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A SUFFICIENT CONDITION FOR COVERING PROJECTION

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ABSTRACT. This paper supplies a sufficient condition for a local homeomorphism to be a covering projection and gives a unified treatment of a series of theorems on the topic of covering space theory.

1. Introduction

In this paper all maps $f : X \to Y$ between topological spaces X and Y are continuous functions. A covering projection is a map $p : X \to Y$ such that each point in Y has a neighborhood V such that $p^{-1}(V)$ is the union of mutually disjoint open sets which map homeomorphically onto V under p. In this instance, X, and sometimes (X, p), is called a covering space of the base space Y. A map $f : X \to Y$ is called a local homeomorphism if for each point $x \in X$ there are a neighborhood U about x and a neighborhood V about $y = f(x)$ such that the restriction $f|U : U \to V$ is a surjective homeomorphism. A map $f : X \to Y$ is called a closed map if for every closed set $A \subset X$ the image $f(A)$ is a close set in Y. A map $f : X \to Y$ is perfect if X is a T_2 space, and f is closed and all fibers $f^{-1}(y)$ is compact. A map $f : X \to Y$ is proper if and only if $f^{-1}(H)$ is compact for each compact subset H of Y. The cardinality of a set A is denoted by $\#(A)$.

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2. Preliminaries

Firstly, we have the following elementary result:

Lemma 1 [5]. *If (X, f) be a covering space of Y, then $f : X \to Y$ is a local homeomorphism.*

On the other hand, we can show that a local homeomorphism needs not be a covering projection.

Example. *p is a map from the open interval $(0, 10)$ onto a unit circle*

$$p(t) = (cost, sint).$$

One can easily check that p is a local homeomorphism, but not a covering projection.

Proof: It is easy to verify that p is a local homeomorphism. The following argument shows that p is not a covering projection.

Consider the point of the unit circle:

$$y = p(10 - 2\pi) = (cos(10 - 2\pi), sin(10 - 2\pi)) = (cos10, sin10) = p(10)$$

For every neighborhood U of y, there exits a component $f^{-1}(U) \cap (10 - \varepsilon, 10)$, $(\varepsilon > 0)$ which contains the point $t = 10$, but $f^{-1}(U) \cap (10 - \varepsilon, 10)$ is not homeomorphic with U.

A natural question is thus raised: When is a local homeomorphism $f : X \to Y$ a covering projection?

Such problems have been investigated by Jungck [2], Lelek and Mycielski [3], and Ho [1], and a number of theorems have been obtained. In this paper, we will give a sufficient condition.

3. Initial results

Lemma 2. *Let f be a local homeomorphism from a T_2 space X onto Y. If for some point $y \in Y$, the inverse image is a finite set, i.e., $f^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$, then there exist a neighborhood W of y and open sets $O_i, i = 1, 2, \cdots, k$, such that $x_i \in O_i$ and $O_i \cap O_j = \emptyset, i \neq j$, and f is a homeomorphism from O_i onto $W, i = 1, 2, \cdots, k$.*

Proof: Since f is a local homeomorphism, for every x_i, there exists a neighborhood $V_i, i = 1, 2, \cdots, k$, such that $f|V_i$ is a homeomorphism from V_i onto $f(V_i)$. And since X is T_2, there exists an open set $U_i, i = 1, 2, \cdots, k$ where $U_i \cap U_j = \emptyset$, $x \in U_i$, and $U_i \subset V_i, i = 1, 2, \cdots, k$, so every $f(U_i)$ is open in Y.

Let $W = \bigcap\limits_{i=1}^{k} f(U_i)$, then W is an open set, and $y \in W$. Let $O_i = f^{-1}(W) \cap U_i$, then each O_i is a neighborhood about x_i, and $f|O_i$ is a homeomorphism from O_i onto $f(O_i)$, and $O_i \subseteq U_i$, so $O_i \cap O_j = \emptyset$.

Note: In Lemma 2, if we take only a part of the inverse image, that is to say, if $\{x_1, x_2, \cdots, x_k\} \subseteq f^{-1}(y)$, we can get the same assertion.

Lemma 3. *If f is a local homeomorphism from a T_2 space X onto Y and k is a finite natural number, then the set $A = \left\{y|\#f^{-1}(y) \geq k\right\}$ is an open set.*

Proof: For every $y \in A$, $\#f^{-1}(y) \geq k$, take k elements $\{x_1, x_2, \cdots, x_k\}$ from $f^{-1}(y)$. By Lemma 2, there exist a neighborhood W of y and open sets $O_i, i = 1, 2, \cdots, k$, where $x_i \in O_i$ and $O_i \cap O_j = \emptyset, i \neq j$, such that f is a homeomorphism from O_i onto W.

Then for every $z \in W$, $f^{-1}(z) \geq k$, so $W \subset A$. This proves that $A = \left\{y|\#f^{-1}(y) \geq k\right\}$ is an open set.

We can also get the result that the set $A = \left\{y|\#f^{-1}(y) \geq k\right\}$ is closed if we add a new condition to Lemma 3.

Lemma 4. *If f is a local homeomorphism from a T_2 space X onto Y, k is a finite natural number, and f is a closed map, then the set $A = \left\{y|\#f^{-1}(y) \geq k\right\}$ is a closed set.*

Proof: We take reduction to absurdity.

Suppose A is not closed, then there exists $y_0 \in d(A)$, but $y_0 \notin A$, that is, $\#f^{-1}(y_0) < k$. Suppose $\#f^{-1}(y_0) = m, m < k$, and $f^{-1}(y_0) = \{x_1, x_2, \cdots, x_m\}$. By Lemma 2 there exist a neighborhood W of y and open sets $O_i, i = 1, 2, \cdots, m$ such that $x_i \in O_i$ and $O_i \cap O_j = \emptyset, i \neq j$, and f is a homeomorphism from each O_i onto W.

Let $F = X \backslash \bigcup\limits_{i=1}^{m} O_i$, then F is a closed set, and $y_0 \notin f(F)$. Since f is closed, $f(F)$ should be a closed set in Y.

On the other hand, for every point $y \in A, \#f^{-1}(y) \geq k$, hence $A \subseteq f(F)$, so $y_0 \in d(A) \subseteq d(f(F)) \subseteq f(F)$. This contradicts $y_0 \notin f(F)$.

4. Main theorem

Theorem 1. *Let f be a local homeomorphism from a T_2 space X onto a connected space Y, and suppose that f is a closed map. If for some finite number k, there exists at least one point $y_0 \in Y$ such that $\#f^{-1}(y_0) = k$, then f must be a covering projection.*

Proof: By lemmas 2 and 3, the set $A = \left\{y|\#f^{-1}(y) \geq k\right\}$ must be clopen. Since Y is connected, so $A = Y$, and for every $y \in Y$, $\#f^{-1}(y) = k$.

That is, for every point $y \in Y$, $f^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$. By Lemma 2, there exist a neighborhood W of y and open sets $O_i, i = 1, 2, \cdots, k$, such that $x_i \in O_i$, and f is a homeomorphism from O_i onto W and $f^{-1}(W) = \bigcup\limits_{i=1}^{k} O_i$. So $f : X \to Y$ is a covering projection.

The theorem clearly implies the following:

Corollary 1. *If f is a local homeomorphism from a T_2 space X onto connected space Y, and f is a perfect map, then f is a covering projection.*

Proof: For every point $y_0 \in Y$, since f is a perfect local homeomorphism, so $\#f^{-1}(y)$ is finite [4, Lemma 3.1]. So the assertion of Corollary 1 follows Theorem 1 immediately.

Corollary 2. *If f is a proper local homeomorphism from the T_2 first countable space X onto a T_2 connected space Y, then f is a covering projection.*

Proof: If f is a proper local homeomorphism from the first countable space X onto T_2 space Y, then f is a perfect map [4, Fact 2.3]. So the assertion of Corollary 2 follows Corollary 1 immediately.

5. About the theorems of Lelek and Mycielski

Problem. Let $p \in S_n$(= the n-dimensional sphere) and let $f : S^n \to S^n$ be a map such that $f(S^n - \{p\}) \subset S^n - \{p\}$, $f(p) = p$, and $f|S^n - \{p\}$ is a local homeomorphism. Must f be a homeomorphism?

In considering this question, Lelek and Mycielski [3] gave the following theorems:

Theorem 2 [3]. *If*

(1) X is connected and X or Y is locally connected,
(2) $f : X \to Y$ is an open local homeomorphism onto Y,
(3) every point $p \in Y$ is an interior point of a set $H \subset Y$ such that $f^{-1}(H)$ is compact,

then (X, f) is a covering of Y.

Theorem 3 [3]. *If*

(1) X is a compact space,
(2) $Q \subset X$ and Q is connected and locally connected,
(3) $f : X \to Y$ is a map and $f|Q$ is an open local homeomorphism,
(4) Q or $f(Q)$ is locally compact,
(5) $f(Q) \cap f(X/Q) = \emptyset$,

then $(Q, f(Q))$ is a covering of $f(Q)$.

We will show that Theorem 2 and Theorem 3 are corollaries of our Theorem 1.

Before the discussion, we must note that the definition of covering projection used in [3] differs from ours. We do not demand the local connectedness in our paper, but Lelek and Mycielski demand in the definition of related spaces not only connectedness but also local connectedness in [3]. The definition is: A pair (X, f) is called a covering space of Y if X is a connected and locally connected space, f is a map of X onto Y, and every point $y \in Y$ has a neighborhood U such that for every connected component of $f^{-1}(U)$ the partial map $f|C$ is a homeomorphism of C onto U. And we adopt the definition of local homeomorphism from [2]; that is, if f is a local homeomorphism, then f should be an open map, but in the definition of [3] local homeomorphism may be not an open map. So they emphasize open local homeomorphism in the theorems 2 and 3. Last, all topological spaces are supposed to be T_2 spaces in [3]. But it does not influence our following discussion.

Lemma 5. *If f is a local homeomorphism from the T_2 space X onto a T_2 space Y, and if for every point $p \in Y$ there exists a* *neighborhood of $H \subset Y$ such that $f^{-1}(H)$ is compact, then f is a perfect map.*

Proof: Let F be a closed set in X. Suppose that $y \in \overline{f(F)}$, and that H is a neighborhood of y such that $f^{-1}(H)$ is a compact set, then $f^{-1}(H) \cap F$ is also a compact set. Since the continuous image of a compact set is compact, $f\left(f^{-1}(H) \cap F\right) = H \cap f(F)$ is compact, and since Y is a T_2 space, $H \cap f(F)$ is a closed set. Since $y \in \overline{H \cap f(F)} = H \cap f(F) \subset f(F)$, then $y \in f(F)$, which proves that f is a closed map.

For every point p, Y is a T_2 space, the one-point-set $\{p\}$ is a closed set, so $f^{-1}(p)$ is also closed in X. And from the hypothesis of Lemma 5, for every $p \in Y$, there exists a neighborhood of $H \subset Y$, such that $f^{-1}(H)$ is compact; hence, $f^{-1}(p)$ is compact as a closed subset of $f^{-1}(H)$, so f is a perfect map.

Lemma 6. *We dispose of conditions (1) and (4) in Theorem 3; we can also attain the result that f is a perfect map as in Lemma 5.*

From Lemma 5 and Lemma 6 and Corollary 1, we can see that Theorem 2 and Theorem 3 both are consequences of Theorem 1.

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