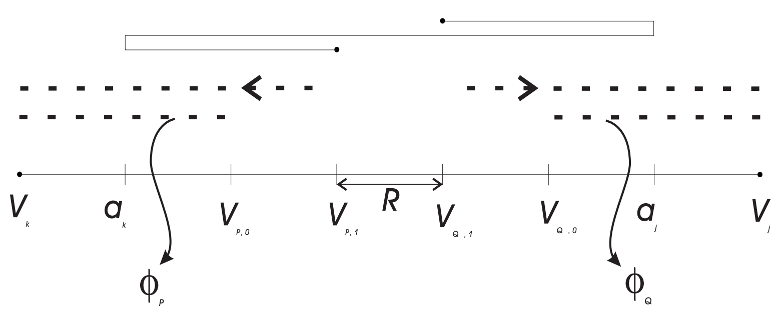
Proof. It is not very difficult to verify that the sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, … θ-converges to $0 \in X$ but, on the other hand, it has no cluster point in X. For every $n \in \mathbb{N}$ let

$$X_n = \left\{ \frac{1}{2^{n-1}}, \frac{1}{3.2^{n-1}}, \frac{1}{5.2^{n-1}}, \frac{1}{7.2^{n-1}}, \dots \right\}.$$

The sets of the family $\{X_n\}_{n=1}^{\infty}$ are pairwise disjoint, non-compact and closed in X. Let Φ be a (not necessarily open) filter in X containing the set $L = \bigcup_{n=1}^{\infty} X_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. Suppose that Φ has no cluster point in X. Then $L \cap F$ is infinite for every $F \in \Phi$ which implies that Φ has a θ-cluster point $0 \in X$. Therefore, if rX is the weak reflection of X in θ-regular spaces, Φ certainly has a cluster point in rX. But, this is a contradiction to Proposition 2.4. □

Lemma 2.8. *Let* $\mathsf{Comp} \subseteq \mathsf{K} \subseteq \Theta\mathsf{Reg}$ *and let* X *have a weak reflection* $j : X \to kX$ *in* K*. Let* $f : X \to Y$ *be a continuous mapping of* X *into a* θ*-regular space* Y*. Then there exists a* θ*-regular subspace* X_f *of* kX *such that* $j(X) \subseteq X_f \subseteq kX$ *and a continuous mapping* $g_f : X_f \to Y$ *such that* $f = g_f \circ j$*.*

Proof. Let $Z = \omega Y$. Since $Z \in \mathsf{K}$ it follows that there exists a continuous mapping $g : kX \to Z$ such that $f = g \circ j$. Put $X_f = g^{-1}(Y)$ and $g_f = g_{/X_f}$. It follows that $f(X) = g(j(X)) \subseteq Y$ which implies that $j(X) \subseteq g^{-1}(Y) = X_f$ and so $f = g_f \circ j$, hence the following diagram commutes:

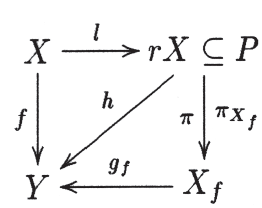


We will show that X_f is θ-regular. Let $x \in X_f$ and $y \in kX \setminus X_f$. Then $g(x) \in Y$ and $g(y) \in Z \setminus Y$ which implies that there exist U, V open in Z such that $g(x) \in U$, $g(y) \in V$ and $U \cap V = \varnothing$. Then $g^{-1}(U)$, $g^{-1}(V)$ are open and disjoint neighborhoods of the points $x, y \in kX$. Since kX obviously is θ-regular, it follows that both the subspaces X_f and $kX \setminus X_f$ of kX are θ-regular as well. $\square$

Theorem 2.9. *Let* $\mathsf{Comp} \subseteq \mathsf{K} \subseteq \Theta\mathsf{Reg}$. *Suppose that* X *has a weak reflection* $j : X \to kX$ *in* K. *Then* X *has a weak reflection in* θ*-regular spaces.*

Proof. Let $M = \{S |\, j(X) \subseteq S \subseteq kX\,,\ S \text{ is } \theta\text{-regular}\}$. Denote $P = \prod_{S \in M} S$ and $j_S = j$ for every $S \in M$. We define a mapping $l : X \to P$ by $l(x)(S) = j_S(x) = j(x)$ for $x \in X$ and $S \in M$. Obviously, l is continuous. Denote $rX = \mathrm{cl}_P\, l(X)$. Then rX is θ-regular because P is a product of θ-regular spaces and rX is its closed subspace. We will show that rX is a weak reflection of X in θ-regular spaces. Let Y be a θ-regular space and let $f : X \to Y$ be a continuous mapping. By Lemma 2.8 there exist a θ-regular subspace $X_f \subseteq kX$ such that $j(X) \subseteq X_f \subseteq kX$ and a continuous mapping $g_f : X_f \to Y$ such that $f = g_f \circ j$. Obviously

$X_f \in M$. Let $\pi : rX \to X_f$ be the restriction of the X_f-th canonical projection $\pi_{X_f} : P \to X_f$ on $rX \subseteq P$. The situation is illustrated by the diagram:



We put $h = g_f \circ \pi$. Then $h : rX \to Y$ is a continuous mapping and for every $x \in X$ it follows $(h \circ l)(x) = (g_f \circ \pi \circ l)(x) = (g_f \circ$

$\pi_{X_f} \circ l)(x) = g_f(l(x)(X_f)) = g_f(j(x)) = f(x)$. Then $f = h \circ l$, the diagram commutes, which implies that h is a continuous extension of f over rX.

Hence, $l : X \to rX$ is the desired weak reflection of X in θ-regular spaces. □

Among others, with Example 2.7 the theorem shows that there is no weakly reflective subclass of **Top** between compact and θ-regular spaces.

Corollary 2.10. *No subclass* **K** *of* **Top** *with* **Comp** $\subseteq$ **K** $\subseteq$ Θ**Reg** *is weakly reflective.*

Remark 2.11. Since the corresponding constructions (of 2.5, 2.6, 2.8 and 2.9) are compatible with T_1 separation axiom, analogous results to 2.9 and 2.10 hold also for the class $\mathbf{Comp}_1$ of compact T_1 spaces and the class $\mathbf{\Theta Reg}_1$ of θ-regular T_1 spaces.

Acknowledgment. I am thankful to Professor Miroslav Hušek for many helpful suggestions, comments and interesting questions regarding the topic. I also thank to Professor Dikran Dikranjan for sending a lot of papers regarding the θ-operator which helped me to improve my θ-related terminology. Finally, I thank to all other colleagues for the discussion related to the topic after my contributed talk in Istanbul 2000.

References

[AR] Adámek J., Rosický J., *On injectivity in locally presentable categories.* Trans. Amer. Math. Soc. **336,2** (1993), 785-804.

[Cs] Császár A., General Topology, published by Akademiai Kiadó, Budapest, 1978.

[DG_1] Dikranjan D., Giuli E., *$S(n)$-θ-closed spaces* Topology Appl. **28** (1988), 59-74.

[DG_2] Dikranjan D., Giuli E., *Closure operators I*, Topology Appl. **27** (1987), 129-143.

[DGT] Dikranjan D., Giuli E., Tholen W., *Closure operators II*, (Adámek J., McLane S., eds), Proceedings of the International Conference on Categorial Topology, Prague (1988), 297-335.

[DW] Dow A., Watson S., *Universal spaces*, preprint (June 1989).

[He_1] Herrlich H., *Almost reflective subcategories of Top*, Topology and its Appl. **49** (1993), 251-264.

[He_2] Herrlich H., *Compact T_0-Spaces and T_0-Compactifications*, Applied Categorial Structures **1** (1993), 111-132.

[Hu] Hušek M., *Čech-Stone-like compactifications for general topological spaces*, Comment. Math. Univ. Carolinae **33,1** (1992), 159-163.

[Ja] Janković D. S., *θ-regular spaces*, Internat. J. Math. Sci. **8** (1985), no.3, 615-619.

[Ko_1] Kovár M. M., *On θ-regular spaces*, Internat. J. Math. Sci. **17** (1994),no. 4, 687-692.

[Ko_2] Kovár M. M., *Which topological spaces have a weak reflection in compact spaces?*, Comment. Math. Univ. Carolinae **36,3** (1995), 529-536.

[Ko_3] Kovár M. M., *A remark on θ-regular spaces*, Internat. J. Math. Sci. **21** (1998), no.1, 199-200.

Department of Mathematics, Faculty of Electrical Engineering and Computer Science, Technical University of Brno, Technická 8, 616 69 Brno, Czech Republic

E-mail address: `kovar@dmat.fee.vutbr.cz`