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COVERING PROPERTIES VIA ELEMENTARY SUBMODELS

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ABSTRACT. We show how elementary submodels can be applied to simplify proofs of results of covering property theorems.

1. Introduction

Elementary submodels have become important tools in topology. An early treatment of applications of elementary submodels in topology was published by Dow [3] in 1988, and focused on what is now called the submodel topology, i.e., the one generated by the basis $\{U \cap M : U \text{ open in } X\}$ on the set $X \cap M$.

The aim of this paper is to show a different way of applying elementary submodels, a way that is helpful in covering property (or generalized metrizability) proofs. Let us recall at this point the structure of the proof of a typical covering property result.

Step 1. Start with the topology of the space, an open cover and other given data (base, sequence of covers, etc.). Use the hypotheses to construct new families of subsets of the space "as many times as needed." Your work leads to a heavily indexed collection of subsets.

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†Sadly, Professor Balogh passed away shortly after this paper was prepared for publication. A memorial article will appear in a future issue of *Topology Proceedings.*

Step 2. Prove that the end result of the work in Step 1 is as required by the conclusion of the theorem (e.g., it is a σ-discrete refinement).

Let M be a countable elementary submodel of $H(\kappa)$ with a space X, the topology $\mathcal{T}$ of X, an open cover $\mathbf{U} = \langle \mathbf{U}_\alpha \rangle_{\alpha<\lambda}$ of X all elements of M.

(From now on throughout the paper we will assume that κ is a cardinal big enough that the whole proof dealt with takes place inside $H(\kappa)$.) The basis for our applications is the following simple observation.

Proposition 1.1. *The following conditions are equivalent.*

(a) $\mathbf{U}$ *has a σ-discrete refinement* $\langle \mathcal{A}_n \rangle_{n\in\omega}$*;*

(b) $\mathfrak{A} = \{\mathcal{A} \in M : \mathcal{A}$ *is a discrete partial refinement of* $\mathbf{U}\}$ *covers* X*.*

Proof: (b) $\Rightarrow$ (a) is obvious. To see (a) $\Rightarrow$ (b), note first that as $\mathbf{U}, \mathbf{X}, \mathcal{T} \in \mathbf{M}$, by replacing $\langle \mathcal{A}_n \rangle_{n\in\omega}$ if needed, we may assume $\langle \mathcal{A}_n \rangle_{n\in\omega} \in M$. Then $\mathcal{A}_n \in \mathfrak{A}$ for every $n \in \omega$. □

Suppose now that in a covering property proof, we want to show that $\mathbf{U}$ has a σ-discrete refinement. Then we can just skip Step 1 and go directly to $\mathfrak{A}$ in Proposition 1.1 as $\mathfrak{A}$ contains "everything we could have possibly constructed in Step 1." Then we can follow Step 2 to show that $\boldsymbol{\mathfrak{A}}$ is as required, i.e., covers X.

It is worth noting that Step 1 consists of writing out concrete Skolem functions that we can get from $\mathbf{U}, \mathcal{T}$, etc. The more complex the covering property proof is, the more Skolem functions are needed and the more we stand to gain by taking an elementary submodel M instead.

Of course, in Proposition 1.1 we can replace σ-discrete by σ-disjoint, σ-point-finite, σ-$\langle$ your favorite property$\rangle$. A (generalized) metric property, such as the existence of a σ-disjoint or σ-discrete base, can be similarly handled.

In the rest of the paper we implement the ideas above in two complex proofs, one due to Junnila, one to Jiang.

Finally, the reader may ask whether taking a single elementary submodel M is always enough in these proofs or one might need more complex model theory, e.g., an ϵ-chain of elementary submodels. Taking a single submodel seems to be enough in most proofs the author has seen (although some analysis of e.g., the para-Lindelöf property may require more).

2. σ-cushioned refinement implies subparacompact

We say that a system $\mathbf{H} = \langle \mathbf{H_i} \rangle_{\mathbf{i} \in \mathbf{I}}$ of subsets of a space X is *cushioned* in another system $\mathbf{G} = \langle \mathbf{G_i} \rangle_{\mathbf{i} \in \mathbf{I}}$ of subsets, if for every $J \subset I$, $\overline{\bigcup_{i \in J} H_i} \subset \bigcup_{i \in J} G_i$. Note that if $\mathbf{H}$ is closed discrete and $H_i \subset G_i$ for $i \in I$, then $\mathbf{H}$ is cushioned in $\mathbf{G}$. X is said to be subparacompact if every open cover has a σ-discrete refinement (which can be assumed to be closed by regularity of X).

Our first example is the following remarkable result of Junnila [5] [6].

Theorem 2.1. *If every open cover of X has a σ-cushioned refinement, then X is subparacompact.*

Proof: Let $\mathbf{U} = \langle \mathbf{U}_\alpha \rangle_{\alpha < \lambda}$ be an open cover of X.

Let M be a countable elementary submodel of $H(\kappa)$ such that $X, \mathbf{U}$ and the topology $\mathcal{T}$ of X are elements of M. Let us set $P_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$ for every $\alpha < \lambda$. We will show that the sets H_α in $\{\mathbf{H} \in \mathbf{M} : \mathbf{H} = \langle \mathbf{H}_\alpha \rangle_{\alpha < \lambda}$ are discrete and $H_\alpha \subset U_\alpha$ for every $\alpha < \lambda\}$ cover X. To prove this, pick and fix $x \in X$. Let us say that x is ***δ**-exclusive* for $\mathbf{G}$ if $\mathbf{G} = \langle \mathbf{G}_\alpha \rangle_{\alpha < \lambda} \in \mathbf{M}$ is a sequence of open sets with $P_\alpha \subset G_\alpha \subset U_\alpha$ for every $\alpha < \lambda$ and whenever $\mathbf{A} \in \mathbf{M}$ is cushioned in $\mathbf{G}$ with $x \in \bigcup_{\alpha < \lambda} A_\alpha$, then $x \in A_\delta$.

Claim. *There is a δ and a $\mathbf{G}$ such that x is δ-exclusive for $\mathbf{G}$.*

To prove the Claim, let $\delta_{x,\mathbf{A}} = \min\{\alpha : x \in A_\alpha\}$ if $x \in \bigcup_{\alpha < \lambda} A_\alpha$, and let $\delta_{x,\mathbf{A}} = \lambda$ otherwise. Let us set $\delta = \min\{\delta_{x,\mathbf{A}} : \mathbf{A} \in \mathbf{M}$ is cushioned in $\mathbf{U}\}$. Fix such a cushioned $\mathbf{A}^* \in \mathbf{M}$ in $\mathbf{U}$ with $\delta_{x,A^*} = \delta$, and set, for every $\alpha < \lambda$,

$$G_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} \overline{A^*_\beta} \ .$$

Then δ and $G = \langle G_\alpha \rangle_{\alpha < \lambda}$ are as required. Note $\mathbf{G} \in \mathbf{M}$ and $x \in A^*_\delta$.

Suppose that $\mathbf{A} \in \mathbf{M}, \mathbf{A}$ is cushioned in $\mathbf{G}$ and $x \in \bigcup_{\alpha < \lambda} A_\alpha$. Then $x \notin \bigcup_{\alpha < \delta} A_\alpha$ by the definition of $\delta_{x,A} \geq \delta$. Since A^*_δ is taken away from every G_β with $\beta > \delta$, it follows that $x \notin \bigcup_{\beta > \delta} G_\beta$ and thus, $x \notin \bigcup_{\beta > \delta} A_\beta$. Hence, $x \in A_\delta$. We have proved the Claim.

Passing now to the end of the proof of Theorem 1.1, let us pick a $\mathbf{G}$ and δ such that x is δ-exclusive for $\mathbf{G}$. Since $\mathbf{G} \in \mathbf{M}$, $\mathbf{G}$ has a σ-cushioned refinement in M. Thus, there is an $\mathbf{A} \in \mathbf{M}$ such that $\mathbf{A}$ is cushioned in $\mathbf{G}$ and $x \in \bigcup_{\alpha<\lambda} A_\alpha$. Note that $x \in A_\delta$.

By passing from $\mathbf{G}$ to $\langle G_\alpha \backslash \overline{\bigcup_{\gamma<\alpha} A_\gamma} \rangle_{\alpha<\lambda}$ and working with this $\mathbf{G}$ if necessary we may assume that $x \notin \bigcup_{\beta>\delta} G_\beta$.

Let us set, for every $\alpha < \lambda, W_\alpha = G_\alpha \bigcap (\bigcup_{\beta>\alpha} G_\beta)$ and $V_\alpha = G_\alpha \backslash \overline{\bigcup_{\beta\neq\alpha} A_\beta}$. Note that $\langle W_\alpha, V_\alpha \rangle_{\alpha<\lambda}$ is an open cover of X as $P_\alpha \bigcap (\bigcup_{\beta>\alpha} G_\beta) \subset W_\alpha$ and $P_\alpha \backslash (\bigcup_{\beta>\alpha} G_\beta) \subset V_\alpha$ for $\alpha < \lambda$.

Since every open cover has a σ-cushioned refinement, there is a $\langle K_\alpha, H_\alpha \rangle_{\alpha<\lambda} \in M$ cushioned in $\langle W_\alpha, V_\alpha \rangle_{\alpha<\lambda}$ (indexed by $\lambda x 2$ such that $x \in \bigcup_{\alpha<\lambda} (H_\alpha \cup K_\alpha)$). By δ-exclusiveness applied to $\langle W_\alpha \cup V_\alpha \rangle_{\alpha<\lambda}$, it follows that $x \in H_\delta \cup K_\delta$. Since $x \notin \bigcup_{\beta>\delta} G_\beta$, $x \notin W_\delta \supset K_\delta$. Thus, $x \in H_\delta$. Now $\langle H_\alpha \rangle_{\alpha<\lambda} \in M$ is discrete in $\bigcup_{\alpha<\lambda} V_\alpha$ by its definition. It is also discrete in X since it is cushioned in $\bigcup_{\alpha<\lambda} V_\alpha$.

3. Strict p-spaces are submetacompact

A strict p-space introduced by Arhangel'skii and characterized in [2] is a "Moore space with compact subsets substituting points"; i.e., X is a *strict* **p**-*space* if it has a sequence of open covers such that for every $x \in X$, $C_x = \bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{G}_n)$ is compact and $\langle \mathrm{st}(x, \mathcal{G}_n) \rangle_{n\in\omega}$ forms a decreasing open neighborhood base for C_x. A space X is called *submetacompact* if for every open cover $\mathcal{U}$ of X there is a sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for every $x \in X$, $(\mathcal{H}_n)_x = \{H \in \mathcal{H}_n : x \in H\}$ is finite for some $n \in \omega$. Such a sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ is called a *θ-refining sequence*. By a result of Junnila, it is enough to consider well-ordered increasing open covers $\mathcal{U}$ in the definition above (see [1]).

For a long time it had been a famous unsolved problem whether strict p-spaces were submetacompact. The problem was eventually solved by Jiang [4], a student of Rudin. It is the goal of this section to give a rendering of this proof via elementary submodels.

Theorem 3.1 [Jiang]. *If X is a strict p-space, then X is submetacompact.*

Proof: Let X be a strict p-space and let $\mathbf{U} = \langle \mathbf{U}_\alpha \rangle_{\alpha<\lambda}$ be an increasing open cover of X. We need to show that $\mathbf{U}$ has a θ-refining sequence of open covers. Set $P_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. Fix a sequence $\langle \mathcal{G}_n \rangle_{n\in\omega}$ of open covers witnessing that X is a strict p-space, i.e., for every $x \in X$, $\langle \mathrm{st}(x, \mathcal{G}_n) \rangle_{n\in\omega}$ forms a decreasing neighborhood base for $C_x = \bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{G}_n)$ which is compact.

For $A \subset X$, the *height* of A is $h(A) = \sup\{\alpha < \lambda : A \cap P_\alpha \neq \emptyset\}$. The *rank* of $x \in X$ is $r(x) = h(C_x)$. Note that we can fix an $n(x) \in \omega$ such that $r(x) = h(\mathrm{st}(x, \mathcal{G}_n))$ for every $n \geq n(x)$. Since finitely many members of $(\mathcal{G}_n)_x = \{G \in \mathcal{G}_n : x \in \mathcal{G}_n\}$ cover C_x, we can fix, for $n \geq n(x)$, a $G_n(x)$ with $x \in G_n(x) \in \mathcal{G}_n$ and $r(x) = h(G_n(x))$.

Having the preliminary set-up behind us, it is time to start the proof.

Let M be a countable elementary submodel of $H(\kappa)$ with

$$X, \langle U_\alpha \rangle_{\alpha<\lambda},\ \langle \mathcal{G}_n \rangle_{n\in\omega},\ \langle G_n(x) \rangle_{n\in\omega,\ x\in X} \in M.$$

Set

$$\mathfrak{H} = \{\mathcal{H} \in M : \mathcal{H} \text{ is an open refinement of } \mathbf{U}\}.$$

Our proof will show that $\mathfrak{H}$ is a θ-refining sequence of open covers, i.e.,

$$Y = \{x \in X : \exists\, \mathcal{H} \in \mathfrak{H}, |\mathcal{H}_x| < \omega\} \text{ is equal to } X.$$

Suppose indirectly that this is not the case, and let us fix $x^* \in X \setminus Y$ with $r(x^*)$ minimal. Hence, for every point of $S = \{x \in X : r(x) < r(x^*)\}$ there is an $\mathcal{H} \in \mathfrak{H}$ with $|\mathcal{H}_x| < \omega$.

At this point, we interrupt the proof of the theorem and present a pair of useful lemmas.

Let us now define $\mathfrak{V} = \{\mathcal{V} \in M : \mathcal{V}$ is a partial open refinement of $\mathfrak{U}$ and $0 \leq |\mathcal{V}_{x^*}| < \omega\}$. Let $\langle \mathcal{V}_j \rangle_{j\in\omega}$ list the members of $\mathfrak{V}$ and let $V_j = \bigcup \mathcal{V}_j,\ V_j^* = \bigcup_{i\leq j} V_i,\ V = \bigcup_{j\in\omega} V_j$.

Lemma 1. (a) $S \subset V$ *and* (b) $V \neq X$.

Proof: (a) For $\mathcal{H} \in \mathfrak{H}$ and $n \geq 1$ let $F_n(\mathcal{H}) = \{x \in X : |\mathcal{H}_x| \leq n\}$. Note that $S \subset \bigcup\{F_n(\mathcal{H}) : n \geq 1,\ \mathcal{H} \in \mathfrak{H}\}$, so it is enough to prove that each $F_n(\mathcal{H})$ can be covered by a $\mathcal{V} \in \mathfrak{V}$. We prove this by induction on n. Suppose there is a $\mathcal{V} \in \mathfrak{V}$ with $F_n(\mathcal{H}) \subset \bigcup \mathcal{V}$.

Then $F_{n+1}(\mathcal{H})\backslash(\bigcup\mathcal{V})$ is the union of a discrete family $\langle F_\gamma\rangle_{\gamma<\mu}\in M$ of closed subsets partially refining $\mathcal{H}$. Thus, all we need to show is that there is an open expansion of $\langle F_\gamma\rangle_{\gamma<\mu}$ defined in M which is point-finite at x^*. Since $\langle \mathrm{st}(x,\mathcal{G}_n)\rangle_{n\in\omega}$ forms a neighborhood base for the compact set C^*_x, one of the $\mathrm{st}(x,\mathcal{G}_n)$ intersects only finitely many F_γ, and then $\{\mathrm{st}(F_\gamma,\mathcal{G}_n):\gamma<\mu\}$ is such an open expansion.

(b) If $V=X$, then since $\mathfrak{V}$ is closed under finite unions there is a $\mathcal{V}\in\mathfrak{V}$ such that $C_{x^*}\subset\bigcup\mathcal{V}$. Pick n with $\mathrm{st}(C_{x^*},\mathcal{G}_n)\subset\bigcup\mathcal{V}$. Then $\mathcal{H}=\{\mathrm{st}(X\backslash\bigcup\mathcal{V},\mathcal{G}_n)\cap U_\alpha:\alpha<\lambda\}\cup\{\mathcal{V}\}\in\mathfrak{H}$ and $|\mathcal{H}_{x^*}|=|\mathcal{V}_{x^*}|<\omega$, in contradiction with $x^*\notin Y$. This completes the proof of Lemma 1. □

Next, let us pick and fix an $x^{**}\in X\backslash V$ and set $n=n(x^{**})$. For $\alpha<\lambda$ and $j\in\omega\backslash n$, let $W_{\alpha j}=\bigcup\{G_j(z):n(z)\leq n,\ r(z)=\alpha,\ z\notin V^*_j\}$, and $\mathcal{W}_j=\{W_{\alpha j}:\alpha<\lambda\}$. Note $W_{\alpha j}\subset U_\alpha,\mathcal{W}_j\in M$ and $x^{**}\in\bigcup\mathcal{W}_j$.

Lemma 2. *There is a $j\in\omega\backslash n$ such that $|(\mathcal{W}_j)x^*|\leq 1$.*

Proof: Let $r(x^*)=\alpha^*$. Suppose indirectly that for each $j\in\omega\backslash n$ there is an $\alpha_j\neq\alpha^*$ with $x^*\in W_{\alpha_j j}$. By the definition of $W_{\alpha j}$, we can pick $\langle z_j\rangle_{j\in\omega}$ such that $r(z_j)=\alpha_j,\ z_j\notin V^*_j$, $n(z_j)\leq n$ and $x^*\in G_j(z_j)$ for every $j\in\omega$.

Claim 1. *$j\geq n(x^*)$ implies $\alpha_j<\alpha^*$.*

Indeed, $\alpha^*=h(\mathrm{st}(x^*,\mathcal{G}_{n(x^*)}))=h(\mathrm{st}(x^*,\mathcal{G}_j))\geq h(G_j(z_j))=\alpha_j$. Note $\alpha_j\neq\alpha^*$.

Claim 2. *There is a $z\in X$ with $z_j\to z$.*

Indeed, $z_j\in\mathrm{st}(x^*,G_j)$ and $\langle\mathrm{st}(x^*,\mathcal{G}_j)\rangle_{j\in\omega}$ is a neighborhood base for the compact set C^*_x.

Claim 3. *$r(z)<\alpha^*$.*

Indeed, let $m=\max\{n(z),n\}$ and pick $z_j\in G_m(z)$ with $j\geq m$. Then $r(z)=h(G_m(z))\leq h(\mathrm{st}(z_j,\mathcal{G}_m))\leq h(\mathrm{st}(z_j,\mathcal{G}_n))=\alpha_j<\alpha^*$.

By Claim 3, $z\in S\subset V$, so $z\in V_{j_0}$ for some $j_0\in\omega\backslash n$. But $z_j\notin V_{j_0}$ for $j\geq j_0$, in contradiction with $z_j\to z$. This completes the proof of Lemma 2.

We now complete the proof of the Theorem.

By Lemma 2, $\mathcal{W}_j \in \mathfrak{V}$ for some $j \in \omega$. Hence, $x^{**} \in \bigcup \mathcal{W}_j \subset V$, in contradiction with our assumption that $x^{**} \in X \setminus V$. $\square$

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