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ADJUNCTION SPACES AND UNIONS OF G-ANE'S

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Dedicated to Professor Yuri M. Smirnov
on the occasion of his 80th birthday

ABSTRACT. Let G be a locally compact Hausdorff group. We study adjunction spaces and unions of equivariant absolute neighborhood extensors (G-ANE's) in the category of all proper G-spaces that are metrizable by G-invariant metrics. We establish equivariant versions of the Borsuk-Whitehead-Hanner Theorem and of the Kodama Theorem. As an application, we prove that every proper G-CW complex is a G-ANE provided G is a Lie group.

1. INTRODUCTION

In the present paper we study proper group actions from the point of view of equivariant theory of retracts. The concept of a proper G-space under consideration (G is a locally compact Hausdorff group) was introduced in 1961 by Palais [29] with the purpose to extend a substantial part of the theory of compact transformation groups to locally compact ones. A G-space X is said to be proper [29, Definition 1.2.2] if every point $x \in X$ has a neighborhood V_x such that for any point $y \in X$ there is a neighborhood V_y with the property that the set $\langle V_x, V_y\rangle = \{g \in G |\ gV_x \cap V_y \neq \emptyset\}$ has

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compact closure in G. Clearly, if G is compact, every G-space is proper. In case G is discrete and X is locally compact, the notion of a proper action is the same as the classical notion of a *properly discontinuous* action. When $G=\mathbb{R}$, the additive group of the reals, proper G-spaces are precisely the *dispersive* dynamical systems with a regular orbit space. Such systems on separable, metrizable phase spaces are just the parallelizable ones (cf. [1]).

In the last decade the interest to proper group actions rose again. Here we should refer first of all to the remarkable work of Illman [21], [22]. In particular, in [22] Illman proved that each smooth proper G-manifold possesses a G-CW complex structure provided G is a Lie group. Under the same assumption on G, Elfving proved in [16] that if a topological manifold is equipped with a locally linear (or locally smooth, in terminology of Bredon [10]) proper action of G, then it has the G-homotopy type of a G-CW complex. Retraction properties of the orbit space and universal proper G-spaces were considered in [7] and [8], respectively.

In the non-equivariant case one of the most important advantages of CW complexes is their property to be an ANE (absolute neighborhood extensor for metrizable spaces). Is a G-CW complex a G-ANE? This important question up to the moment has remained open. Investigating this question is one of the purposes of the present paper.

In this connection, in Section 3 we prove the equivariant version of the classical Borsuk-Whitehead-Hanner Theorem on the adjunction space $X \cup_f Y$ of a map $f : A \to Y$, where A is a closed subset of X. The version of this theorem in [17, Theorem 1.2, Ch. VI], states that if $X \cup_f Y$ is metrizable then it is an ANR, provided X and Y are ANR's. It was Hyman [18] who proved that $X \cup_f Y$ is an ANE, without assuming the metrizability of $X \cup_f Y$. However, Hyman's theorem still assumes the metrizability of both X and Y. It turns out that here, in fact, the metrizability of Y can also be dropped. We establish in Section 3 at once the corresponding equivariant result, by proving that $X \cup_f Y$ is a G-ANE for the class G-$\mathcal{M}$ of all proper G-spaces that are metrizable by an invariant metric, provided $X \in G$-$\mathcal{M}$ and all the three G-spaces X, A and Y are G-ANE's for G-$\mathcal{M}$. So, we do not assume here metrizability of Y. In our proofs we follow the general idea of Hyman's paper [18] and also use some methods from the equivariant theory of retracts.

In Section 4, analogous results for G-AE's are established. In Section 5, we prove the equivariant version of Kodama's theorem [24] on the union of ANE's for proper G-spaces, where G is any locally compact Hausdorff group. In Section 6, we apply these results to prove that every proper G-CW complex is a G-ANE provided G is a Lie group.

2. Preliminaries

Throughout the paper the letter G will always denote a locally compact Hausdorff group. All topological spaces and topological groups are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in Bredon [10] and in Palais [28]. Our basic reference on proper group actions is the Palais article [29]. Other good sources are [1], [2], [25] and [21]. For equivariant theory of retracts the reader can see, for instance, [4] and [5].

For the convenience of the reader, we recall, however, some more special definitions and facts below.

By a G-space we mean a triple (G, X, α), where X is a topological space, and $\alpha : G \times X \to X$ is a continuous action of the group G on X. If Y is another G-space, a continuous map $f : X \to Y$ is called a G-map or an equivariant map if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. If G acts trivially on Y then we will use the term "invariant map" instead of "equivariant map."

If X is a G-space then for a subset $S \subset X$, $G(S)$ denotes the saturation of S, i.e., $G(S)= \{gs|\ g \in G,\ s \in S\}$. In particular, $G(x)$ denotes the G-orbit $\{gx \in X|\ g \in G\}$ of x. If $G(S)$=S then S is said to be an invariant or G-invariant set. The orbit space is denoted by X/G.

A G-pair is a couple (X, A) where X is a G-space and A is a closed invariant subset of X.

For any $x \in X$, we denote $G_x = \{g \in G|\ gx = x\}$, the stabilizer (or stationary subgroup, or isotropy subgroup) of x.

A compatible metric ρ on a G-space X is called invariant or G-invariant if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$.

A cover $\mathcal{V} = \{V_\alpha\}$ of a G-space X is called a G-cover iff the index set $\mathcal{A}$ is a G-set which satisfies the condition: $V_{g\alpha} = gV_\alpha$ for all $\alpha \in \mathcal{A}$ and $g \in G$; in particular, $gV_\alpha \in \mathcal{V}$.

Let X be a G-space. Two subsets U and V in X are called thin relative to each other [29, Definition 1.1.1] if the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G. A subset U of a G-space X is called *small* if every point in X has a neighborhood thin relative to U. A G-space X is called *proper* (in the sense of Palais) if every point in X has a small neighborhood. Each orbit in a proper G-space is closed, and each stabilizer is compact [29, Proposition 1.1.4]. Furthermore, if X is a compact proper G-space, then G has to be compact too.

Important examples of proper G-spaces are the coset spaces G/H with H a compact subgroup of a locally compact group G. Other interesting examples the reader can find in [1], [2], [6], [25], [21].

In the present paper we are especially interested in the class G-$\mathcal{M}$ of all metrizable proper G-spaces X that admit a G-invariant metric. Observe that in this case the orbit space is metrizable. Indeed, if ρ is a G-invariant metric on the metrizable proper G-space X, then the formula

$$\widetilde{\rho}\big(G(x), G(y)\big) = inf\{\rho(x', y') \mid \ x' \in G(x), y' \in G(y)\}$$

defines a metric $\widetilde{\rho}$, compatible with the quotient topology of X/G [29, Theorem 4.3.4].

It is well-known that for G a compact group, the class G-$\mathcal{M}$ coincides with the class of *all* metrizable G-spaces. A fundamental result of Palais, [29, Theorem 4.3.4], states that G-$\mathcal{M}$ includes all *separable*, metrizable proper G-spaces provided G is a Lie group. The question of whether the separability in Palais' result can be omitted still remains open (even for G=$\mathbb{R}$, the additive group of the reals).

The twisted products $G \times_K Y$, where G is a locally compact, compact subgroup of G, and Y is a metrizable K-space, constitute another important subclass of G-$\mathcal{M}$ (see [8, Lemma 1.1]). It is known from [1] that if G is a locally compact group having a compact space of connected components, then each $X \in G$-$\mathcal{M}$ has the form $G \times_K Y$ with K a maximal compact subgroup of G. Recall that $G \times_K Y$ is the orbit space of the K-space $G \times Y$, where K acts on $G \times Y$ by $k(g, y) = (gk^{-1}, ky)$. Further, there is a natural action of G on $G \times_K Y$ given by $g'[g, y] = [g'g, y]$, where $[g, y]$ denotes the K-orbit of $(g, y) \in G \times Y$.

A G-space Y is called a G-ANE (for the class G-$\mathcal{M}$) (notation: $Y \in G$-ANE), if for any G-pair (X, A) with $X \in G$-$\mathcal{M}$ and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi \colon U \to Y$ such that $\psi|A = f$. If, in addition, we can always take $U = X$,then we say that Y is a G-AE (notation: $Y \in G$-AE). The map ψ is called a G-extension of f.

Let Y be a G-space. Then Y is called a G-ANR (notation: $Y \in G$-ANR) provided $Y \in G$-$\mathcal{M}$, and for any G-space X from G-$\mathcal{M}$ and any closed G-embedding $Y \hookrightarrow X$, there exist an invariant neighborhood U of Y in X and a G-retraction $r \colon U \to Y$. If in addition we can always take $U = X$, then we say that Y is a G-AR (notation: $Y \in G$-AR).

We notice that in general a metrizable G-ANE space Y need not be a G-ANR because it may not belong to the class G-$\mathcal{M}$. But if $Y \in G$-$\mathcal{M}$ and $Y \in G$-ANE, then clearly, $Y \in G$-ANR. The converse, for G a locally compact group with a compact space of connected components, is proved in [7, Remark 5].

Let $f_0, f_1 \colon X \to Y$ be two G-maps. A G-homotopy of f_0 into f_1 is a homotopy in the ordinary sense which is a G-map at each stage of the deformation. An invariant subset A of a G-space X is a strong neighborhood G-deformation retract of X if there exist an invariant neighborhood W of A in X and a G-homotopy $f_t : W \to X$, $t \in I$, such that f_0 is the inclusion $W \hookrightarrow X$, f_1 is a G-retraction of W onto A, and $f_t(a) = a$ for all $a \in A$ and $t \in I$. If we can take $W = X$, then A is called a strong G-deformation retract of X. As usual, here and in what follows, the letter I stands for the closed segment $[0, 1]$.

3. Adjunction spaces of G-ANE's

Let (X, A) be a G-pair, Y be a G-space, and $f : A \to Y$ a G-map. We shall denote by $X \sqcup Y$ the topological sum of X and Y, which naturally becomes a G-space. As usual, let us identify each point $a \in A$ with its image $f(a)$ in Y. Then, the quotient space obtained from $X \sqcup Y$ by this topological identification, and denoted by $X \cup_f Y$, is called the adjunction space obtained by adjoining X to Y by means of the given map f. Let $p : X \sqcup Y \to X \cup_f Y$ denote the natural projection. Then one can easily verify that the restriction $p|Y$ is a closed embedding and the restriction $p|(X \setminus A)$ is an open embedding. From the G-action α on $X \sqcup Y$, the adjunction space inherits a natural G-action β. Continuity of this action follows from the fact that in the commutative diagram

$$\begin{array}{ccc} G \times (X \sqcup Y) & \xrightarrow{\alpha} & X \sqcup Y \\ \downarrow id \times p & & \downarrow p \\ G \times (X \cup_f Y) & \xrightarrow{\beta} & X \cup_f Y \end{array}$$

the map $id \times p$ is a quotient map, since p is a quotient map and G is locally compact (see [11, Lemma V. 2.13]). Thus, $X \cup_f Y$ is a G-space and $p : X \sqcup Y \to X \cup_f Y$ is a G-map.

Note that the adjunction space of two Tychonoff spaces is not in general a Tychonoff space.

Let (X, A) be a pair. Following Hyman [18], we will call an open cover $\{V_\alpha\}$ of $X \setminus A$ a semicanonical cover for the pair (X, A), provided for each $a \in A$ and each neighborhood U of a in X, there is a neighborhood W of a in X such that $V_\alpha \subset U$ whenever V_α meets W. Notice that locally finite semicanonical covers, known as canonical covers, were first introduced in Dugundji [12].

Definition 3.1. A G-semicanonical pair is a G-pair (X, A) such that $X \setminus A$ admits a semicanonical G-cover.

Lemma 3.2. *If (X, A) is a G-pair with $X \in G$-$\mathcal{M}$, then (X, A) is G-semicanonical.*

Proof: Fix a G-invariant metric d on X. For each point $x \in X \setminus A$, let V_x denote the open neighborhood of x in X defined by

$$V_x = \{y \in X |\ d(y, x) < (1/2)d(x, A)\}.$$

It is clear that $V_{gx} = gV_x$ for all $x \in X \setminus A$ and $g \in G$. Let us check that the G-cover $\mathcal{V} = \{V_x |\ x \in X \setminus A\}$ is semicanonical. Let U be any neighborhood of a point $a \in A$ in X. There exists a positive real r such that for every $y \in X$, $d(a, y) < 2r$ implies $y \in U$. Let W be the neighborhood of the point a defined by

$$W = \{y \in X |\ d(a, y) < (1/2)r\}.$$

Assume that $V_x \in \mathcal{V}$ meets W at some point $y \in X$. By the definition of V_x, we have

$$d(a,x) \le d(a,y) + d(x,y)$$
$$< (1/2)r + (1/2)d(x,A) \le (1/2)r + (1/2)d(a,x).$$

This gives $d(a,x) < r$. Consequently, for an arbitrary point $z \in V_x$, one has

$$d(a,z) \le d(a,x) + d(x,z)$$
$$< d(a,x) + (1/2)d(x,A) \le (3/2)d(x,a) < 2r.$$

This implies that $z \in U$. Hence $V_x \subset U$.

Thus we obtain an open semicanonical G-cover $\mathcal{V}$ of $X \setminus A$, indexed by the G-set $X \setminus A$; hence, (X,A) is a G-semicanonical pair. □

Proposition 3.3. *Let (X,A) be a G-semicanonical pair such that every stabilizer G_y with $y \in X \setminus A$ is compact. Then there exists a semicanonical G-cover $\mathcal{U} = \{U_x |\ x \in X \setminus A\}$ such that $gU_x = U_{gx}$ for every $x \in X \setminus A$, $g \in G$. In particular, each U_x is G_x-invariant.*

Proof: Let $\mathcal{V} = \{V_\alpha\}$ be a semicanonical G-cover of $X \setminus A$. On each orbit $G(y) \subset X \setminus A$, we fix a point - say $y \in G(y)$. Let $y \in V_\alpha$ for some $V_\alpha \in \mathcal{V}$. Due to the compactness of the stabilizer G_y, there exists a G_y-invariant neighborhood U_y such that $U_y \subset V_\alpha$. For every $x = hy$, where $h \in G$, we define $U_x{=}hU_y$. If x has yet another representation of the form $x = h_1y$, $h_1 \in G$, then $h^{-1}h_1 \in G_y$. This yields that $h^{-1}h_1U_y = U_y$ or $h_1U_y = hU_y$, which means that the set U_x is well-defined. Let us check that $gU_x = U_{gx}$ for every $x \in X \setminus A$ and $g \in G$. Indeed, if $x = hy$ for some $h \in G$ then $gx = (gh)y$; so $U_{gx} = (gh)U_y = g(hU_y) = gU_x$. Now the family $\mathcal{U}$ defined by $\mathcal{U} = \{U_x |\ x \in X \setminus A\}$ is an open G-cover of $X \setminus A$, indexed by the G-set $X \setminus A$, that satisfies the condition $gU_x = U_{gx}$. As $hU_y \subset hV_\alpha \in \mathcal{V}$, we see that $\mathcal{U}$ is a refinement of $\mathcal{V}$, which in turn yields that $\mathcal{U}$ also is a semicanonical cover. The proof is complete. □

Proposition 3.4 ([18])**.** *Suppose that $\{V_\alpha\}$ is a semicanonical cover for a pair (Y,B). Let $\{x_\nu\}$ and $\{y_\nu\}$ be two nets in $Y \setminus B$, and suppose that for each ν, x_ν and y_ν lie in a common element V_ν of $\{V_\alpha\}$. Then $\{x_\nu\}$ converges to a point $b \in B$ if and only if $\{y_\nu\}$ converges to b.*

Theorem 3.5. *Let (Y,B) be a G-semicanonical pair such that every stabilizer G_y with $y \in Y \setminus B$ is compact (in particular, if $Y \setminus B$ is* *a proper G-space), and B is a strong neighborhood G-deformation retract of Y. If both B and $Y \setminus B$ are G-ANE, then Y is a G-ANE.*

Proof: Let $h : W \times I \to Y$ be a strong G-deformation retraction onto B, where W is a G-neighborhood of B in Y. Let $\{V_y | \ y \in Y \setminus B\}$ be a semicanonical G-cover for (Y, B) as in Proposition 3.3 above.

To prove that Y is a G-ANE, it suffices to show that for any G-pair (X, A) with $X \in G$-$\mathcal{M}$, each equivariant map $f : A \to W$ has a G-extension $F : U \to Y$ over some invariant neighborhood U of A in X. From this it follows first that $F|F^{-1}(W) : F^{-1}(W) \to W$ is a neighborhood G-extension of f, so that W is a G-ANE; and then Y, being the union of the two open G-ANE subspaces W and $Y \setminus B$, is itself a G-ANE. In this connection we refer to [15, Proposition 2.12], where it was proved that if a G-space Z is the union of two open invariant subsets O_1 and O_2, which are G-ANE for the class G-$\mathcal{P}$ of all paracompact proper G-spaces that have paracompact orbit space, then Z is itself a G-ANE for the class G-$\mathcal{P}$. The same proof in [15, Proposition 2.12] is valid also in the case when O_1 and O_2 are G-ANE for the class G-$\mathcal{M}$.

Given (X, A) and $f : A \to W$, we proceed to construct F.

Let $A_0 = f^{-1}(B)$, $A_1 = A \setminus A_0$ and $X_1 = X \setminus A_0$. Since A_1 is a closed invariant subset of X_1, $f(A_1) \subset Y \setminus B$ and $Y \setminus B \in G$-ANE, there are an invariant neighborhood C_1 of A_1 in X_1 and an equivariant map $\phi_1 : C_1 \to Y \setminus B$ such that $\phi_1|A_1 = f|A_1$. Let d be an invariant metric on X. For each $a \in A_1$, let C_a be the set of points x in C_1 such that

(a) $d(x, A_0) > \frac{1}{2}d(a, A_0)$,
(b) $d(x, a) < d(a, A_0)$,
(c) $x \in \phi_1^{-1}(V_{\phi_1(a)})$,
(d) $x \in \phi_1^{-1}(W)$.

Then the set $C_2 = \bigcup\{C_a | a \in A_1\}$ is an open subset of X_1 and contains A_1. It is also invariant. In fact, if $x \in C_a$ and $g \in G$ then $gx \in C_{ga} \subset C_2$, because due to the invariance of the metric d and the equivariance of the map ϕ_1, one has

(a′) $d(gx, A_0) = d(x, A_0) > \frac{1}{2}d(a, A_0) = \frac{1}{2}d(ga, A_0)$,
(b′) $d(gx, ga) = d(x, a) < d(a, A_0) = d(ga, A_0)$,

(c') $gx \in g\phi_1^{-1}(V_{\phi_1(a)}) = \phi_1^{-1}(gV_{\phi_1(a)}) = \phi_1^{-1}(V_{g\phi_1(a)})$
$= \phi_1^{-1}(V_{\phi_1(ga)})$,

(d') $gx \in g\phi_1^{-1}(W) = \phi_1^{-1}(gW) = \phi_1^{-1}(W)$.

Let C be an invariant neighborhood of A_1 in X_1 whose closure K in X_1 is contained in C_2, and let $\lambda : X_1 \to [0,1]$ be an invariant map such that $\lambda(A_1) = \{0\}$ and $\lambda(X_1 \setminus C) = \{1\}$. This is possible because the orbit space X_1/G is metrizable (and hence, normal). Define $\phi_2 : K \cup A_0 \to Y$ by

$$\phi_2(x) = \begin{cases} h(\phi_1(x), \lambda(x)), & \text{if} \quad x \in K, \\ f(x), & \text{if} \quad x \in A_0. \end{cases}$$

ϕ_2 is well-defined, equivariant and extends f.

Furthermore, ϕ_2 is clearly continuous except possibly at those points of A_0 which are limit points of $K \setminus A_1$. To prove its continuity at these points also, we suppose that $a \in A_0$ is the limit of a sequence $\{x_n\}$ in $K \setminus A_1$ and show that $\{\phi_2(x_n)\}$ converges to $\phi_2(a)$. For each n, choose $a_n \in A_1$ such that $x_n \in C_{a_n}$. Since $\{x_n\}$ converges to $a \in A_0$, it follows from the property (a) above that $\{d(a_n, A_0)\}$ converges to 0, and from (b) that $\{d(x_n, a_n)\}$ converges to 0. Therefore, $\{a_n\}$ converges to a. Since $\{\phi_1(a_n)\} = \{f(a_n)\}$ converges to $f(a)$, we find by (c) and Proposition 3.4, that $\{\phi_1(x_n)\}$ converges to $f(a)$. Given a neighborhood V of $f(a)$ in Y, there is a neighborhood V_1 of $f(a)$ such that $h(V_1 \times I) \subset V$. Since $\{\phi_1(x_n)\}$ converges to $f(a)$, the sequence $\{\phi_1(x_n)\}$ is eventually in V_1, and by the definition of ϕ_2, the sequence $\{\phi_2(x_n)\}$ is eventually in V. Therefore, ϕ_2 is continuous at a, and hence, it is continuous on $K \cup A_0$.

Since $\lambda = 1$ on the boundary (in X_1) of C, and since h maps $W \times 1$ into B, it follows that ϕ_2 maps the boundary $\partial(K \cup A_0)$ of $K \cup A_0$ in X into B. Since B is a G-ANE, it follows that ϕ_2 has an equivariant extension $F_1 : U_1 \to Y$ to some invariant neighborhood U_1 of $\partial(K \cup A_0)$ in X. Then the set $U = U_1 \cup (K \cup A_0)$ is an invariant neighborhood of $K \cup A_0$ in X, which is the union of the sets $U_2 = U_1 \setminus int(K \cup A_0)$ and $K \cup A_0$, both closed and invariant in U. As $U_2 \cap (K \cup A_0)$ is just the boundary $\partial(K \cup A_0)$, we conclude that the map $F : U \to Y$, defined by

$$F(x) = \begin{cases} F_1(x), & \text{if} \quad x \in U_2 \\ \phi_2(x) & \text{if} \quad x \in K \cup A_0, \end{cases}$$

is a well-defined, equivariant extension of f, and the proof is complete. □

Lemma 3.6. *Let (X, A) be a G-pair, Y be a G-space, and $f : A \to Y$ be a G-map. If $\{V_\alpha | \alpha \in \mathcal{A}\}$ is a semicanonical G-cover for $(X \sqcup Y, A \sqcup Y)$, then $\{p(V_\alpha) = V_\alpha |\ \alpha \in \mathcal{A}\}$ is a semicanonical G-cover for $(X \cup_f Y, p(Y))$.*

Proof: It is proved in [18] that $\{p(V_\alpha) | \alpha \in \mathcal{A}\}$ is a semicanonical cover for the pair $(X \cup_f Y, p(Y))$. On the other hand, since $p(V_\alpha) = V_\alpha$ for all V_α, it follows that $\{p(V_\alpha) | \alpha \in \mathcal{A}\}$ is a G-cover. □

Lemma 3.7. *Let (X, A) be a G-semicanonical pair and Y be any G-space. Then the pair $(X \sqcup Y, A \sqcup Y)$ is G-semicanonical too.*

Proof: Let $\{V_\alpha |\ \alpha \in \mathcal{A}\}$ be a semicanonical G-cover for the G-pair (X, A). Since $(X \sqcup Y) \setminus (A \sqcup Y) = X \setminus A$, it is easy to check that the same cover $\{V_\alpha | \alpha \in \mathcal{A}\}$ is a semicanonical G-cover for the G-pair $(X \sqcup Y, A \sqcup Y)$. □

Now Lemma 3.2, Lemma 3.6, and Lemma 3.7 immediately imply the following:

Corollary 3.8. *Let (X, A) be a G-pair with $X \in G\text{-}\mathcal{M}$, Y be any G-space and let $f : A \to Y$ be a G-map. Then $(X \cup_f Y, p(Y))$ is a G-semicanonical pair.*

Lemma 3.9. *Let $X \in G\text{-}\mathcal{M} \bigcap G\text{-}ANE$, let Y be a $G\text{-}ANE$, and let $f : A \to Y$ be an equivariant map, where A is an invariant closed subset of X. Then $X \cup_f Y$ is a $G\text{-}ANE$ if and only if $p(Y)$ is a strong neighborhood G-deformation retract of $X \cup_f Y$.*

Proof: Suppose that $X \cup_f Y$ is a $G\text{-}ANE$. Since Y is a $G\text{-}ANE$ and $X \in G\text{-}\mathcal{M}$, f has an equivariant extension $F : U' \to Y$, where U' is some invariant neighborhood of A in X. Since the orbit space X/G is normal, there is an invariant neighborhood U of A in X such that $\overline{U} \subset U'$. Define an equivariant map

$$h : Z = (X \times \{0\}) \cup (A \times I) \cup (\overline{U} \times \{1\}) \to X \cup_f Y$$

by

$$h(x,t) = \begin{cases} p(x), & \text{if } \ x \in X, \ \ t = 0, \\ p(x), & \text{if } \ x \in A, \ \ 0 \le t \le 1, \\ pF(x), & \text{if } \ x \in \overline{U}, \ \ t = 1. \end{cases}$$

Since Z is a closed invariant subset of $X \times I \in G$-$\mathcal{M}$ and $X \cup_f Y$ is a G-ANE, h has an equivariant extension $H : V \to X \cup_f Y$ over some invariant neighborhood V of Z in $X \times I$. Let W' be a neighborhood of A in X such that $W' \times I \subset V$. By the invariance of V, the invariant neighborhood $W = G(W')$ of A in X also satisfies the condition $W \times I \subset V$. The equivariant map $k : p(W \sqcup Y) \times I \to X \cup_f Y$, defined by

$$k(z,t) = \begin{cases} H\big((p|X)^{-1}(z), t\big) & \text{if } \ z \in p(W), \ 0 \le t \le 1, \\ z & \text{if } \ z \in p(Y), \ \ 0 \le t \le 1, \end{cases}$$

is the desired strong neighborhood G-deformation of the neighborhood $W \cup_f Y = p(W \sqcup Y)$ onto $p(Y)$.

We prove that the map k is continuous. First, we define the map $k'\colon (W \sqcup Y) \times I \to X \cup_f Y$ by the formulas $k'(x,t) = H(x,t)$, if $x \in W$, and $k'(x,t) = p(x)$, if $x \in Y$. Then, we have $k \circ ((p|) \times \mathrm{id}) = k'$, where $p|\colon W \sqcup Y \to p(W \sqcup Y)$ is the restriction of p. Denote $Q = p(W \sqcup Y)$. Since $A \subset W$, we have $p^{-1}\big(p(W \sqcup Y)\big) = W \sqcup Y$ and thus, Q is open in $X \cup_f Y$. By [12, Theorem VI. 2.1], the restriction $p|\colon p^{-1}(Q) \to Q$ is an identification map. Thus, $(p|) \times \mathrm{id}$ is an identification map, and we conclude that k is continuous.

Now we assume that $p(Y)$ is a strong neighborhood G-deformation retract of $X \cup_f Y$. Then $p(Y)$, being G-homeomorphic to Y, is a G-ANE. The complement $(X \cup_f Y) \setminus p(Y)$, being G-homeomorphic to $X \setminus A$, is in G-$\mathcal{M} \cap G$-ANE. Since, by Corollary 3.8, $(X \cup_f Y, p(Y))$ is a G-semicanonical pair, the claim follows from Theorem 3.5, and the proof is complete. □

Lemma 3.10. *Let $X \in G$-$\mathcal{M} \bigcap G$-ANE and A be a closed invariant G-ANR subset of X. Then A is a strong neighborhood G-deformation retract of X. Moreover, there exists a G-homotopy $h_t : X \to X$, $t \in I$, satisfying the conditions:*

(1) *h_0 is the identity map of X;*
(2) *$h_t(a) = a$ for every $t \in I$ and $a \in A$;*

(3) *there exists an open invariant neighborhood U of A in X such that $h_1(U) = A$.*

Proof: Since $A \in G\text{-}ANR$ and $X \in G\text{-}\mathcal{M}$, there exist a closed invariant neighborhood V of A in X and a G-retraction $r : V \to A$. Consider the closed invariant subset

$$Z = (X \times \{0\}) \cup (A \times I) \cup (V \times \{1\})$$

of the G-space $X \times I \in G\text{-}\mathcal{M}$. Define a G-map $f : Z \to X$ by taking

$$f(x,t) = \begin{cases} x, & \text{if} \quad x \in X, \quad t = 0, \\ x, & \text{if} \quad x \in A, \quad 0 \le t \le 1, \\ r(x), & \text{if} \quad x \in V, \quad t = 1. \end{cases}$$

Since $X \in G\text{-}ANE$, f has a G-extension $\varphi : N \to X$ over a G-neighborhood N of Z in $X \times I$. Let W' be a neighborhood of A in X such that $W' \times I \subset N$. By the invariance of N, the invariant neighborhood $W = G(W')$ of A in X also satisfies the condition $W \times I \subset N$. We may choose W so that $W \subset V$. Since the orbit space X/G is normal, we can choose an open invariant subset U of X such that $A \subset U \subset \overline{U} \subset W$. Let $\lambda : X \to I$ be an invariant map such that $\lambda(\overline{U}) = \{1\}$ and $\lambda(X \setminus W) = \{0\}$. The desired G-homotopy $h_t : X \to X$, $t \in I$ is then defined by taking

$$h_t(x) = \varphi\big(x, t\lambda(x)\big)$$

for every $t \in I$ and every $x \in X$. □

An application of Theorem 3.5 gives the following equivariant generalization of the Borsuk-Whitehead-Hanner Theorem:

Theorem 3.11. *Let (X, A) be a G-pair with $X \in G\text{-}\mathcal{M}$, and let $f : A \to Y$ be an equivariant map. If X, A, and Y are G-ANE's, then $X \cup_f Y$ is a G-ANE.*

Proof: Let $p : X \sqcup Y \to X \cup_f Y$ be the natural projection. To prove that $X \cup_f Y$ is a $G\text{-}ANE$, it suffices, by Lemma 3.9, to show that $p(Y)$ is a strong neighborhood G-deformation retract of $X \cup_f Y$. By Lemma 3.10, A is a strong neighborhood G-deformation retract of X. Hence, there exist an invariant neighborhood U of A in X and a G-homotopy $h_t : U \sqcup Y \to X \sqcup Y$, $t \in I$, satisfying the conditions:

(1) h_0 is the identity map of $U \sqcup Y$;
(2) $h_t(w) = w$ for every $t \in I$ and $w \in A \sqcup Y$;

(3) h_1 is a G-retraction of $U \sqcup Y$ onto $A \sqcup Y$.

Define a G-homotopy $k_t : U \cup_f Y \to X \cup_f Y$, $t \in I$, by taking

$$k_t(z) = p\big(h_t(p^{-1}(z))\big)$$

for every $t \in I$ and every $z \in U \cup_f Y$. That k_t is single-valued follows from the second property above. The continuity of k_t can be proved similarly as in Lemma 3.9. It is easy to verify that k_t is a strong G-deformation retraction of the neighborhood $U \cup_f Y = p(U \sqcup Y)$ onto $p(Y)$, and the proof is complete. □

If we take Y in Theorem 3.11 to be a single point, we obtain the following:

Corollary 3.12. *Let (X, A) be a G-pair with $X \in G$-$\mathcal{M} \bigcap G$-ANE and $A \in G$-ANE. Then $X/A \in G$-ANE.*

When G is the trivial group, Theorem 3.11 gives the following more general version of the Borsuk-Whitehead-Hanner Theorem, where no metrizability of the space Y is required:

Corollary 3.13. *Let (X, A) be a metrizable pair, and let $f : A \to Y$ be a continuous map. If X and A are ANR's and Y is an ANE, then $X \cup_f Y$ is an ANE.*

4. Results for G-AE's

Lemma 4.1. *If Y is a G-ANE and if Y can be G-deformed into a G-AE subspace $B \subset Y$, then Y is a G-AE.*

Proof: Let $h : Y \times I \to Y$ be a G-deformation such that $h_1(Y) \subset B$. Suppose that (X, A) is a G-pair from G-$\mathcal{M}$ and let $f : A \to Y$ be an equivariant map. Since Y is a G-ANE, f has an equivariant extension $F : U' \to Y$, where U' is some invariant neighborhood of A in X. Since the orbit space X/G is normal, there is an invariant neighborhood U of A in X such that $\overline{U} \subset U'$. Let $\lambda : X \to [0, 1]$ be an invariant map such that $\lambda(A) = \{0\}$ and $\lambda(X \setminus U) = \{1\}$. Since B is a G-AE, there is an equivariant map $H : X \setminus U \to B$ such that $H|\partial U = h_1 F|\partial U$, where ∂U denotes the boundary of U in X. Define an equivariant map $\phi : X \to Y$ by

$$\phi(x) = \begin{cases} h(F(x), \lambda(x)) & \text{if}, \quad x \in \overline{U}, \\ H(x) & \text{if}, \quad x \in X \setminus U. \end{cases}$$

Then ϕ equivariantly extends f, and the lemma is proved. □

Theorem 4.2. *Let (Y, B) be a G-semicanonical pair such that every stabilizer G_y with $y \in Y \setminus B$ is compact (in particular, if $Y \setminus B$ is a proper G-space), and B is a strong G-deformation retract of Y. If B is a G-AE and if $Y \setminus B$ is a G-ANE, then Y is a G-AE.*

Proof: By Theorem 3.5, Y is a G-ANE. Since by hypothesis Y is G-deformable into B, Y is a G-AE by Lemma 4.1. □

The next lemma is the analog of Lemma 3.9 and can be proved analogously.

Lemma 4.3. *Let $X \in G$-$\mathcal{M} \bigcap G$-AE, Y be a G-AE, and $f : A \to Y$ be an equivariant map where A is an invariant closed subset of X. Then $X \cup_f Y$ is a G-AE if and only if $p(Y)$ is a strong G-deformation retract of $X \cup_f Y$.*

The following lemma is the analog of Lemma 3.10 and can be proved analogously.

Lemma 4.4. *Let $X \in G$-$\mathcal{M} \bigcap G$-AE and A be a closed invariant G-AR subset of X. Then A is a strong G-deformation retract of X.*

We now establish the analog of Theorem 3.11.

Theorem 4.5. *Let (X, A) be a G-pair with $X \in G$-$\mathcal{M}$, and let $f : A \to Y$ be an equivariant map. If X, A, and Y are G-AE's, then $X \cup_f Y$ is a G-AE.*

Proof: Let $p : X \sqcup Y \to X \cup_f Y$ be the natural projection. To prove that $X \cup_f Y$ is a G-AE, it suffices, by Lemma 4.3, to show that $p(Y)$ is a strong G-deformation retract of $X \cup_f Y$.

By Lemma 4.4, A is a strong G-deformation retract of X. Hence, there exists a G-homotopy $h_t : X \sqcup Y \to X \sqcup Y$, $t \in I$, satisfying the conditions:

(1) h_0 is the identity map of $X \sqcup Y$;
(2) $h_t(w) = w$ for every $t \in I$ and $w \in A \sqcup Y$;
(3) h_1 is a G-retraction of $X \sqcup Y$ onto $A \sqcup Y$.

Define a G-homotopy $k_t : X \cup_f Y \to X \cup_f Y$, $t \in I$, by taking

$$k_t(z) = p\big(h_t(p^{-1}(z))\big)$$

for every $t \in I$ and every $z \in X \cup_f Y$. That k_t is single-valued follows from the second property. The continuity of k_t can be proved similarly as in Lemma 3.9.

Then k_t is a strong G-deformation retraction of $X \cup_f Y$ onto $p(Y)$, and the proof is complete. □

By taking Y to be a single point, we obtain the following from Theorem 4.5:

Corollary 4.6. *Let (X, A) be a G-pair with $X \in G$-$\mathcal{M} \bigcap G$-AE and $A \in G$-AE. Then $X/A \in G$-AE.*

Recall that the cone over X, $con(X)$ is obtained from $[0,1] \times X$ by collapsing the closed subset $\{0\} \times X$ to a point (the vertex of the cone), i.e., $con(X) = [0,1] \times X/\{0\} \times X$. As usual, we consider the quotient topology on $con(X)$.

Now, if $X \in G$-$\mathcal{M} \bigcap G$-ANE, then it is easy to see that $[0,1] \times X \in G$-$\mathcal{M} \bigcap G$-ANE, and hence, by Corollary 3.12, $con(X) \in G$-ANE. But $con(X)$ is G-equivariantly contractible to its vertex, and therefore, Theorem 4.2 implies that $con(X) \in G$-AE. Thus, we have proved the following:

Corollary 4.7. *If $X \in G$-$\mathcal{M} \bigcap G$-ANE, then $con(X) \in G$-AE.*

An analogous result for the metric cone was proved in [8] (see also [7, Remark 3]).

When G is the trivial group, we get the following more general version of the *global* Borsuk-Whitehead-Hanner Theorem, where no metrizability of the space Y is assumed:

Corollary 4.8. *Let (X, A) be a metrizable pair, and let $f : A \to Y$ be a continuous map. If X and A are AR's and Y is an AE, then $X \cup_f Y$ is an AE.*

5. Unions of G-ANE's

In this section G denotes any locally compact Hausdorff topological group. We shall prove the following equivariant analogue of Kodama's Theorem in [24]:

Theorem 5.1. *Let X be a G-space having the weak* topology with respect to a closed invariant cover $\{A_\alpha\}_{\alpha \in \Omega}$. Assume that for each finite subcollection $\{A_{\alpha_i}\}_{i=1}^n$ of $\{A_\alpha\}_{\alpha \in \Omega}$, the intersection $\bigcap_{i=1}^n A_{\alpha_i}$ is a G-ANE. Then X is itself a G-ANE.*

Before passing to the proof, let us mention a related result in [23], where it was proved that the property of a G-space being a G-ANE for the class of all paracompact G-spaces is a local property, provided G is a *compact* group.

Definition 5.2. Let X be a topological space and $\{A_\alpha\}_{\alpha\in\Omega}$ a closed covering of X. Then X is said to have the *weak* topology* with respect to $\{A_\alpha\}_{\alpha\in\Omega}$, if the following holds for any $\Omega' \subset \Omega$:

(i) the union $\cup\{A_\beta \mid \beta \in \Omega'\}$ is closed in X;

(ii) any subset of $\cup\{A_\beta \mid \beta \in \Omega'\}$ whose intersection with each A_β, $\beta \in \Omega'$ is closed relative to the subspace topology of A_β, is closed in the subspace $\cup\{A_\beta \mid \beta \in \Omega'\}$.

Remark 5.3. The concept defined in Definition 5.2, see also [24], is not exactly the same as the concept of weak topology as defined for example in [13, p. 131]. That is why we call this concept the weak* topology.

Lemma 5.4 ([24])**.** *Let X be a topological space having the weak* topology with respect to a closed covering $\{A_\alpha\}_{\alpha\in\Omega}$ and Y a metric space. Let f be a continuous mapping of Y into X. Put $Y_\alpha = f^{-1}(A_\alpha)$, $\alpha \in \Omega$. Then there exists a closed covering $\{B_\alpha\}_{\alpha\in\Omega}$ of Y which satisfies the following conditions:*

i) $B_\alpha \subset Y_\alpha$, $\alpha \in \Omega$;

ii) $\{B_\alpha \mid \alpha \in \Omega\}$ is locally finite.

Proof: We assume that the set Ω of indices α consists of all ordinals α less than a fixed ordinal η. Put

$$B_\alpha = \overline{Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta}, \quad \alpha < \eta.$$

Then by the proof of Lemma 1 in [24], the family $\{B_\alpha \mid \alpha < \eta\}$ has the required properties. □

The following definition can be found, for example, in [3, p. 118]:

Definition 5.5. Suppose that $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Omega}$ and $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Omega}$ are two families of subsets of a space X with the same set of indices. It is said that $\mathcal{U}$ is *similar* to $\mathcal{V}$ if, for every $\Omega' \subset \Omega$,

$$\bigcap_{\alpha\in\Omega'} U_\alpha = \emptyset \text{ if and only if } \bigcap_{\alpha\in\Omega'} V_\alpha = \emptyset.$$

In case the families $\mathcal{U}$ and $\mathcal{V}$ are coverings of X, the similarity of $\mathcal{U}$ and $\mathcal{V}$ implies that the nerves $N(\mathcal{U})$ and $N(\mathcal{V})$ are isomorphic as simplicial complexes.

Lemma 5.6. *Assume that $Y \in G$-$\mathcal{M}$, B is a closed invariant subset of Y, and $\{B_\alpha\}_{\alpha\in\Omega}$ is a locally finite closed invariant covering of B. Then there exists a closed invariant neighborhood F of B in Y and a locally finite closed invariant covering $\{F_\alpha\}_{\alpha\in\Omega}$ of F which satisfies the following conditions:*

i) $F_\alpha \cap B = B_\alpha$, $\alpha \in \Omega$;

ii) $\{F_\alpha\}_{\alpha\in\Omega}$ is similar to $\{B_\alpha\}_{\alpha\in\Omega}$.

Proof: Denote by π the projection $Y \to Y/G$ onto the orbit space. Since π is an open map and the sets B and B_α, $\alpha \in \Omega$, are G-invariant and closed in Y, the images $\pi(B)$ and $\pi(B_\alpha)$, $\alpha \in \Omega$, are closed in Y/G. Also by G-invariance of the sets B_α, and by openness of π, the family $\{\pi(B_\alpha)\}_{\alpha\in\Omega}$ is locally finite. By applying [24, Lemma 2] to the sets $\pi(B)$ and $\{\pi(B_\alpha)\}_{\alpha\in\Omega}$ in Y/G, we find a closed neighborhood $\tilde{F}$ of $\pi(B)$ in Y/G and a locally finite closed covering $\{\tilde{F}_\alpha \mid \alpha \in \Omega\}$ of $\tilde{F}$ such that $\tilde{F}_\alpha \cap \pi(B) = \pi(B_\alpha)$ for every $\alpha \in \Omega$, and $\{\tilde{F}_\alpha\}_{\alpha\in\Omega}$ is similar to $\{\pi(B_\alpha)\}_{\alpha\in\Omega}$.

Now choose $F = \pi^{-1}(\tilde{F})$ and $F_\alpha = \pi^{-1}(\tilde{F}_\alpha)$, $\alpha \in \Omega$. These sets satisfy the required conditions. □

Lemma 5.7. *Assume that $Y \in G$-$\mathcal{M}$, B is a closed invariant subset of Y, and F is a closed invariant neighborhood of B in Y. Moreover, let $\{F_\alpha\}_{\alpha\in\Omega}$ be a locally finite closed invariant covering of F. Suppose that for each α there is a closed invariant neighborhood C_α of $F_\alpha \cap B$ in F_α. Then*

$$C = \bigcup_{\alpha\in\Omega} C_\alpha$$

is a closed invariant neighborhood of B in Y.

Proof: By Lemma 3 in [24], C is a closed neighborhood of B in Y. As a union of invariant sets, C is also invariant. □

Lemma 5.8. *Let X be a G-space and $\{A_i\}_{i=1}^n$ a closed invariant covering of X. Assume that $\cap_{j=1}^p A_{i_j}$ is a G-ANE for each $i_1, \ldots, i_p \in \{1, ..., n\}$. Moreover, let $Y \in G$-$\mathcal{M}$, B be a closed invariant subset of Y, and $\{Y_i\}_{i=1}^n$ a closed invariant covering of Y.*

Put $B_i = B \cap Y_i, i = 1, ..., n$. Let f be a G-map of B into X such that $f(B_i) \subset A_i$, $i = 1, ..., n$. Then there exist a closed invariant neighborhood F of B in Y and an equivariant extension $h : F \to X$ of f such that $h(F \cap Y_i) \subset A_i$ for all $i = 1, ..., n$.

Proof: Put

$$H = \bigcup \{\cap_{j=1}^{p} Y_{i_j} \mid (\cap_{j=1}^{p} Y_{i_j}) \cap B = \emptyset,\ i_1, ..., i_p \in \{1, ..., n\}\}.$$

Since $H \cap B = \emptyset$, H and B are G-invariant and closed subsets of Y, and Y/G is normal, we can find a closed invariant neighborhood D of B in Y such that $D \cap H = \emptyset$. Put $D_i = D \cap Y_i$, $i = 1, ..., n$. Then $\{D_i\}_{i=1}^{n}$ is similar to $\{B_i\}_{i=1}^{n}$.

Denote by K the nerve of $\{D_i\}$. A simplex of K is denoted by $(i_0, ..., i_p)$, $i_0, ..., i_p \in \{1, ..., n\}$. For each simplex $s = (i_0, ..., i_p)$ of K put $|s| = \cap_{j=0}^{p} D_{i_j}$. Give a simple order to the simplexes of K as follows: first, give same dimensional simplexes any order; next, if $\dim s > \dim s'$, define s less than s', i.e., $s < s'$.

Fix $\tilde{s} \in K$ and assume that for each simplex $s < \tilde{s}$ the following closed invariant set $M(s)$ and a G-map f_s are constructed:

$i)_s$ $M(s)$ is a closed invariant neighborhood of $|s| \cap B$ in $|s|$.

$ii)_s$ f_s is a G-map of $M(s)$ into $\cap_{j=0}^{p} A_{i_j}$, where $s = (i_0, ..., i_p)$ such that

$$f_s|(B \cap M(s)) = f|(B \cap M(s)).$$

$iii)_s$ Let s_1 and s_2 be two simplexes such that $s_1 \le s_2 \le s$ and s_1, s_2 span a simplex s_3 of K. Then we have

$$(M(s_1) \cap |s_3|) \cup (M(s_2) \cap |s_3|) \subset M(s_3)$$

and

$$f_{s_1}|(M(s_1) \cap M(s_2)) = f_{s_2}|(M(s_1) \cap M(s_2)).$$

We shall construct a closed invariant neighborhood $M(\tilde{s})$ of $|\tilde{s}| \cap B$ in $|\tilde{s}|$ and a G-map $f_{\tilde{s}}$ satisfying $i)_{\tilde{s}}$, $ii)_{\tilde{s}}$, and $iii)_{\tilde{s}}$.

Let $\tilde{s} = (k_0, ..., k_l)$, $k_j \in \{1, ..., n\}$, $j = 0, ..., l$.

At first let $\tilde{s}$ be a principal simplex, i.e., $\tilde{s}$ is not a face of any higher dimensional simplex. Note that in this case $|\tilde{s}| \cap (\cup\{|s| \mid s < \tilde{s}\}) = \emptyset$. Consider $f|(|\tilde{s}| \cap B)\colon |\tilde{s}| \cap B \to \cap_{j=0}^{l} A_{k_j}$. Since $\cap_{j=0}^{l} A_{k_j}$ is a G-ANE, there exist a closed invariant neighborhood $M(\tilde{s})$ of $|\tilde{s}| \cap B$ in $|\tilde{s}|$ and a G-extension $f_{\tilde{s}}\colon M(\tilde{s}) \to \cap_{j=0}^{l} A_{k_j}$ extending $f|(|\tilde{s}| \cap B)$. It is obvious that the conditions $i)_{\tilde{s}}$, $ii)_{\tilde{s}}$, and $iii)_{\tilde{s}}$ are satisfied.

Next, let $\tilde{s}$ be a face of s_i^{l+1}, $i = 1, 2, ..., m$. Then since $M(s_i^{l+1})$ is a closed neighborhood of $|s_i^{l+1}| \cap B$ in $|s_i^{l+1}|$ and $|s_i^{l+1}| \subset |\tilde{s}|$, $i = 1, 2, ..., m$, we have

$$\left(\bigcup_{i=1}^{m} \overline{|s_i^{l+1}| \setminus M(s_i^{l+1})} \right) \cap (|\tilde{s}| \cap B) = \emptyset.$$

Since the two sets in the above formula are closed and G-invariant and $|\tilde{s}|/G$ is a normal space, there exists a closed invariant neighborhood N of $|\tilde{s}| \cap B$ in $|\tilde{s}|$ such that

$$N \cap \left(\bigcup_{i=1}^{m} \overline{|s_i^{l+1}| \setminus M(s_i^{l+1})} \right) = \emptyset.$$

Then define a G-map

$$\alpha : \bigcup_{i=1}^{m} M(s_i^{l+1}) \cup (|\tilde{s}| \cap B) \to \bigcap_{j=0}^{l} A_{k_j}$$

by $\alpha | M(s_i^{l+1}) = f_{s_i^{l+1}},\ i = 1, 2, .., m,\ \ \alpha|(|\tilde{s}| \cap B) = f$.

Then α is a single-valued continuous map by the inductive assumption. Since $\cap_{j=0}^{l} A_{k_j}$ is a G-ANE, there exist a closed invariant neighborhood $M(\tilde{s})$ of

$$\bigcup_{i=1}^{m} M(s_i^{l+1}) \cup (|\tilde{s}| \cap B) \ \text{ in } \ \bigcup_{i=1}^{m} M(s_i^{l+1}) \cup N$$

and an equivariant extension $f_{\tilde{s}}$ of α over $M(\tilde{s})$. Since $\cup_{i=1}^{m} M(s_i^{l+1}) \cup N$ is a closed invariant neighborhood of $|\tilde{s}| \cap B$ in $|\tilde{s}|$, it is easy to verify that the conditions $i)_{\tilde{s}}$, $ii)_{\tilde{s}}$, and $iii)_{\tilde{s}}$ are satisfied.

Hence, we have constructed $M(s)$ and f_s satisfying $i)_s$, $ii)_s$, and $iii)_s$ for each $s \in K$.

If we put

$$F = \bigcup_{s \in K} M(s),$$

F is a closed invariant neighborhood of B in Y by Lemma 5.7. Define $h : F \to X$ by $h|M(s) = f_s$. Since the condition $iii)_s$ is satisfied for each $s \in K$, F and h are the closed invariant neighborhood and the equivariant extension which we require. □

Lemma 5.9. *Suppose $F \in G\text{-}\mathcal{M}$ and $\mathcal{F}$ is a locally finite closed invariant covering of F. Then there exists a locally finite open invariant covering $\mathcal{V} = \{V_\beta\}_{\beta\in\Lambda}$ of F such that for each $\beta \in \Lambda$ the set $\overline{V_\beta}$ meets only a finite number of elements of $\mathcal{F}$.*

Proof: Denote by $\pi\colon F \to F/G$ the projection onto the orbit space. Note that F/G is metrizable, hence also paracompact and normal. Then $\widetilde{\mathcal{F}} = \{\pi(F') \mid F' \in \mathcal{F}\}$ is a locally finite closed covering of F/G. For each $\tilde{x} \in F/G$ choose a neighborhood $\widetilde{U}_{\tilde{x}}$ which meets only a finite number of elements of $\widetilde{\mathcal{F}}$. By normality of F/G we find a neighborhood $\widetilde{W}_{\tilde{x}}$ of $\tilde{x}$ whose closure is contained in $\widetilde{U}_{\tilde{x}}$. By paracompactness of F/G we find a locally finite open refinement $\{\widetilde{V}_\beta\}_{\beta\in\Lambda}$ for the covering $\{\widetilde{W}_{\tilde{x}} \mid \tilde{x} \in F/G\}$. Now $\mathcal{V} = \{\pi^{-1}(\widetilde{V}_\beta) \mid \beta \in \Lambda\}$ has the required properties. □

Proof of Theorem 5.1: Let $Y \in G\text{-}\mathcal{M}$, B be a closed invariant subset of Y, and f be a G-map from B into X. We shall show that there exists a closed invariant neighborhood U of B in Y and an equivariant extension $h : U \to X$ of f.

We assume that the set Ω of indices α consists of all ordinals α less than a fixed ordinal η. Put $C_\alpha = f^{-1}(A_\alpha)$ and $B_\alpha = \overline{C_\alpha \setminus \cup_{\beta<\alpha} C_\beta}$ for each $\alpha < \eta$. Then $\mathcal{F}_1 = \{B_\alpha \mid \alpha < \eta\}$ is a locally finite closed covering of B by the proof of Lemma 5.4. Clearly the sets B_α are G-invariant.

By applying Lemma 5.6, we find a closed invariant neighborhood F of B in Y and a locally finite closed invariant covering $\mathcal{F}_2 = \{F_\alpha\}_{\alpha<\eta}$ of F such that $F_\alpha \cap B = B_\alpha$ for every $\alpha < \eta$, and $\mathcal{F}_2$ is similar to $\mathcal{F}_1$.

Since $\mathcal{F}_2$ is a locally finite closed invariant covering of F, there exists by Lemma 5.9 a locally finite open invariant covering $\{V_\beta\}_{\beta\in\Lambda}$ of F such that for each $\beta \in \Lambda$, the set $\overline{V_\beta}$ meets only a finite number of elements of $\mathcal{F}_2$.

Put $\mathcal{B} = \{\overline{V_\beta} \mid V_\beta \cap B \neq \emptyset\}$. Then $\mathcal{B}$ is a locally finite closed invariant covering of B, and $F' = \cup\{\overline{V_\beta} \mid V_\beta \in \mathcal{B}\}$ is a closed invariant neighborhood of B in Y.

We assume that the set of indices β for the covering $\mathcal{B}$ consists of all ordinals β less than a fixed ordinal δ and put, for each $\theta < \delta$,

$$P_\theta = \bigcup_{\beta\le\theta} \overline{V_\beta}.$$

By local finiteness of $\mathcal{B}$, each P_θ is closed in Y.

Let $\mu < \delta$. Assume that for each $\theta < \mu$, the following closed invariant set N_θ and a G-map f_θ are constructed:

$i)_\theta$ N_θ is a closed invariant neighborhood of $P_\theta \cap B$ in P_θ;
$ii)_\theta$ f_θ is a G-map of $N_\theta \cup B$ into X;
$iii)_\theta$ $f_\theta|B = f$;
$iv)_\theta$ if $\gamma < \theta$, we have $N_\gamma \subset N_\theta$ and $f_\theta|N_\gamma = f_\gamma$;
$v)_\theta$ $\overline{N_\theta \setminus \bigcup_{\gamma<\theta} N_\gamma} \subset \overline{V_\theta}$;
$vi)_\theta$ for each $\alpha < \eta$, we have $f_\theta(N_\theta \cap F_\alpha) \subset A_\alpha$.

Put $M = \bigcup\{N_\theta \mid \theta < \mu\} \cup B$. We have

$$M = B \cup \bigcup_{\theta<\mu} \overline{(N_\theta \setminus \bigcup_{\gamma<\theta} N_\gamma)},$$

and using $v)_\theta$ we find that M is closed in Y. Define $g : M \to X$ by $g|N_\theta \cup B = f_\theta$. By $iv)_\theta$ and the local finiteness of $\mathcal{B}$, g is single-valued and continuous.

Let $F_{\alpha_1}, ..., F_{\alpha_n}$ be all elements of $\mathcal{F}_2$ which meet $\overline{V_\mu}$. Now we apply Lemma 5.8 to $\overline{V_\mu}$, $\overline{V_\mu} \cap M$, $\{\overline{V_\mu} \cap F_{\alpha_i}, i = 1, 2, ..., n\}$, $\cup_{i=1}^n A_{\alpha_i}$, and $g| \colon \overline{V_\mu} \cap M \to \cup_{i=1}^n A_{\alpha_i}$. By Lemma 5.8 we can find a closed invariant neighborhood M_μ of $\overline{V_\mu} \cap M$ in $\overline{V_\mu}$ and a G-map $h_\mu \colon M_\mu \to \cup_{i=1}^n A_{\alpha_i}$ such that $h_\mu|(\overline{V_\mu} \cap M) = g|(\overline{V_\mu} \cap M)$ and $h_\mu(M_\mu \cap F_{\alpha_i}) \subset A_{\alpha_i}$, $i = 1, 2, ..., n$.

Put

$$N_\mu = \bigcup_{\theta<\mu} N_\theta \cup M_\mu = \bigcup_{\theta\le\mu} M_\theta$$

and define $f_\mu : N_\mu \cup B \to X$ by $f_\mu|M = g$ and $f_\mu|M_\mu = h_\mu$. Using Lemma 5.7, we can prove $i)_\mu$. The conditions $ii)_\mu, ..., vi)_\mu$ are easily verified.

Put $U = \cup_{\theta<\delta} N_\theta$ and define $h : U \to X$ by $h|(N_\theta \cup B) = f_\theta$, $\theta < \delta$.

Then U is a closed invariant neighborhood of B in Y by Lemma 5.7 and h is the required equivariant extension of f. This completes the proof of the theorem. □

6. G-CW complexes are G-ANE's

In this section, G denotes a Lie group.

As an application of the results obtained in Section 3 and in Section 5, we shall prove here that every proper G-CW complex is a G-ANE. For actions of non-discrete groups, G-CW complexes were defined by Illman in [19] and [20] and by Matumoto in [27]. In the case of a discrete group, a G-CW complex simply consists of an ordinary CW complex X and an action of G on X which permutes the cells; see the work of Bredon [9]. The definitions and basic properties concerning G-CW complexes can also be found in [15]. First recall the following two basic results. In general, compactness of isotropy subgroups is not a sufficient condition for an action to be proper. For G-CW complexes we, however, have

Proposition 6.1. *If X is a G-CW complex such that for every $x \in X$ the isotropy subgroup G_x is compact, then X is a proper G-space.*

Proof: Proposition 4.10. in [15]. □

Proposition 6.2. *If X is a G-CW complex, then X and X/G are paracompact spaces.*

Proof: Proposition 4.16. in [15]. □

The basic building blocks for G-CW complexes are the G-spaces $G/H \times D^n$ and $G/H \times S^{n-1}$, where $D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, $n = 1, 2, ...$ (D^0 is a singleton space, $S^{-1} = \emptyset$), and $H \subset G$ is a closed subgroup. Here G acts by left translations on the coset spaces G/H and the action of G is trivial on the spaces D^n and S^{n-1}. We shall first prove that these spaces are in G-$\mathcal{M} \bigcap G$-ANE.

Lemma 6.3. *If X is an ANE, then X is a G-ANE when considered as a G-space endowed with the trivial action of G.*

Proof: Let $Y \in G$-$\mathcal{M}$, B be a closed invariant subspace of Y, and $f\colon B \to X$ be a G-map. Denote by $\pi\colon Y \to Y/G$ the orbit projection. Then the map f factors through B/G. Denote by $\tilde{f}$ the map $B/G \to X$ for which $\tilde{f} \circ \pi = f$.

Now B/G is a closed subspace of the metric space Y/G and since X is an ANE, there exists a neighborhood $\widetilde{U}$ of B/G in Y/G and an extension $\tilde{h}\colon \widetilde{U} \to X$ of $\tilde{f}$. Now $\pi^{-1}(\widetilde{U})$ is an invariant neighborhood of B in Y and $h = \tilde{h} \circ \pi$ is an equivariant extension of f. □

Proposition 6.4. *If $\{X_\alpha\}_{\alpha\in\Omega}$ is a family of G-ANE's, then the discrete union*

$$X = \bigsqcup_{\alpha\in\Omega} X_\alpha$$

is a G-ANE.

Proof: Let (Y, B) be a G-pair with $Y \in G$-$\mathcal{M}$ and $f\colon B \to X$ a G-map. Put $B_\alpha = f^{-1}(X_\alpha)$. Then $\{B_\alpha\}_{\alpha\in\Omega}$ is a locally finite closed invariant covering of B. By Lemma 5.6, there exist a closed invariant neighborhood F of B in Y and a locally finite closed invariant covering $\{F_\alpha\}_{\alpha\in\Omega}$ of F such that $F_\alpha \cap B = B_\alpha$ and $F_\alpha \cap F_{\alpha'} = \emptyset$ whenever $\alpha \neq \alpha'$. Let U be the interior of F and $U_\alpha = U \cap F_\alpha$. It follows from local finiteness of $\{F_\alpha\}_{\alpha\in\Omega}$ that each U_α is open in U (and hence, in Y). As $X_\alpha \in G$-ANE, there exist an open invariant neighborhood V_α of B_α in U_α and a G-map $\varphi_\alpha\colon V_\alpha \to X_\alpha$ which extends the map $f|B_\alpha$. Now the set $V = \bigcup_{\alpha\in\Omega} V_\alpha$ is an open invariant neighborhood of B in Y and the map $\psi : V \to X$ defined by

$$\psi|V_\alpha = \varphi_\alpha, \quad \alpha \in \Omega$$

is the desired neighborhood G-extension of f. □

Proposition 6.5. *If $\{H_\alpha\}_{\alpha\in\Omega}$ is a family of compact subgroups of G then the G-space*

$$X = \bigsqcup_{\alpha\in\Omega} G/H_\alpha \times Z,$$

where Z is S^{n-1} or D^n, is in G-$\mathcal{M} \bigcap G$-ANE.

Proof: Denote $X_\alpha = G/H_\alpha \times Z$. By Lemma 6.3, Z is a G-ANE, and by [15, Proposition 2.5 and Proposition 2.7], each G/H_α is a G-ANE. Hence, the product $X_\alpha = G/H_\alpha \times Z$ is also a G-ANE, and hence, by Proposition 6.4, X is a G-ANE. Thus, it remains to prove that $X \in G$-$\mathcal{M}$.

Choose on every coset space G/H_α a G-invariant metric ρ_α (see [26] or [29] for the existence of such a metric), and a metric s on Z. Then for each $\alpha \in \Omega$, the formula

$$d'_\alpha(x,y) = \rho_\alpha(x_1,y_1) + s(x_2,y_2),$$

where x_1, $y_1 \in G/H_\alpha$ and x_2, $y_2 \in Z$, defines a G-invariant metric d'_α on X_α.

Now define, for each $\alpha \in \Omega$:

$$d_\alpha(x,y) = \begin{cases} d'_\alpha(x,y), & \text{if} \quad d'_\alpha(x,y) \le 1, \\ 1, & \text{if} \quad d'_\alpha(x,y) > 1. \end{cases}$$

Then the metrics d_α are clearly G-invariant. A compatible G-invariant metric for the G-space X now can be defined by taking

$$d(x,y) = \begin{cases} d_\alpha(x,y), & \text{if} \quad x, y \in X_\alpha \quad \text{for some } \alpha, \\ 1, & \text{otherwise.} \end{cases}$$

□

Proposition 6.6. *Let G be a Lie group and X a finite-dimensional proper G-CW complex. Then X is a G-ANE.*

Proof: The 0-skeleton X^0 of X is a discrete union of coset spaces G/H_α, $\alpha \in \Omega$, where H_α is a compact subgroup of G for each $\alpha \in \Omega$. As mentioned above, each coset space G/H_α is a G-ANE, and it follows from Proposition 6.4 that X^0 is a G-ANE. As X is finite-dimensional it coincides with a skeleton, say $X = X^k$, $k \ge 1$. Suppose $k \ge n \ge 1$ and the $(n-1)$-skeleton X^{n-1} is a G-ANE. Now the n-skeleton X^n can be obtained from X^{n-1} in the following way, see Remark 4.3 in [15]: there exist a family $\{H_j\}_{j\in J}$ of compact subgroups of G and a G-map

$$\varphi\colon \bigsqcup_{j\in J} G/H_j \times S^{n-1} \to X^{n-1}$$

such that

$$X^n \cong_G \left(\bigsqcup_{j\in J} G/H_j \times D^n \right) \cup_\varphi X^{n-1}.$$

Now, by Proposition 6.5, the discrete unions

$$\bigsqcup_{j\in J} G/H_j \times D^n \qquad \text{and} \qquad \bigsqcup_{j\in J} G/H_j \times S^{n-1}$$

are in G-$\mathcal{M}\bigcap G$-ANE. Consequently, according to Theorem 3.11, X^n is a G-ANE. By induction this leads to the desired conclusion that X is a G-ANE. □

Since an arbitrary G-CW complex X has the weak* topology with respect to the family of its skeletons $\{X^n\}_{n\geq 0}$, by applying Theorem 5.1 to X and the family $\{X^n\}_{n\geq 0}$, we get the following main result of this section:

Theorem 6.7. *Let G be a Lie group and X a proper G-CW complex. Then X is a G-ANE.*

In conclusion, we want to note that the proof of Theorem 6.7 can be shortened by using a more simple result instead of Theorem 5.1.

We recall first that a space X is said to be the topological direct limit of the tower of its subspaces

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots$$

if $X=\bigcup_{i=1}^{\infty} X_i$, and a subset $B\subset X$ is closed in X iff $B\cap X_i$ is closed in X_i for each $i\geq 1$.

Now, since each G-CW complex X is the topological direct limit of the tower of its skeletons

$$X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \ldots \hookrightarrow X^n \hookrightarrow \ldots$$

and since by Proposition 6.6 each skeleton is a G-ANE, we see that Theorem 6.7 follows directly from the following:

Theorem 6.8. *Let a G-space X be the topological direct limit of a tower*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots$$

where every X_i is an G-ANE. Then X is a G-ANE.

For the proof we shall need the following simple

Lemma 6.9. *Let X be the topological direct limit of a tower*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots$$

and let $f: A\to X$ be a map of a metric space A. Then for every $a\in A$ there exist an open neighborhood V of a and an index n such that $f(V)\subset X_n$.

Proof: Let ρ be a metric on A. Let $f(a) \in X_{n_0}$ for some n_0. Suppose that the claim is not true. Then there exists $a_n \in A$ with $\rho(a_n, a) < 1/n$, $n = 1, 2, \dots$ such that

$$f(a_n) \in X_{k_n} \setminus X_{k_{n-1}}$$

for some $k_n > k_{n-1}$ with $k_1 > n_0$.

Set $B = \{f(a_n) | n = 1, 2, \dots\}$. Since $f(a) \notin B$ and B is closed, we obtain a contradiction with the continuity of f at the point a. □

Proof of Theorem 6.8: Let $Y \in G\text{-}\mathcal{M}$, A be a closed invariant subset of Y, and $f : A \to X$ be a G-map. It follows from the preceding lemma and from paracompactness of the orbit space Y/G, that

(1) there exists a locally finite family $\mathcal{F}$ of open invariant subsets of Y such that $A \subset \bigcup \mathcal{F}$;
(2) for each $U_\alpha \in \mathcal{F}$ there exists an index $n = n(\alpha)$ such that $f(\overline{U}_\alpha \cap A) \subset X_n$.

Observe that the family $\overline{\mathcal{F}} = \{\overline{U}_\alpha |\ U_\alpha \in \mathcal{F}\}$ is also locally finite. For each fixed index $n \geq 1$, let $\mathcal{F}_n$ be the subset of $\mathcal{F}$ consisting of all those elements U_α for which $f(\overline{U}_\alpha \cap A) \subset X_n$. Then $\mathcal{F}$ is the increasing union of all the subfamilies $\mathcal{F}_n$, $n = 1, 2, \dots$. Let $V_n = \cup\{U_\alpha |\ U_\alpha \in \mathcal{F}_n\}$. Then $\overline{V}_n = \cup\{\overline{U}_\alpha |\ U_\alpha \in \mathcal{F}_n\}$, $f(\overline{V}_n \cap A) \subset X_n$, $V_n \subset V_{n+1}$, and $\cup_{n=1}^{\infty} V_n = \bigcup \mathcal{F}$.

Let $\varphi_1 : W_1 \to X_1$ be an equivariant extension of the G-map $f|_{\overline{V}_1 \cap A}$, where W_1 is a closed invariant neighborhood of $\overline{V}_1 \cap A$ in $\overline{V}_1$. Then the map $f_1 : A_1 \to X$, defined by $f_1|_A = f$ and $f_1|_{W_1} = \varphi_1$, is a G-map.

Let $A_0 = A$, $A_n = A_{n-1} \cup W_n$, $n \geq 1$, and $f_0 = f$.

Assume we have already constructed a G-map $f_n : A_n \to X_n$ such that $f_n|_{A_{n-1}} = f_{n-1}$. Then $f_n(\overline{V}_{n+1} \cap A_n) \subset X_{n+1}$. As $X_{n+1} \in G\text{-}ANE$, the map $f_n|_{\overline{V}_{n+1} \cap A_n}$ has a G-extension $\varphi_{n+1} : W_{n+1} \to X_{n+1}$, where W_{n+1} is a closed invariant neighborhood of $\overline{V}_{n+1} \cap A_n$ in $\overline{V}_{n+1}$. Then the map $f_{n+1} : A_{n+1} \to X$, defined by $f_{n+1}|_{A_n} = f_n$ and $f_{n+1}|_{W_{n+1}} = \varphi_{n+1}$, is a continuous map. Besides, it is equivariant.

Thus, we get a sequence $f_n : A_n \to X$, $n = 0, 1, 2, \dots$ of G-maps such that $A_n \subset A_{n+1}$ and $f_{n+1}|_{A_n} = f_n$. Let $B = \cup_{n=1}^{\infty} A_n$. Then the map $F : B \to X$ defined by $F|_{A_n} = f_n$ is well defined. Since B is a closed neighborhood of A, we conclude that F is continuous on the interior of B. Besides, B, as well as its interior $Int\, B$, is an invariant set, and F is an equivariant map. Thus, $F|_{Int\, B}$ is the desired neighborhood G-extension of f and the proof is completed. □

Remark 6.10. The second short proof of Theorem 6.7 (combination of Proposition 6.6 and Theorem 6.8) is new also in the non-equivariant case, i.e., when G is the trivial group. Yet another short proof of the non-equivariant version of Theorem 6.7 was suggested quite recently by Dydak [14].

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