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RECENT GEOMETRIC DEVELOPEMENTS IN THE THEORY OF ANOSOV DIFFEOMORPHISMS

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Abstract

It is an old conjecture that Anosov diffeomorphisms are topologically conjugate to infranilmanifold automorphisms. This article attempts to survey recent work in this direction which involve the geometric artifact of connections.

1. Introduction

The purpose of this survey article is to report certain geometric developements in the theory of Anosov diffeomorphisms. The use of the adjective "geometric" in connection with smooth manifolds as distinct from purely differential topological is meant to be indicative of the presence of an inherent connection, in other words, of a rule of parallel displacement for tangent vectors. ([K-N], chapter III, section 2.) The word connection will be used in full generality referring to a *connexion infinitésimale linéaire* in the terminology of C. Ehresmann with possibly non-vanishing torsion.([E])

In the theory of dynamical systems which can be regarded with little historical inaccuracy as the direct offspring of H. Poincaré's work in celestial mechanics, much effort goes into understanding systems which exhibit "chaos" , a catchword that has come to connote a seemingly erratic qualitative nature of orbits combined with overall stability. In such systems, the

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qualitative nature of orbits depend sensitively on "initial conditions", yet the ensuing complicated global structure cannot be perturbed away by small changes in the parameters defining the system as a whole. Such tenacity of global behaviour has been referred to by a variety of names like *stability*, *roughness*, *robustness*, *grossièreté*.

A related phenomenon in dynamical systems is that of *hyperbolicity* which stipulates a local splitting of the carrier space as the product of two subspaces, in one of which orbits converge exponentially with time whereas in the other they converge exponentially with the time reversed. The exact correlation between hyperbolicity and "chaos" is one of the fundamental issues in the theory of dynamical systems.

For purposes of the present survey a (topological) *dynamical system* (or a *cascade*) will mean an action of the commutative group $\mathbb{Z}$ of integers on a topological space. Since an action of $\mathbb{Z}$ is entirely determined by the action of $1 \in \mathbb{Z}$, we shall equivalently understand a dynamical system simply to be a homeomorphism $f : X \longrightarrow X$ from a topological space into itself. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$. f will be said to be *topologically transitive* if f has a dense orbit, in other words there exists $x \in X$ such that $\{f^n(x) \mid n \in \mathbb{Z}\}$ is dense in X. Dynamical systems $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ are said to be *topologically conjugate* if there exists a homeomorphism $\varphi : X \longrightarrow Y$ such that $\varphi \circ f = g \circ \varphi$. Systems which are topologically conjugate may thus be regarded as indistinguishable from the topological point of view. If X, Y are smooth manifolds and φ is a diffeomorphism then the systems in question are said to be *differentiably conjugate*.

2. Anosov Diffeomorphisms

Being geometric instances of the Lagrangian formalism of theoretical physics, geodesic flows on the unit tangent bundles of Riemannian spaces have been intensely studied by many mathematicians. ([H], [Mo], [He], [Ho]) During his study of the geodesic flows on the unit tangent bundles of Riemannian spaces of negative curvature D. V. Anosov elucidated the basic stability properties of these objects and made the seminal observation that these properties were embryonically present in a simple class of diffeomorphisms which now bear his name. ([A1], [A2], [A3])

Definition 2.1. Given a smooth, compact manifold M, a diffeomorphism $a : M \longrightarrow M$ of class C^1 is said to be an *Anosov diffeomorphism* if there exists a Riemannian metric G on M, constants $C \geq 1$, $\lambda > 1$, $1 > \mu > 0$ and subbundles E^+, E^- of class C^0 of TM invariant under Ta such that

$$TM = E^+ \oplus E^-$$

and for each $m \in M$, $u \in E^+_m$, $v \in E^-_m$,

$$\| T_m a^n(u) \|_{a^n(m)} \quad \geq C^{-1}\lambda^n \quad \| u \|_m$$

$$\| T_m a^n(v) \|_{a^n(m)} \quad \leq C\mu^n \quad \| v \|_m$$

for all $n \in \mathbb{Z}, n \geq 0$, where $\| \, . \, \|_m$ stands for the norm that stems from the inner product G_m on $T_m M$.

It should be noticed immediately that since M is compact, the validity of the above given growth and decay conditions is independent of the choice of the tensor G. However, the constants C, λ, μ do depend on the choice of G. Indeed, by a very well known and useful result of J. Mather, a suitable choice of G can render $C = 1$ ([M1]).

The most arresting peculiarity of Anosov diffeomorphisms is the non-smooth distribution of the subbundles E^+, E^-. This circumstance is in fact necessary and natural : Consider an Anosov diffeomorphism $a : M \longrightarrow M$. It has been known since the earliest phases in the developement of the theory, that every diffeomorphism $b : M \longrightarrow M$ sufficiently close to $a : M \longrightarrow M$ in the C^1 sense ([Hi2]) is again an Anosov diffeomorphism (by the principal result in [M1] which will be revisited below) and topologically conjugate to $a : M \longrightarrow M$. (This is originaly due to Anosov [A3] and Moser [Mos], a modern version is to be found in [M2], a more recent related result in [W]. An elementary treatment of the toral case which reflects all essentials is in [Ar-Av] .) In other words, the class of Anosov diffeomorphisms are stable under perturbations of class C^1. The crucial point is that even when the unperturbed diffeomorphism has smoothly distributed expanding and contracting subbundles, the slightest perturbation will cause them to wrinkle into non-smoothness.

Example 2.2. Given the (n dimensional) torus $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, as quotient of additive groups, a *toral automorphism* is a map of the form $a_A : \mathbf{T}^n \longrightarrow \mathbf{T}^n$ sending a coset $x + \mathbb{Z}^n$ to the coset $Ax + \mathbb{Z}^n$ where $A \in \mathbb{Z}^{n\times n}$ with $detA = \pm 1$. a_A is said to be *hyperbolic* if A has no eigenvalues of modulus 1. It can be easily checked that hyperbolic toral automorphisms are Anosov diffeomorphisms.

Recall that, given a group G and and subgroups $A, B \leq G$, a *double coset* with respect to A and B in G is a set of the form $AxB = \{axb \mid a \in A, b \in B\}$ for some $x \in G$. Let the set of double cosets with respect to A and B in G be denoted by $A\backslash G/B$ ([Her]).

Given a group G and a subgroup $\Phi \leq \mathrm{Aut}(G)$, let $G \rtimes \Phi$ denote the *semidirect product* of G and Φ. To be precise, $G \rtimes \Phi$ is the group with carrier set $G \times \Phi$ and the binary operation defined by $(g, \varphi)(g', \varphi') = (g\varphi(g'), \varphi\circ\varphi')$ for $g, g' \in G$, $\varphi, \varphi' \in \Phi$.

A non-commutative generalisation of tori, an *infranilmanifold* is a smooth manifold whereof the carrier set is a double coset space $M = \Gamma\backslash(N \rtimes \Phi)/\hat{\Phi}$ where N is a connected, simply connected nilpotent Lie group, Φ is a finite group of automorphisms of N, $\Phi \simeq \hat{\Phi} = \{e_N\} \times \Phi \leq N \rtimes \Phi$, Γ is a subgroup of and acting cocompactly and properly discontinuously by left multiplication on $N \rtimes \Phi$. N can be seen to be diffeomorphic to $(N \rtimes \Phi)/\hat{\Phi}$ with respect to the obvious and trivial differentiable structures on the objects in question and the quotient map $N \longrightarrow M$ is a covering projection with respect to the quotient topology on M and this map defines the smooth structure on M. An infranilmanifold in which Φ is trivial is called a *nilmanifold*.

The Klein bottle constitutes the simplest example of a non-nilmanifold infranilmanifold with $N = \mathbb{R}^2$,

$$\Phi = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\}$$

and

$$\Gamma = (\mathbb{Z}^2 \times \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}) \cup ([\ \mathbb{Z}^2 + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}] \times \{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\}) \ .$$

Example 2.3. An *infranilmanifold automorphism* is a map of the form $a_A : M \longrightarrow M$ where M is an infranilmanifold with $M = \Gamma \backslash (N \rtimes \Phi)/\hat{\Phi}$, sending a double coset $\Gamma x \hat{\Phi}$ into the double coset $\Gamma A(x)\hat{\Phi}$ for each $x \in N \rtimes \Phi$ and $A : N \rtimes \Phi \longrightarrow N \rtimes \Phi$ being a Lie group automorphism that makes a_A well-defined. [1] a_A is said to be a *hyperbolic* infranilmanifold automorphism if T_eA has no eigenvalues of modulus 1. It can be easily checked that hyperbolic infranilmanifold automorphisms are Anosov diffeomorphisms.

It is rather difficult to produce concrete non-trivial examples in this area. The following is an old example on a nilmanifold, produced by A. Borel upon the inquiry of S. Smale ([Sm1]). (It should not be difficult to modify it to obtain an automorphism on a proper infranilmanifold) : Let H be the *Heisenberg group* defined by

$$H = \{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \ \mid \ x, y, z \in \mathbb{R}\}.$$

[1] In the current literature, this well definedness is vouchsafed by requiring $A(\Gamma) \subseteq \Gamma$ and $A \circ \Phi = \Phi \circ A$. This seems to be unduely restrictive and a source of hitherto unnoticed small technical difficulties which I intend to clarify elsewhere.

Being nilpotent, connected and simply connected, this Lie group may be identified with its Lie algebra ([P], [C-G])

$$\mathcal{H} = \{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \mid x, y, z \in \mathbb{R}\}$$

which has generators

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

subject to the sole non-zero commutation rule $[X, Y] = Z$. Now consider the Lie algebra direct sum $\mathcal{N} = \mathcal{H} \oplus \mathcal{H}$, which is cleary the Lie algebra of $N = H \times H$, with generators

$$X_1 = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \; Y_1 = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}, \; Z_1 = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$X_2 = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}, \; Y_2 = \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix}, \; Z_2 = \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}.$$

Let

$$\mathcal{Z} = \{ \begin{bmatrix} 0 & \xi & \eta \\ 0 & 0 & \zeta \\ 0 & 0 & 0 \end{bmatrix} \mid \xi, \eta, \zeta \;\text{ integers in }\; \mathbb{Q}(\sqrt{3})\}$$

If σ stands for the unique non-trivial field automorphism of $\mathbb{Q}(\sqrt{3})$ sending $a + b\sqrt{3}$ into $a - b\sqrt{3}$ for each $a, b \in \mathbb{Q}$ <u>and</u> its obvious componentwise extension to matrices, it is easily seen that

$$\mathcal{L} = \{ \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix} \mid A \in \mathcal{Z}\}$$

is a lattice in $\mathcal{N}$. The linear map $\mathcal{A} : \mathcal{N} \longrightarrow \mathcal{N}$ defined by

$$\begin{aligned} \mathcal{A}(X_i) &= 3^{(-1)^i/2} X_i \\ \mathcal{A}(Y_i) &= 3^{(-1)^i} Y_i \\ \mathcal{A}(Z_i) &= 3^{(-1)^i 3/2} Z_i \end{aligned}$$

for $i = 1, 2$ is a Lie algebra morphism which preserves $\mathcal{L}$. Finally, using the exponential map $exp : \mathcal{N} \longrightarrow N$ it is seen that $\Gamma = exp(\mathcal{L})$ is a discrete, cocompact subgroup of N and $A = exp \circ \mathcal{A} \circ exp^{-1} : N \longrightarrow N$ is a Lie group automorphism with $A(\Gamma) = \Gamma$. Now we can put $M = \Gamma \backslash N$ and consider the map $a_A : M \longrightarrow M$ sending the coset Γx into the coset $\Gamma A(x)$. Clearly a_A is a hyperbolic automorphism of the nilmanifold M.

Remark 2.4. The above examples are smooth Anosov diffeomorphisms with smooth stable and unstable distributions. Yet under the slightest perturbation of whatever smoothness class they will generically degenerate into Anosov diffeomorphisms of class C^1 with non-smooth stable and unstable distributions.

Mainly owing to the non-smoothness of the stable and unstable distributions, Anosov diffeomorphisms have since their very inception been peculiarly inaccessible by direct means. All the more valuable are the few substantial results of which the following two are of primary importance :

(1) Given a smooth compact manifold M with tangent bundle TM and tangential projection $\tau : TM \longrightarrow M$ a vector field on M may be understood to be a map $X : M \longrightarrow TM$ with $\tau \circ X = Id_M$. The set $\mathcal{V}$ of vector fields of class C^0 on M has a natural structure as a Banachable topological vector space. Each diffeomorphism $a : M \longrightarrow M$ of class C^1 induces a bounded $\mathbb{R}$-linear map $a_* : \mathcal{V} \longrightarrow \mathcal{V}$ defined by $a_*(X) = Ta \circ X \circ a^{-1}$ for each $X \in \mathcal{V}$. It is a remarkable fact discovered by J. Mather, that $a : M \longrightarrow M$ is an Anosov diffeomorphism iff 1 does not lie in the spectrum of a_*. ([M1] .)

(2) Since the subbundles E^+, E^- of an Anosov diffeomorphism $a : M \longrightarrow M$ are not smooth, we cannot hope to integrate them by standard means like the Frobenius theorem.

It was therefore by a tremendous feat of delicate global analysis that Hirsch and Pugh proved the stable and unstable subbundles to be integrable into submanifolds immersed as smoothly as the smoothness class of $a : M \longrightarrow M$ but to be in general non-smoothly distributed ([Hi-P]).

Although touched by many a masterly hand, Anosov diffeomorphims retain their enigmatic status. It is, for instance, still not known whether Anosov diffeomorphisms always have fixed points. Topological transitivity, in other words the existence of an orbit which lies densely in the carrier manifold is another unresolved seemingly simple issue.

The central problem in the theory of Anosov diffeomorphisms is the following over-thirty-year-old conjecture:

Conjecture 2.5. *Every Anosov diffeomorphism is topologically conjugate to a hyperbolic infranilmanifold automorphism.*

The question whether there were "non-toral" Anosov diffeomorphisms was brought up by D. V. Anosov himself ([Sm1]), inducing S. Smale to search for an example and happening upon the above presented example of A. Borel. S. Smale gives a weaker form of the conjecture in [Sm1]. To the best of my knowledge the conjecture was first put forward in its final form by J. Franks ([Fr]). The restatements have been at best sporadic,([Hi1]) the last at the writing of this survey being the addendum to [Sm2].

The conjecture seems to have originated by simple inspection of the known instances which consist of diffeomorphisms of very algebraic nature such as described in the above examples and, of course, their perturbations. Furthermore, once algebraic, a hyperbolic map appears to force the carrier algebraic object to be nilpotent. For instance, a real Lie algebra $\mathcal{A}$ which admits a Lie algebra homomorphism $F : \mathcal{A} \longrightarrow \mathcal{A}$ that, as a linear map, has no eigenvalue of unit modulus, has to be a nilpotent Lie algebra. ([B], Exercise 21, Section 4)

The conjecture was settled partially by S. Newhouse, ([N]) in the case of the so called codimension one Anosov diffeomorphisms, that is Anosov diffeomorphisms in which one of the stable and unstable subbundles is one dimensional.

The strongest favourable evidence is the work of M. Gromow which affirmatively settles the corresponding conjecture in the case of the closely allied expanding maps ([G] and for a related result [Hi1]) .

It should be remembered that we cannot hope to improve the statement of the conjecture by replacing the topological conjugacy with differentiable conjugacy. This is known to be false since there exist Anosov diffeomorphisms on smooth manifolds homeomorphic but not diffeomorphic to an infranilmanifold ([F-J2] and for the relatively easier version with expanding maps [F-J1]).

3. Invariant Connections for Anosov Diffeomorphisms

As has been amply pointed out above, Anosov diffeomorphisms present features which are not directly accessible by geometric means. Furthermore, the fact, arising from the above cited work of F. T. Farrell and L. E. Jones, that Anosov diffeomorphisms may be compatible with several differentiable structures on the same topological manifold, indicates manifestly that an unmixedly geometric approach is bound to fall short of complete success.

Notwithstanding these objections there have recently been attempts at relieving the absence of the geometric note by investigating Anosov diffeomorphisms which leave a connection invariant.

Consider a smooth manifold M, let $\mathcal{X}(M) \subset \mathcal{V}$ be the set of smooth vector fields on M. Clearly $\mathcal{X}(M)$ consitutes a module over the ring $C^\infty(M)$ of real valued smooth functions on M. A (smooth) connection on M is an $\mathbb{R}$-bilinear map

$$\bigtriangledown : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

such that

$$\begin{aligned} \bigtriangledown(fX,Y) &= \quad f\bigtriangledown(X,Y) \\ \bigtriangledown(X,gY) &= XgY + g\bigtriangledown(X,Y) \end{aligned}$$

for any $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$, where Xg stands for the directional derivative of g along X. It can be routinely checked that a connection makes sense even on locally defined vector fields. In the presence of a chart $x = (x^1, x^2, \ldots, x^n)$ the functions $\Gamma^k_{ij} : \mathrm{dom}(x) \subseteq M \longrightarrow \mathbb{R}$ for $1 \leq i, j, k \leq n = \dim(M)$ defined by

$$\Gamma^k_{ij} = dx^k(\bigtriangledown|_{dom(x)}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}))$$

are called the Christoffel symbols (of the second kind, to be precise) associated with $\bigtriangledown$. Clearly, a connection determines uniquely and is uniquely determined by its Christoffel symbols with respect to some smooth atlas on M.

Now, given a smooth diffeomorphism $\varphi : M \longrightarrow M$ it is easy to check that the map

$$\bigtriangledown^\varphi : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

defined by

$$\bigtriangledown^\varphi(X,Y) = \varphi_*^{-1} \bigtriangledown (\varphi_*(X), \varphi_*(Y))$$

for $X, Y \in \mathcal{X}(M)$ is a connection on M. $\bigtriangledown$ is said to be *invariant under* φ or φ is said to be *affine* with respect to $\bigtriangledown$ if $\bigtriangledown^\varphi = \bigtriangledown$. Let us call a diffeomorphism which leaves <u>some</u> connection invariant an *affinable* diffeomorphism. Isometries of Riemannian manifolds are clearly affinable since they leave the associated Levi-Civita connections.

A word of caution is necessary before proceeding further : For reasons of stability we have defined Anosov diffeomorphisms to be of class C^1. So, if we insist on this larger class we must readjust the concept of connection by replacing $\mathcal{X}(M)$ and $C^\infty(M)$ with the larger versions $\mathcal{X}^{(r)}(M)$ and $C^r(M)$ of class C^r, $r \geq 0$.

This is a perfectly legitimate procedure that defines a connection of class C^r, $r \geq 0$ as an $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{X}^{(r)}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}^{(r)}(M)$$

with the accompanying conditions adequately modified. Clearly, for each $r \geq 0$, a connection of class C^r is exactly a connection with Christoffel symbols of class C^r. On the other hand, remembering that each diffeomorphism of class C^r, $r \geq 1$ can be approximated (in the C^r sense) by a smooth diffeomorphism and in view of the fact that Anosov diffeomorphisms are stable under perturbations of class C^r, we may assume the Anosov diffeomorphism we work with to be smooth. We shall indeed do both : The Anosov diffeomorphism $a : M \longrightarrow M$ will be assumed to be smooth and we shall talk freely of connections of class $C^r, r \geq 0$.

The first work which I would like to mention addresses the smooth version of the Conjecture, giving an affirmative answer under the condition of topological transitivity, smoothness of stable and unstable subbundles and smooth affinability.

Theorem 3.1. ([Be-La]) *A topologically transitive, smooth Anosov diffeomorphism with smooth stable and unstable subbundles, admitting a smooth invariant connection is smoothly conjugate to a hyperbolic infranilmanifold automorphism.*

Of course, hyperbolic infranilmanifold automorphisms admit invariant connections, namely those of the Cartan-Schouten type ([Hel], [Kam-T]). Yet the presence of an invariant connection for a diffeomorphism is a strong assumption. In fact, even on relatively simple spaces one can prove the existence of large classes of diffeomorphisms that admit no invariant connection whatsoever! ([T1])

Furthermore it must be remembered that not even the affirmative answer to the Conjecture vouchsafes the existence of an invariant connection for an arbitrary Anosov diffeomorphism. Evidently, an Anosov diffeomorphism which is topologically but not diffeomorphically conjugate to a hyperbolic infranilmanifold automorphism may well fail to be affinable!

Therefore, producing an invariant connection for a given Anosov diffeomorphism is not an easy task. A partial result has been derived under a curious condition on the growth and decay rates of the diffeomorphism :

Let G be an arbitrary but fixed Riemannian tensor on M. Given an Anosov diffeomorphism $a : M \longrightarrow M$ there exist, by compactness of M constants $0 < A < 1$, and $1 < B$ such that

$$A \parallel u \parallel_m \leq \parallel T_m a(u) \parallel_{a(m)} \leq B \parallel u \parallel_m$$

where $\parallel . \parallel_m$ stands for the norm that stems from the inner product G_m on $T_m M$. Putting

$$\begin{aligned} \alpha &= \log \max(A^{-1}, B) \\ \beta &= \log \min(\lambda, \mu^{-1}) \end{aligned}$$

the growth and decay conditions characterising Anosov diffeomorphisms can be reformulated in a more symmetric form :

$$C^{-1} e^{-n\alpha} \parallel u \parallel_m \quad \leq \quad \parallel T_m a^n(u) \parallel_{a^n(m)} \quad \leq \quad C e^{-n\beta} \parallel u \parallel_m$$

with $n \leq 0$ for $u \in E_m^+$ and with $n \geq 0$ for $u \in E_m^-$. Clearly $\beta < \alpha$.

An Anosov diffeomorphism $a : M \longrightarrow M$ is said to satisfy the *1/2-pinching condition* ([Fer]) if the above equations hold for $\alpha, \beta > 0$ with $\alpha < 2\beta$.

Theorem 3.2. ([Fer]) *A smooth Anosov diffeomorphism satisfying the* 1/2-pinching condition *admits a unique torsion free invariant connection of class* C^0 *with respect to which the curvature tensor makes sense along stable and unstable leaves which are thus rendered complete and flat.*

4. Canonical Connections for Anosov Diffeomorphisms

The purpose of this section is to present a brief exposition of some new results of the author which appear to indicate that, within the theory of Anosov diffeomorphisms, the search for a relevant connection should probably proceed in an altogether different direction.

In order to simplify the survey of the ideas, it is perhaps best to start by reminding the reader of a standard result due to Cauchy : By the *Polar Decomposition Theorem* (any textbook on linear algebra should do, for instance [Hof-K]) every non-singular matrix A decomposes uniquely as the product $A = US$ of a unitary matrix U and a symetric, positive definite matrix S. I am astonished to find that the following straightforward adaptation of this result to Riemannian manifolds is widely unknown:

Let (M, G) be a smooth Riemannian manifold.

Theorem 4.1. ([T2]) *To each non-singular smooth map $f : M \longrightarrow M$ there exists a canonically (but not functorially) associated vector bundle morphism*

$$Uf : TM \longrightarrow TM$$

and a tensor field S of bidegree $(1,1)$ with the following properties :

(a) $\tau \circ Uf = f \circ \tau$ (i. e. "Uf lies over f") .

(b) $Tf = Uf \circ S$.

(c) S is self-adjoint with respect to G. To be precise,

$$\mathsf{G}_m(S_m(u), v) = \mathsf{G}_m(u, S_m(v))$$

for each $m \in M$ and each $u, v \in T_mM$.

(d) S is positive definite with respect to G. To be precise,

$$\mathsf{G}_m(S_m(u), u) \geq 0$$

and $\mathsf{G}_m(S_m(u), u) = 0$ only if $u = 0_m$ for each $m \in M$ and each $u \in T_mM$.

(e) Uf acts as an isometry on the tangent spaces. To be precise,

$$\mathsf{G}_{f(m)}(U_mf(u), U_mf(v)) = \mathsf{G}_m(u, v)$$

for each $m \in M$ and each $u, v \in T_m M$.

Given an Anosov diffeomorphism $a : M \longrightarrow M$, let $\varepsilon : \mathcal{V} \longrightarrow \mathcal{V}$ be defined by

$$\varepsilon(X) = Ua \circ X \circ a^{-1}.$$

It can be readily checked that

$$\varepsilon(fX) = (f \circ a^{-1})\varepsilon(X)$$

and

$$\mathsf{G}(\varepsilon(X), \varepsilon(Y)) = \mathsf{G}(X, Y) \circ a^{-1}$$

for any $X, Y \in \mathcal{V}$ and $f \in C^0(M)$ ([T2]) .

Now, given a connection

$$\bigtriangledown : \mathcal{V} \times \mathcal{X}(M) \longrightarrow \mathcal{V}$$

of class C^0, the map $\bigtriangledown^{[a]} : \mathcal{V} \times \mathcal{X}(M) \longrightarrow \mathcal{V}$ defined by

$$\bigtriangledown^{[a]}(X, Y) = \varepsilon^{-1} \bigtriangledown (a_*(X), \varepsilon(Y))$$

for each $X \in \mathcal{V}, Y \in \mathcal{X}(M)$, is routinely checked to be a connection of class C^0. By proving that the map sending each connection $\bigtriangledown$ of class C^0 into $\bigtriangledown^{[a]}$ has a unique fixed point, the following theorem can be established :

Theorem 4.2. ([T2]) *Given an Anosov diffeomorphism $a : M \longrightarrow M$ and a Riemannian tensor G on M, there exists a <u>unique</u> connection $\square$ of class C^0 on M with the following properties :*

(a) $\square(a_*(X), \varepsilon(Y)) = \varepsilon\square(X, Y)$ *for each $X \in \mathcal{V}, Y \in \mathcal{X}(M)$.*

(b) $\square\mathsf{G} = 0$.

Although the connection $\square$ is of class C^0 only and thus gives rise to no curvature tensor, it does make sense to talk about parallel translation along curves with respect to $\square$. Given a curve $\gamma : [0, 1] \longrightarrow M$ of class C^1, let $Tr_{\square,\gamma} : T_{\gamma(0)} \longrightarrow T_{\gamma(1)}$ be the linear map that sends each $u \in T_{\gamma(0)}$ into its image under parallel displacement along γ with respect to $\square$. Given any set

$A \subseteq M$, we shall call $\square$ *flat along* A if $Tr_{\square,\gamma} = Id(T_{\gamma(0)}M)$ for any homotopically trivial closed curve γ lying in A. With this terminology another strong property of $\square$ can be articulated :

Theorem 4.3. ([T2]) $\square$ *is flat along the stable and unstable submanifolds.*

I wish to end this survey on an optimistic note : As much a stranger in topological dynamics as it is in differential geometry, the canonical connection introduced above is nonetheless a very natural object with strong properties. It is for this reason that I hope it may play a crucial role in unraveling the enigmata surrounding Anosov diffeomorphisms.

References

[A1] D. V. Anosov, *Roughness of geodesic flows on compact Riemannian manifolds of negative curvature.* Soviet Mathematics Doklady **3** (1962), 1068-1070.

[A2] D. V. Anosov, *Ergodic properties of geodesic flows on compact Riemannian manifolds of negative curvature.* Soviet Mathematics Doklady **4** (1963), 1153-1156.

[A3] D. V. Anosov, *Geodesic Flows on Closed Riemannian Manifolds of Negative Curvature.* Proceedings of the Steklov Institut of Mathematics, volume **90**, (1969).

[Ar-Av] V. I. Arnold, A. Avez, *Problèmes ergodiques de la mecanique classique.* Gauthier-Villars, (1967).

[Be-La] Y. Benoist, F. Labourie, *Sur les difféomorphismes d'Anosov affines à feuilletages stables et instables différentiables.* Inventiones Mathematicae **111** (1993), 285-308.

[B] N. Bourbaki, *Groups et algèbres de Lie.* Fascicule XXVI of Éléments de Mathématique, (1960).

[C-G] L. J. Corwin, F. P. Greenleaf, *Representations of nilpotent Lie Groups and Their Applications.* Cambridge University Press (1990).

[E] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable.* Colloque de Topologie (Espaces Fibrés), 29-55. Centre Belge de Recherches Mathématiques (1950).

[F-J1] F. T. Farrell, L. E. Jones, *Examples of expanding endomorphisms on exotic tori.* Inventiones Mathematicae **45** (1978), 175-179.

[F-J2] F. T. Farrell, L. E. Jones, *Anosov diffeomorphisms constructed from* $\pi_1(\mathrm{Diff}(\mathbf{S}^n))$. Topology **17** (1978), 273-282.

[Fer] R. Feres, *The invariant connection of a 1/2-pinched Anosov diffeomorphism and rigidity.* Pacific Journal of Mathematics **171** (1995), 139-155.

[Fr] J. Franks, *Anosov diffeomorphisms.* Proceedings of Symposia in Pure Mathematics, American Mathematical Society **14** (1970), 61-93.

[G] M. Gromow, *Groups of polynomial growth and expanding maps.* Institut des Hautes Études Scientifiques, Publications Mathématiques **53** (1981), 53-73. (With an Appendix by J. Tits 74-78.)

[H] J. Hadamard, *Les surfaces à courbures opposés et leurs lignes géodèsiques.* Journal de Mathématiques Pures et Appliquées **4** (1898), 27-73.

[He] G. Hedlund, *The dynamics of geodesic flows.* Bulletin of the American Mathematical Society **45** (1939), 241-246.

[Hel] S. Helgason, *Differential Geometry of Lie Groups and Symmetric Spaces.* Academic Press (1978).

[Her] I. N. Herstein, *Topics in Algebra.* Wiley and Sons (1975).

[Hi1] M. W. Hirsch, *Expanding maps and transformation groups.* Proceedings of Symposia in Pure Mathematics, American Mathematical Society **14** (1970), 125-131.

[Hi2] M. W. Hirsch, *Differential Topology.* Springer Verlag (1976).

[Hi-P] M. W. Hirsch, C. Pugh, *Stable manifolds and hyperbolic sets.* Proceedings of Symposia in Pure Mathematics, American Mathematical Society **14** (1970), 133-163.

[Hof-K] K. Hoffman, R. Kunze, *Linear Algebra.* Prentice-Hall (1961).

[Ho] E. Hopf, *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung.* Berichte der Verhandlungen sächsischer Akademie der Wissenschaften. Leipzig **91** (1939), 261-304.

[Kam-T] F. W. Kamber, Ph. Tondeur, *Flat manifolds with parallel torsion.* Journal of Differential Geometry **2** (1968), 385-389.

[K-N] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry.* John Wiley and Sons (1963).

[M1] J. N. Mather, *Characterisations of Anosov diffeomorphisms.* Indagationes Mathematicae **30** (1968), 479-483.

[M2] J. N. Mather, *Anosov diffeomorphisms*, appendix to Part I of [Sm1].

[Mo] M. Morse, *A one to one representation of geodesics on a surface of negative curvature.* American Journal of Mathematics **43** (1921), 33-51.

[Mos] J. Moser, *On a theorem of Anosov.* Journal of Differential Equations **5** (1969), 411-440.

[N] S. Newhouse, *On codimension one Anosov diffeomorphisms.* American Journal of Mathematics **92** (1969), 761-770.

[P] L. Pukanszky, *Leçons sur les représentations des groups.* Dunod (1963).

[Sm1] S. Smale, *Differentiable dynamical systems.* Bulletin of the American Mathematical Society **73** (1967), 747-817.

[Sm2] S. Smale, *Mathematical problems for the next century.* In Mathematics, Frontiers and Perspectives, pp. 271-294. Editors : V. I. Arnold, M. Atiyah, P. Lax, B. Mazur. American Mathematical Society (2000).

[T1] C. Tezer, *Sur les transformations de la sphère admettant une connexion invariante.* Indagationes Mathematicae **11** (2000), 467-475.

[T2] C. Tezer, *Sur une connexion canonique pour les difféomorphismes d'Anosov.* Submitted.

[W] P. Walters, *Anosov diffeomorphisms are topologically stable.* Topology **9** (1970), 71-78.

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