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SIZE LEVELS OF HYPERSPACES

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Abstract. Let X be a metric continuum and $C(X)$ the hyperspace of subcontinua of X. A *size map* is a continuous function $\sigma : C(X) \to [0, \infty)$ such that $\sigma(\{x\}) = 0$ for each $x \in X$ and, if $A, B \in C(X)$ and $A \subset B$, then $\sigma(A) \leq \sigma(B)$. A *size level* is a set of the form $\sigma^{-1}(t)$, where σ is a size map and $t \in [0, \sigma(X)]$. It is known that size levels are subcontinua of $C(X)$, so we consider the space $SL(X)$ of size levels as a subspace of $C(C(X))$. In this paper we study the space $SL(X)$ and obtain an intrinsic characterization of size levels. As a consequence, we show that $SL([0, 1])$ is not homeomorphic to the Hilbert space l_2 and we obtain topological characterizations of size levels of the hyperspaces of $[0, 1]$ and the unit circle in the plane.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X, the hyperspaces of X considered in this paper are $2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\}$ and $C(X) = \{A \in 2^X : A \text{ is connected}\}$, both with the Hausdorff metric H. A *map* is a continuous function.

A *Whitney map for* 2^X (respectively, $C(X)$) is a map $\mu : 2^X \to [0, \infty)$ (respectively, $\mu : C(X) \to [0, \infty)$) such that $\mu(\{p\}) = 0$ for each $p \in X$ and, if $A, B \in 2^X$ (respectively, $A, B \in C(X)$) and $A \subset B \neq A$, then $\mu(A) < \mu(B)$. A *Whitney level* of $C(X)$

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is a set of the form $\mu^{-1}(t)$ and a *Whitney block* (this terminology was introduced in [7]) is a set of the form $\mu^{-1}([s,t])$, where μ is a Whitney map for $C(X)$ and $0 \leq s \leq t \leq \mu(X)$. It is known that, for each continuum X, there are Whitney maps for 2^X (see [8, Theorem 13.4]). Whitney levels of $C(X)$ are continua ([8, Theorem 19.9]) and have been considered by many authors who have mainly studied the similarities of Whitney levels of $C(X)$ with the continuum X. A very complete description of the research that has been done in this direction can be found in Chapter VIII of [8].

A *size map for* $C(X)$ is a map $\sigma : C(X) \to [0,\infty)$ such that $\sigma(\{p\}) = 0$ for each $p \in X$ and, if $A, B \in C(X)$ and $A \subset B$, then $\sigma(A) \leq \sigma(B)$. A *size level* of $C(X)$ is a set of the form $\sigma^{-1}(t)$, where σ is a size map and $t \in [0, \sigma(X)]$. Examples of size maps are Whitney maps and diameter maps. The diameter of a set A will be denoted by $\operatorname{diam}(A)$. Size maps and size levels have been considered in [2] and [9]. Following the proof of Theorem 19.9 of [8], it can be shown that size levels are subcontinua of $C(X)$. Then the space of size levels, $SL(X)$, can be considered as a subspace of $C(C(X))$. In [8, Question 83.16], it was asked if, for each non-degenerate continuum X, $SL(X)$ is homeomorphic to the Hilbert space l_2.

The main result of [9] says that a continuum Z is a size level of $C([0,1])$ if and only if (1) Z is a planar AR, (2) cut points of Z have component number two, and (3) any true cyclic element of Z contains at most two cut points of Z. In the same paper, the authors propose the problem of characterizing those continua that are size levels of $C(X)$, where X is a simple closed curve (see also [8, Question 83.15]). In [10], size levels for simple closed curves are studied.

Consider the square $C_0 = [0,1] \times [0,1]$ and the annulus $A_0 = \{z \in \mathbb{R}^2 : 1 \leq |z| \leq 2\}$. Let the projections $\pi_1 : C_0 \to [0,1]$ and $\pi_2 : A_0 \to S^1$ be given by $\pi_1(x,y) = x$ and $\pi_2(z) = \frac{z}{|z|}$. A *pinched square* (respectively, *pinched annulus*) is a space homeomorphic to one that can be obtained from C_0 (respectively, A_0) by taking a (possibly empty) closed subset C of $[0,1]$ (respectively, of S^1) and identifying each one of the fibers of $\pi_1^{-1}(C)$ (of $\pi_2^{-1}(C)$, respectively) to a point. In the case that C is a one-point set, a pinched annulus is called a *singular pinched annulus.* A *bunch of* 2*-cells* is a space which is homeomorphic to the union of a nonempty and at most countable family $\mathcal{F}$ of 2-cells such that the intersection of any two different elements of $\mathcal{F}$ is exactly one degenerate set $\{q_0\}$ and q_0 belongs to the manifold boundary of any element of $\mathcal{F}$ and, in the case that the family is countable, the 2-cells tend to $\{q_0\}$; then the bunch is called a *a countable bunch of* 2-*cells*. Notice that any two countable bunches of 2-cells are homeomorphic.

In this paper we study the space of size levels $SL(X)$ and we obtain an intrinsic characterization of size levels (Theorem 1). As a consequence we prove that $SL([0,1])$ is not homeomorphic to the Hilbert space l_2 and that the nondegenerate size levels of $C([0,1])$ are exactly the pinched squares and that the nondegenerate size levels of $C(S^1)$ are exactly the pinched annulus and the bunches of 2-cells.

The letter X always denotes a continuum with metric d. The set of positive integers is denoted by $\mathbb{N}$. If $\varepsilon > 0$ and $p \in X$, the ε-ball around p is denoted by $B(\varepsilon, p)$; if $A \in 2^X$, let $N(\varepsilon, A) = \{q \in X :$ there exists a point $a \in A$ such that $d(a,q) < \varepsilon\}$. The Hausdorff metric, induced by H, on $C(C(X))$ is denoted by H^2. For $\varepsilon > 0$ and $A \in 2^X$ the ε-ball in $C(X)$ around A is denoted by $B^2(\varepsilon, A)$. If $\varepsilon > 0$ and $\mathcal{A} \in C(C(X))$, let $N^2(\varepsilon, \mathcal{A}) = \{B \in C(X) :$ there exists an element $A \in \mathcal{A}$ such that $H(A,B) < \varepsilon\}$. Given a sequence $\{A_n\}_{n=1}^{\infty}$ in 2^X, let

$$\liminf A_n = \{p \in X : \text{for each } \varepsilon > 0,\ B(\varepsilon, p) \cap A_n \neq \emptyset \text{ for all but finitely many } n\},$$

$$\text{and } \limsup A_n = \{p \in X : \text{for each } \varepsilon > 0,\ B(\varepsilon, p) \cap A_n \neq \emptyset \text{ for infinitely many } n\}.$$

It is known that $A_n \to A$ (with the Hausdorff metric) if and only if $\liminf A_n = \limsup A_n = A$ (see [8, Theorem 4.7]).

Given two size levels $\mathcal{A}$ and $\mathcal{B}$ of $C(X)$, we define $\mathcal{A} \prec \mathcal{B}$ provided that each element of $\mathcal{A}$ is contained in some element of $\mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Recall that an order arc (see [8, Definition 14.1]) is a one-to-one map $\alpha : [0,1] \to C(X)$ such that, if $0 \leq s \leq t \leq 1$, then $\alpha(s) \subset \alpha(t)$. In this case we say that α joins $\alpha(0)$ and $\alpha(1)$. The order arc α is said to be a *large order arc* provided that $\alpha(0) \in F_1(X)$ and $\alpha(1) = X$. Given a subcontinuum $\mathcal{A}$ of $C(X)$, we say that $\mathcal{A}$ is *convex* provided that, if α is an order arc joining two elements of

$\mathcal{A}$, then $\alpha([0,1]) \subset \mathcal{A}$ or equivalently, if $A \subset B \subset C$ and $A, C \in \mathcal{A}$, then $B \in \mathcal{A}$.

A given subset $\mathcal{A}$ of $C(X)$ is called an *antichain* provided that if $A, B \in \mathcal{A}$ and $A \subset B$, then $A = B$.

2. An intrinsic characterization

In [5, Theorem 1.2] Whitney levels $\mathcal{A}$ differ from $F_1(X)$ and $\{X\}$ are characterized as compact subsets $\mathcal{A}$ of $C(X) - (F_1(X) \cup \{X\})$ such that $\mathcal{A}$ is an antichain and for every order arc α, $\alpha([0,1]) \cap \mathcal{A} \neq \emptyset$. We are ready to give a similar intrinsic characterization for size levels.

Theorem 1. *Let $\mathcal{A}$ be a subcontinuum of $C(X)$. Then $\mathcal{A}$ is a size level if and only if $\mathcal{A}$ is convex, $\mathcal{A} \cap \alpha([0,1]) \neq \emptyset$ for each large order arc α and either $F_1(X) \subset \mathcal{A}$ or $\mathcal{A} \cap F_1(X) = \emptyset$.*

Proof: The necessity is clear. In order to prove the sufficiency, assume that $\mathcal{A}$ is convex, $\mathcal{A} \cap \alpha([0,1]) \neq \emptyset$ for each large order arc and either $F_1(X) \subset \mathcal{A}$ or $\mathcal{A} \cap F_1(X) = \emptyset$.

Let $\mathcal{D} = \{D \in C(X) - \mathcal{A}$: there exists an element $A \in \mathcal{A}$ such that $A \subset D\}$ and $\mathcal{E} = \{D \in C(X) - \mathcal{A}$: there exists an element $A \in \mathcal{A}$ such that $D \subset A\}$. Given $D \in C(X) - \mathcal{A}$, take a large order arc $\alpha : [0,1] \to C(X)$ such that $\alpha(t) = D$ for some $t \in [0,1]$. Then there exists $s \in [0,1]$ such that $\alpha(s) \in \mathcal{A}$. Thus $s \neq t$. If $s < t$, then $D \in \mathcal{D}$ and, if $t < s$, then $D \in \mathcal{E}$. This proves that $C(X) - \mathcal{A} = \mathcal{D} \cup \mathcal{E}$. The convexity of $\mathcal{A}$ implies that $\mathcal{D} \cap \mathcal{E} = \emptyset$.

Fix a Whitney map $\omega : 2^X \to [0,1]$ such that $\omega(X) = 1$ and if $A, B, C \in 2^X$ and $A \subset B$, then

$$\omega(B \cup C) - \omega(A \cup C) \leq \omega(B) - \omega(A)$$

(such a map exists as it was observed in [1]).

Let $\varphi : C(X) \times (0,1] \times C(X) \to \mathbb{R}$ be given by

$$\varphi(A, t, B) = \omega(A \cup B) - \omega(B) - t(\omega(B) - \omega(A)).$$

The map φ was considered in [6, Definition 3.1] where it was proved that if $A \in C(X) - (\{X\} \cup F_1(X))$ and $t \in (0,1]$ are fixed, then the set $L(A,t) = \{B \in C(X) : \varphi(A,t,B) = 0\}$ is a Whitney level of $C(X)$ and $A \in L(A,t)$ ([6, Lemma 3.4 (a)]). □

Claim 1. *If $D \in \mathcal{D}$, then there exists a* Whitney *level $\mathcal{B}_D$ of $C(X)$ which separates $\{D\}$ and $\mathcal{A}$ in $C(X)$ and $\mathcal{A} \prec \mathcal{B}_D$.*

In order to prove Claim 1, take a large order arc $\alpha : [0,1] \to C(X)$ such that $\alpha(r) = D$ for some $r \in [0,1]$. Then there exists $s \in [0,1]$ such that $\alpha(s) \in \mathcal{A}$. Since $D \notin \mathcal{E}$, $s < r$. Since $\mathcal{A}$ is compact, there exists a number $r_0 \in (s,r)$ such that $\alpha([r_0,r]) \cap \mathcal{A} = \emptyset$. Let $A = \alpha(r_0)$ and, for each $t \in (0,1]$ consider the Whitney level $L(A,t)$. We claim that there exists $n \in \mathbb{N}$ such that $L(A,\frac{1}{n}) \cap \mathcal{A} = \emptyset$. Suppose to the contrary that, for each $n \in \mathbb{N}$, there exists an element $B_n \in L(A,\frac{1}{n}) \cap \mathcal{A}$. We may assume that $B_n \to B$ for some $B \in \mathcal{A}$. Since $\omega(A \cup B_n) - \omega(B_n) - \frac{1}{n}(\omega(B_n) - \omega(A)) = 0$ for each $n \in \mathbb{N}$, taking limit as $n \to \infty$, we obtain that $\omega(A \cup B) = \omega(B)$. Since ω is a Whitney map, $A \cup B = B$ and $A \subset B$. Then $\alpha(s) \subset A \subset B$ and $\alpha(s), B \in \mathcal{A}$. The convexity of $\mathcal{A}$ implies that $A \in \mathcal{A}$. This contradicts the choice of r_0 and proves that there exists $n \in \mathbb{N}$ such that $L(A,\frac{1}{n}) \cap \mathcal{A} = \emptyset$. Let $\mu : C(X) \to [0,1]$ be a Whitney map and let $v \in [0,1]$ be such that $\mu^{-1}(v) = L(A,\frac{1}{n})$. Since $s < r_0$ by the definition of r_0, it follows that $\alpha(s) \subset A$. Further, $\alpha([r_0,r]) \cap \mathcal{A} = \emptyset$ implies that $\alpha(s) \neq A$, whence $\mu(\alpha(s)) < v$. Since $\mathcal{A}$ is connected, $\mathcal{A} \subset \mu^{-1}([0,v))$. On the other hand, A is a proper subset of D, so $D \in \mu^{-1}((v,1])$. Thus $\mathcal{B}_D = \mu^{-1}(v)$ has the required properties. This completes the proof of Claim 1.

Claim 2. *If $E \in \mathcal{E}$, then there exists a* Whitney *level $\mathcal{B}_E$ of $C(X)$ which separates $\{E\}$ and $\mathcal{A}$ in $C(X)$ and $\mathcal{B}_E \prec \mathcal{A}$.*

Take a large order arc $\alpha : [0,1] \to C(X)$ such that $\alpha(r) = E$ for some $r \in [0,1]$. Then there exists $s \in [0,1]$ such that $\alpha(s) \in \mathcal{A}$. Since $E \notin \mathcal{D}$, $r < s$. Since $\mathcal{A}$ is compact, there exists a number $r_0 \in (r,s)$ such that $\alpha([r,r_0]) \cap \mathcal{A} = \emptyset$. Let $A = \alpha(r_0)$. If $\mathcal{A} \cap F_1(X) \neq \emptyset$, then $F_1(X) \subset \mathcal{A}$. Take $x \in E$. Then $\{x\} \subset E \subset \alpha(s)$ and $\{x\}, \alpha(s) \in \mathcal{A}$. By the convexity of $\mathcal{A}$, we get $E \in \mathcal{A}$. This contradiction proves that $\mathcal{A} \cap F_1(X) = \emptyset$. Hence, there exists $u \in (0,1)$ such that $\mathcal{A} \cap \omega^{-1}([0,u]) = \emptyset$ and $u < \omega(A)$.

For each $t \in (0,1]$, let $\mathcal{F}_t = \{B \in C(X) : \varphi(B,t,A) = 0\}$ and $\mathcal{G}_t = (\mathcal{F}_t \cap \omega^{-1}([u,1])) \cup \{B \in \omega^{-1}(u) \cap C(X) : \varphi(B,t,A) \geq 0\}$. Notice that $A \in \mathcal{G}_t$.

Next we show that $\mathcal{G}_t$ is a Whitney level of $C(X)$. Clearly, $\mathcal{G}_t$ is compact. Since $u > 0$, $\mathcal{G}_t \cap F_1(X) = \emptyset$. Since $\varphi(X,t,A) = 1-\omega(A)-t(\omega(A)-1) > 0$, $X \notin \mathcal{G}_t$. According to [5, Theorem 1.2] we only need to show that $\mathcal{G}_t$ is an antichain and, if $\beta : [0,1] \to C(X)$

is a large order arc, then $\mathcal{G}_t \cap \beta([0,1]) \neq \emptyset$. Note that there exists $v \in [0,1]$ such that $\beta(v) \in \omega^{-1}(u)$. If $\varphi(\beta(v),t,A) \geq 0$, then $\beta(v) \in \mathcal{G}_t$. Suppose then that $\varphi(\beta(v),t,A) \leq 0$. Since $\varphi(\beta(1),t,A) = \varphi(X,t,A) > 0$, there exists $v_1 \in [v,1]$ such that $\varphi(\beta(v_1),t,A) = 0$. Since $\beta(v) \subset \beta(v_1)$, $u = \omega(\beta(v)) \leq \omega(\beta(v_1))$. Thus $\beta(v_1) \in \mathcal{G}_t$. This completes the proof that $\beta([0,1]) \cap \mathcal{G}_t \neq \emptyset$. Now, suppose that $\mathcal{G}_t$ is not an antichain. Then there exist $B, C \in \mathcal{G}_t$ such that B is a proper subcontinuum of C. Thus $B \notin \omega^{-1}(u)$ or $C \notin \omega^{-1}(u)$ and $\varphi(B,t,A) = \omega(B \cup A) - \omega(A) - t(\omega(A) - \omega(B)) < \omega(C \cup A) - \omega(A) - t(\omega(A) - \omega(C)) = \varphi(C,t,A)$. This last inequality implies that either $B \notin \mathcal{F}_t$ or $C \notin \mathcal{F}_t$. If $B \in \mathcal{F}_t \cap \omega^{-1}([u,1])$, then $C \notin \mathcal{F}_t$ and $C \notin \omega^{-1}(u)$. This is impossible since $C \in \mathcal{G}_t$. If $B \in \omega^{-1}(u)$ and $\varphi(B,t,A) \geq 0$, then $C \notin \omega^{-1}(u)$ and $C \notin \mathcal{F}_t$. This is also impossible since $C \in \mathcal{G}_t$. Therefore, $\mathcal{G}_t$ is an antichain. We have proved that $\mathcal{G}_t$ is a Whitney level.

Now, we show that there exists $n \in \mathbb{N}$ such that $\mathcal{G}_{\frac{1}{n}} \cap \mathcal{A} = \emptyset$. Suppose to the contrary that, for each $n \in \mathbb{N}$, there exists an element $B_n \in \mathcal{G}_{\frac{1}{n}} \cap \mathcal{A}$. We may assume that $B_n \to B$ for some $B \in \mathcal{A}$. By the choice of u, $B_n \in \mathcal{F}_{\frac{1}{n}}$ and thus $0 = \varphi(B_n, \frac{1}{n}, A) = \omega(B_n \cup A) - \omega(A) - \frac{1}{n}(\omega(A) - \omega(B_n))$. Taking limit as $n \to \infty$ we get $\omega(B \cup A) = \omega(A)$. Thus $A \cup B = A$ and $B \subset A$. Hence $B \subset A \subset \alpha(s)$ and $B, \alpha(s) \in \mathcal{A}$. By the convexity of $\mathcal{A}$, $A \in \mathcal{A}$. This contradicts the choice of r_0 and proves that there exists $n \in \mathbb{N}$ such that $\mathcal{G}_{\frac{1}{n}} \cap \mathcal{A} = \emptyset$.

Let $\mu : C(X) \to [0,1]$ be a Whitney map and let $v_0 \in [0,1]$ be such that $\mu^{-1}(v_0) = \mathcal{G}_{\frac{1}{n}}$. Since $E \subsetneq A \subsetneq \alpha(s)$ and $A \in \mathcal{G}_{\frac{1}{n}}$, $E \in \mu^{-1}([0,v_0))$ and $\alpha(s) \in \mu^{-1}((v_0,1]) \cap \mathcal{A}$. Since $\mathcal{A}$ is connected, $\mathcal{A} \subset \mu^{-1}((v_0,1])$. Thus $\mathcal{B}_E = \mu^{-1}(v_0)$ has the required properties. This proves Claim 2.

Claim 3. *If $X \notin \mathcal{A}$, then there exists a sequence of* Whitney *levels $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $C(X)$ such that ... $\prec \mathcal{A}_3 \prec \mathcal{A}_2 \prec \mathcal{A}_1 \prec \{X\}$, $\mathcal{A} \prec \mathcal{A}_n$, $\mathcal{A}_n \subset N^2(\frac{1}{n}, \mathcal{A})$ and $\mathcal{A}_n$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(\frac{1}{n}, \mathcal{A})$ in $C(X)$ for each $n \in \mathbb{N}$.*

In order to prove Claim 3, we first show that, for each $\varepsilon > 0$, there exists a Whitney level $\mathcal{B}$ of $C(X)$ such that $\mathcal{A} \prec \mathcal{B}$, $\mathcal{B} \subset N^2(\varepsilon, \mathcal{A})$ and $\mathcal{B}$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(\varepsilon, \mathcal{A})$ in $C(X)$.

Since $X \notin \mathcal{A}$, $\mathcal{D} \neq \emptyset$. Let $\varepsilon_1 \in (0, \varepsilon)$ be such that $X \notin N^2(\varepsilon_1, \mathcal{A})$. Then $\mathcal{D}_1 = \mathcal{D} - N^2(\varepsilon_1, \mathcal{A}) \neq \emptyset$.

Let $\{D_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}_1$. Then, for each $n \in \mathbb{N}$ there exists an element $A_n \in \mathcal{A}$ such that $A_n \subset D_n$. Since $\mathcal{A}$ is compact, there exist subsequences $\{A_{n_k}\}_{k=1}^{\infty}$ and $\{D_{n_k}\}_{k=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$, respectively, such that $A_{n_k} \to A$ for some $A \in \mathcal{A}$ and $D_{n_k} \to D$ for some $D \in C(X) - N^2(\varepsilon_1, \mathcal{A})$. Notice that $A \subset D$. Therefore, $D \in \mathcal{D}_1$. We have proved that $\mathcal{D}_1$ is compact.

By Claim 1, for each $D \in \mathcal{D}_1$, there exists a Whitney map $\mu_D : C(X) \to [0,1]$ such that $\mu_D(X) = 1$, $\mathcal{A} \subset \mu_D^{-1}([0, \frac{1}{2}))$ and $D \in \mu_D^{-1}((\frac{1}{2}, 1])$. Since $\mathcal{D}_1$ is compact, there exist $m \in \mathbb{N}$ and $D_1, ..., D_m \in \mathcal{D}_1$ such that $\mathcal{D}_1 \subset \mu_{D_1}^{-1}((\frac{1}{2}, 1]) \cup ... \cup \mu_{D_m}^{-1}((\frac{1}{2}, 1])$.

Define $\mu : C(X) \to [0,1]$ by $\mu(A) = \max\{\mu_{D_1}(A), ..., \mu_{D_m}(A)\}$. It is easy to check that μ is a Whitney map, $\mathcal{A} \subset \mu^{-1}([0, \frac{1}{2}))$ and $\mathcal{D}_1 \subset \mu^{-1}((\frac{1}{2}, 1])$.

Put $\mathcal{B} = \mu^{-1}(\frac{1}{2})$. Let $A \in \mathcal{B}$ and take a large order arc $\alpha : [0,1] \to C(X)$ such that $\alpha(r) = A$ for some $r \in [0,1]$. Then there exists $s \in [0,1]$ such that $\alpha(s) \in \mathcal{A}$. Since $\mu(\alpha(s)) < \frac{1}{2} = \mu(\alpha(r))$, $s < r$ and $\alpha(s) \subset \alpha(r) = A$. Thus $A \in \mathcal{D} - (\mathcal{A} \cup \mathcal{D}_1) \subset N^2(\varepsilon_1, \mathcal{A}) \subset N^2(\varepsilon, \mathcal{A})$. We have shown that $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \subset N^2(\varepsilon, \mathcal{A})$.

Now we are ready to prove Claim 3. Let $\mathcal{A}_1$ be a Whitney level of $C(X)$ such that $\mathcal{A} \prec \mathcal{A}_1$, $\mathcal{A}_1 \subset N^2(1, \mathcal{A})$ and $\mathcal{A}_1$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(1, \mathcal{A})$ in $C(X)$. Let $\mu_1 : C(X) \to [0,1]$ and $t \in [0,1]$ be such that $\mathcal{A}_1 = \mu_1^{-1}(t)$. Then $\mathcal{A} \subset \mu_1^{-1}([0,t))$ and $\mathcal{D} - N^2(1, \mathcal{A}) \subset \mu_1^{-1}((t, 1])$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{2}$, $\mathcal{A}_1 \cap N^2(\varepsilon, \mathcal{A}) = \emptyset$ and $N^2(\varepsilon, \mathcal{A}) \subset \mu_1^{-1}([0,t))$. Then there exists a Whitney level $\mathcal{A}_2$ such that $\mathcal{A} \prec \mathcal{A}_2$, $\mathcal{A}_2 \subset N^2(\varepsilon, \mathcal{A}) \subset \mu_1^{-1}([0,t))$ and $\mathcal{A}_2$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(\varepsilon, \mathcal{A})$ in $C(X)$. Thus $\mathcal{A}_2 \prec \mathcal{A}_1$, $\mathcal{A}_2 \subset N^2(\frac{1}{2}, \mathcal{A})$ and $\mathcal{A}_2$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(\frac{1}{2}, \mathcal{A})$ in $C(X)$. Proceeding in this way, it is possible to construct the required sequence in Claim 3.

With similar arguments to those used in Claim 3, but changing min by max in the definition of μ, the following claim can be proved.

Claim 4. *If $\mathcal{A} \cap F_1(X) = \emptyset$, then there exists a sequence of* Whitney *levels $\{\mathcal{B}_n\}_{n=1}^{\infty}$ of $C(X)$ such that $F_1(X) \prec \mathcal{B}_1 \prec \mathcal{B}_2 \prec \mathcal{B}_3 \prec ...$, $\mathcal{B}_n \prec \mathcal{A}$, $\mathcal{B}_n \subset N^2(\frac{1}{n}, \mathcal{A})$ and $\mathcal{B}_n$ separates $\mathcal{A}$ and $\mathcal{E} - N^2(\frac{1}{n}, \mathcal{A})$ in $C(X)$ for each $n \in \mathbb{N}$.*

We are ready to finish the proof of the theorem. Assume that $\mathcal{A} \cap F_1(X) = \emptyset$ and $X \notin \mathcal{A}$. The cases $F_1(X) \subset \mathcal{A}$ and $X \in \mathcal{A}$ are easier. Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ and $\{\mathcal{B}_n\}_{n=1}^{\infty}$ be as in Claims 3 and 4. Let $\mathcal{C}_1 = F_1(X) \cup \mathcal{B}_1 \cup \mathcal{A}_1 \cup \{X\}$. Since $F_1(X) \prec \mathcal{B}_1 \prec \mathcal{A}_1 \prec \{X\}$, the function $\nu_1 : \mathcal{C}_1 \to [0,1]$ defined by the conditions: $\nu_1(F_1(X)) = \{0\}$, $\nu_1(\mathcal{B}_1) = \{\frac{1}{2} - \frac{1}{3}\}$, $\nu_1(\mathcal{A}_1) = \{\frac{1}{2} + \frac{1}{3}\}$ and $\nu_1(X) = 1$ is a well defined Whitney map for $\mathcal{C}_1$. Applying the extension theorem for Whitney maps proved by L. E. Ward, Jr. (see Theorem 16.10 of [8]), there exists a Whitney map $\mu_1 : C(X) \to [0,1]$ which extends ν_1. Notice that $\mathcal{A}_1 = \mu_1^{-1}(\frac{1}{2} + \frac{1}{3})$ and $\mathcal{B}_1 = \mu_1^{-1}(\frac{1}{2} - \frac{1}{3})$. Let $\mathcal{C}_2 = \mu_1^{-1}([0, \frac{1}{2} - \frac{1}{3}] \cup [\frac{1}{2} + \frac{1}{3}, 1] \cup \mathcal{A}_2 \cup \mathcal{B}_2$. Since $\mathcal{B}_1 \prec \mathcal{B}_2 \prec \mathcal{A}_2 \prec \mathcal{A}_1$, the function $\nu_2 : \mathcal{C}_2 \to [0,1]$ defined by the condition $\nu_2(A) = \mu_1(A)$ for each $A \in \mu_1^{-1}([0, \frac{1}{2} - \frac{1}{3}] \cup [\frac{1}{2} + \frac{1}{3}, 1])$, $\mathcal{B}_2 = \nu_2^{-1}(\frac{1}{2} - \frac{1}{4})$ and $\mathcal{A}_2 = \nu_2^{-1}(\frac{1}{2} + \frac{1}{4})$ is a well defined Whitney map for $\mathcal{C}_2$. Applying again Theorem 16.10 of [8] there exists a Whitney map $\mu_2 : C(X) \to [0,1]$ such that $\mu_2 | \mathcal{C}_2 = \nu_2$. Notice that $\mathcal{A}_2 = \mu_2^{-1}(\frac{1}{2} + \frac{1}{4})$ and $\mathcal{B}_2 = \mu_2^{-1}(\frac{1}{2} - \frac{1}{4})$. By similar successive applications of Theorem 16.10 of [8] it is possible to construct a sequence of Whitney maps $\mu_1, \mu_2, \mu_3, \ldots$ such that, for each $n \in \mathbb{N}$, $\mathcal{A}_{n+1} = \mu_{n+1}^{-1}(\frac{1}{2} + \frac{1}{n+2})$, $\mathcal{B}_{n+1} = \mu_{n+1}^{-1}(\frac{1}{2} - \frac{1}{n+2})$ and , $\mu_{n+1}^{-1} | \mu_n^{-1}([0, \frac{1}{2} - \frac{1}{n+1}] \cup [\frac{1}{2} + \frac{1}{n+1}, 1]) = \mu_n^{-1} | \mu_n^{-1}([0, \frac{1}{2} - \frac{1}{n+1}] \cup [\frac{1}{2} + \frac{1}{n+1}, 1])$.

In order to see that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence, let $\varepsilon > 0$ and let $M \in \mathbb{N}$ be such that $\frac{1}{M+1} < \frac{\varepsilon}{2}$. Let $A \in C(X)$ and $n \geq m \geq M$. If $A \in \mu_M^{-1}([0, \frac{1}{2} - \frac{1}{M+1}] \cup [\frac{1}{2} + \frac{1}{M+1}, 1])$, then $\mu_n(A) = \mu_M(A) = \mu_m(A)$. If $A \notin \mu_M^{-1}([0, \frac{1}{2} - \frac{1}{M+1}] \cup [\frac{1}{2} + \frac{1}{M+1}, 1])$, then there exist $C, B \in C(X)$ such that $C \subset A \subset B$, $\mu_M(C) = \frac{1}{2} - \frac{1}{M+1}$ and $\mu_M(B) = \frac{1}{2} + \frac{1}{M+1}$. Thus $\frac{1}{2} - \frac{1}{M+1} = \mu_n(C) = \mu_m(C) \leq \mu_n(A), \mu_m(A) \leq \mu_n(B) = \mu_m(B) = \frac{1}{2} + \frac{1}{M+1}$. Hence $|\mu_n(A) - \mu_m(A)| < \frac{2}{M+1} < \varepsilon$. In both cases $|\mu_n(A) - \mu_m(A)| < \varepsilon$. Therefore, $\{\mu_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence.

Hence, there exists a map $\sigma : C(X) \to [0,1]$ such that $\lim \mu_n = \sigma$. By Theorem 3 below, σ is a size map.

Given $A \in \mathcal{A}$ and $n \in \mathbb{N}$, $\mu_n^{-1}(\frac{1}{2} - \frac{1}{n+1}) = \mathcal{B}_n \prec \mathcal{A} \prec \mathcal{A}_n = \mu_n^{-1}(\frac{1}{2} + \frac{1}{n+1})$. So, $\frac{1}{2} - \frac{1}{n+1} \leq \mu_n(A) \leq \frac{1}{2} + \frac{1}{n+1}$. Thus $\sigma(A) = \frac{1}{2}$. Therefore, $\mathcal{A} \subset \sigma^{-1}(\frac{1}{2})$. Now, if $A \in C(X) - \mathcal{A}$, then there exists $n \in \mathbb{N}$ such that $A \notin N^2(\frac{1}{n}, \mathcal{A})$. Suppose, for example, that $A \in \mathcal{D}$.

Then $A \in \mathcal{D} - N^2(\frac{1}{n}, \mathcal{A})$. Since $\mathcal{A}_n = \mu_n^{-1}(\frac{1}{2} + \frac{1}{n+1})$ separates $\mathcal{A}$ and $\mathcal{D} - N^2(\frac{1}{n}, \mathcal{A})$ in $C(X)$ and $\mathcal{A} \subset \mu_n^{-1}([0, \frac{1}{2} + \frac{1}{n+1}])$, we have that $\frac{1}{2} + \frac{1}{n+1} < \mu_n(A)$. Thus, $\frac{1}{2} + \frac{1}{n+1} < \mu_n(A) = \mu_{n+1}(A) = \mu_{n+2}(A) = \ldots$. Hence, $\frac{1}{2} + \frac{1}{n+1} < \sigma(A)$ and $A \notin \sigma^{-1}(\frac{1}{2})$. Therefore, $\mathcal{A} = \sigma^{-1}(\frac{1}{2})$. This proves that $\mathcal{A}$ is a size level and completes the proof of the theorem. □

3. The space of size levels

In the space $M(X) = [0, \infty)^{C(X)}$ of maps $f : C(X) \to [0, \infty)$ with the "sup" metric consider the subspaces $S(X) = \{\sigma : C(X) \to [0, \infty) : \sigma \text{ is a size map}\}$ and $W(X) = \{\mu : C(X) \to [0, \infty) : \mu \text{ is a Whitney map}\}$. The space $W(X)$ was considered in [5] where it was proved that $W(X)$ is homeomorphic to the Hilbert space l_2 ([5, Theorem 5.6]).

Whitney blocks, Whitney levels and size levels are subcontinua of $C(X)$, so they can be considered as elements in $C(C(X))$. So, we can consider the subspaces:

$WB(X) = \{\mathcal{A} \in C(C(X)) : \mathcal{A} \text{ is a Whitney block of } C(X)\}$,
$WL(X) = \{\mathcal{A} \in C(C(X)) : \mathcal{A} \text{ is a Whitney level of } C(X)\}$,
$SL(X) = \{\mathcal{A} \in C(C(X)) : \mathcal{A} \text{ is a size level of } C(X)\}$

of $C(C(X))$. The space $WL(X)$ was considered in [6] where it was proved that $WL(X)$ is homeomorphic to the Hilbert space l_2.

Lemma 2. *Let $\sigma : C(X) \to [0, \infty)$ be a size map. Then there exists a sequence of* Whitney *maps $\{\mu_n\}_{n=1}^{\infty}$ such that $\{\mu_n\}_{n=1}^{\infty}$ uniformly converges to σ and $\sigma(A) < \mu_n(A)$ for each $n \in \mathbb{N}$ and for each $A \in C(X) - F_1(X)$.*

Proof: Fix a Whitney map $\omega : C(X) \to [0, 1]$, where $\omega(X) = 1$. Define $\mu_n : C(X) \to [0, \infty)$ by $\mu_n(A) = \sigma(A) + \frac{\omega(A)}{2^n}$. It is easy to show that the sequence $\{\mu_n\}_{n=1}^{\infty}$ has the required properties. □

Theorem 3. $\mathrm{cl}_{M(X)}(W(X)) = S(X)$.

Proof: The inclusion $S(X) \subset \mathrm{cl}_{M(X)}(W(X))$ is proved in Lemma 2. Since $W(X) \subset S(X)$, we only need to show that $S(X)$ is closed in $M(X)$. Take a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $S(X)$ and suppose that $\{\sigma_n\}_{n=1}^{\infty}$ uniformly converges to an element $\sigma \in M(X)$. Given

$A \in C(X)$, $\sigma_n(A) \to \sigma(A)$. If $p \in X$, $\sigma_n(\{p\}) = 0$ for each $n \in \mathbb{N}$, so $\sigma(\{p\}) = 0$. If $A, B \in C(X)$ and $A \subset B$, then $\sigma_n(A) \leq \sigma_n(B)$, thus $\sigma(A) \leq \sigma(B)$. Therefore, $\sigma \in S(X)$. This completes the proof that $S(X)$ is closed in $M(X)$ and ends the proof of the theorem. □

Theorem 4. $WL(X) \subset WB(X) \subset SL(X) \subset \mathrm{cl}_{C(C(X))}(WB(X))$.

Proof: Clearly, $WL(X) \subset WB(X)$. Let $\mathcal{A} \in WB(X)$. Then there exist a Whitney map $\mu : C(X) \to [0, \infty)$ and numbers $0 \leq s \leq t \leq \mu(X)$ such that $\mathcal{A} = \mu^{-1}([s,t])$. Define $\sigma : C(X) \to [0, \infty)$ by:

$$\sigma(A) = \begin{cases} \mu(A), & \text{if } \mu(A) \leq s, \\ s, & \text{if } s \leq \mu(A) \leq t, \\ s + \mu(A) - t, & \text{if } t \leq \mu(A). \end{cases}$$

Clearly, σ is a size map and $\sigma^{-1}(s) = \mathcal{A}$. Thus $\mathcal{A} \in SL(X)$. This proves that $WB(X) \subset SL(X)$.

In order to prove that $SL(X) \subset \mathrm{cl}_{C(C(X))}(WB(X))$, let $\mathcal{B} \in SL(X)$. Then there exist a size map $\eta : C(X) \to [0, \infty)$ and a number $r \in [0, \eta(X)]$ such that $\mathcal{B} = \eta^{-1}(r)$. Fix a Whitney map $\omega : C(X) \to [0, 1]$, where $\omega(X) = 1$. For each $n \in \mathbb{N}$, let $\mu_n : C(X) \to [0, \infty)$ be given by $\mu_n(A) = \eta(A) + \frac{\omega(A)}{2^n}$. Clearly, μ_n is a Whitney map. Let $\mathcal{C}_n = \mu_n^{-1}([r, r + \frac{1}{2^n}])$. Since $\mu_n(X) = \eta(X) + \frac{1}{2^n} \geq r + \frac{1}{2^n}$, the set $\mathcal{C}_n$ is a Whitney block.

We claim that $\lim \mathcal{C}_n = \mathcal{B}$ (in $C(C(X))$). Given $B \in \mathcal{B}$ and $n \in \mathbb{N}$, $\mu_n(B) = r + \frac{\omega(B)}{2^n} \in [r, r + \frac{1}{2^n}]$. Thus $\mathcal{B} \subset \mathcal{C}_n$ for each $n \in \mathbb{N}$, whence $\mathcal{B} \subset \liminf \mathcal{C}_n$.

In order to prove that $\limsup \mathcal{C}_n \subset \mathcal{B}$, let $C \in \limsup \mathcal{C}_n$. Then there exist a sequence $n_1 < n_2 < \ldots$ and elements $C_k \in \mathcal{C}_{n_k}$ such that $C_k \to C$ (in $C(X)$). Then $r \leq \eta(C_k) + \frac{\omega(C_k)}{2^{n_k}} \leq r + \frac{1}{2^{n_k}}$. Taking limits we obtain that $\eta(C) = r$. Thus $C \in \mathcal{B}$. We have shown that $\limsup \mathcal{C}_n \subset \mathcal{B}$. Therefore, $\mathcal{B} \in \mathrm{cl}_{C(C(X))}(WB(X))$. This completes the proof that $SL(X) \subset \mathrm{cl}_{C(C(X))}(WB(X))$ and the proof of the theorem. □

Theorem 5. *If $\mathcal{A} \in SL(X)$ is such that $F_1(X) \subset \mathcal{A}$, then $SL(X)$ is not locally compact at $\mathcal{A}$.*

Proof: We may assume that $\mathrm{diam}(X) \leq 1$. Let $\sigma : C(X) \to [0, \infty)$ be a size map such that $\sigma^{-1}(0) = \mathcal{A}$. Let $\varepsilon > 0$. Fix a Whitney map $\mu : 2^X \to [0, 1]$ such that $\mu(X) = 1$. It is easy to prove that there exists a positive number $\delta < 1$ such that, if $A \in C(X)$

and $\mu(A) \leq \delta$, then $\operatorname{diam}(A) < \varepsilon$. Fix a point $p \in X$ and a sequence of nondegenerate subcontinua $A_1 \supset A_2 \supset A_3 \supset ...$ such that $A_n \to \{p\}$, $p \in A_n$ and $\mu(A_n) < \delta$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $\mathcal{C}_n = \{A \in C(X) : A_n \subset A\} \cup \mu^{-1}([\delta, 1])$. Notice that $\mathcal{C}_n$ is a closed subset of $C(X)$. Let $\varsigma_n : C(X) \to [0,1]$ be given by

$$\varsigma_n(A) = \inf\{\lambda > 0 : \text{there exists } C \in \mathcal{C}_n \text{ such that } C \subset N(\lambda, A)\}.$$

Then ς_n has the following properties:
(a) ς_n is continuous,
(b) if $A, B \in C(X)$ and $A \subset B$, then $\varsigma_n(A) \geq \varsigma_n(B)$,
(c) $\varsigma_n(A) = 0$ if and only if $A \in \mathcal{C}_n$,
(d) $\varsigma_n(\{x\}) > 0$ for each $x \in X$.

Properties (a) and (b) are easy to check, (d) follows from (c). Sufficiency in (c) follows from the fact that $A \subset N(\lambda, A)$ for each $A \in C(X)$. In order to prove the necessity in (c), suppose that $\varsigma_n(A) = 0$. Then there exist sequences $\{\lambda_m\}_{m=1}^{\infty}$ and $\{C_m\}_{m=1}^{\infty}$ of positive numbers and elements of $\mathcal{C}_n$ such that $\lambda_m \to 0$ and $C_m \subset N(\lambda_m, A)$ for each $m \in \mathbb{N}$. We may assume that $C_m \to C$ for some $C \in \mathcal{C}_n$. Clearly, $C \subset A$. This implies that $A \in \mathcal{C}_n$.

Let $\rho = \max\{\delta - \delta \cdot \varsigma_n(\{x\}) : x \in X\}$. Then $0 \leq \rho < \delta$.

Define $\tau_n : C(X) \to [0, \infty)$ by

$$\tau_n(A) = \inf\{\lambda > 0 : \text{there exists } C \in \mathcal{A} \cup \mu^{-1}([0,\delta]) \text{ such that } A \subset N(\lambda, C)\}.$$

Then τ_n has the following properties:
(e) τ_n is continuous,
(f) if $A, B \in C(X)$ and $A \subset B$, then $\tau_n(A) \leq \tau_n(B)$,
(g) $\tau_n(A) = 0$ if and only if $A \in \mathcal{A} \cup \mu^{-1}([0, \delta])$,
(h) $\tau_n(\{x\}) = 0$ for each $x \in X$.

The proof of the properties (e)-(h) is similar to the proof of properties (a)-(d).

Let $\eta_n : C(X) \to [0, \infty)$ be given by

$$\eta_n(A) = \begin{cases} \delta - \rho + \tau_n(A), & \text{if } \mu(A) \geq \delta, \\ \max\{\rho, \delta + (\mu(A) - \delta)\varsigma_n(A)\} - \rho, & \text{if } \mu(A) \leq \delta. \end{cases}$$

We will show that η_n is a size map. By properties (g) and (c), if $\mu(A) = \delta$, then both definitions of $\eta_n(A)$ give the same value $\delta - \rho$. Thus η_n is well defined and continuous. Notice that if $\mu(A) \geq \delta$, then $\eta_n(A) \geq \delta - \rho$ and if $\mu(A) \leq \delta$, then $\eta_n(A) \leq \delta - \rho$. By the choice of ρ, for each $x \in X$, $\eta_n(\{x\}) = 0$.

Let $A, B \in C(X)$ be such that $A \subset B$. We want to see that $\eta_n(A) \leq \eta_n(B)$. If $\mu(A) \leq \delta \leq \mu(B)$, then $\eta_n(A) \leq \delta - \rho \leq \eta_n(B)$. If $\mu(A) \leq \mu(B) \leq \delta$, then $\varsigma_n(A) \geq \varsigma_n(B)$, whence $\eta_n(A) \leq \eta_n(B)$. Finally, if $\delta \leq \mu(A) \leq \mu(B)$, property (f) implies that $\eta_n(A) \leq \eta_n(B)$. This completes the proof that η_n is a size map.

Let $\mathcal{B}_n = \eta_n^{-1}(\delta - \rho)$. Then $\mathcal{B}_n$ is a size level. Now, we see that $H^2(\mathcal{A}, \mathcal{B}_n) < \varepsilon$. Let $A \in \mathcal{A}$. If $\mu(A) \geq \delta$, then $\tau_n(A) = 0$. So $\eta_n(A) = \delta - \rho$ and $A \in \mathcal{B}_n$. If $\mu(A) \leq \delta$, then using order arcs it is possible to construct an element $B \in \mu^{-1}(\delta)$ such that $A \subset B$. By the choice of δ, $\operatorname{diam}(B) < \varepsilon$. Thus, $H(A, B) < \varepsilon$. Furthermore, $\eta_n(B) = \delta - \rho$. Hence, $B \in \mathcal{B}_n$. In both cases we have found an element $B \in \mathcal{B}_n$ such that $H(A, B) < \varepsilon$. Therefore, $\mathcal{A} \subset N^2(\varepsilon, \mathcal{B}_n)$. Now, take an element $C \in \mathcal{B}_n$. If $C \in \mu^{-1}([0, \delta])$, fix a point $x \in C$. By the choice of δ, $\operatorname{diam}(C) < \varepsilon$. Thus, $H(C, \{x\}) < \varepsilon$. Notice that $\{x\} \in \mathcal{A}$. Hence, $C \in N^2(\varepsilon, \mathcal{A})$. If $\mu(C) \geq \delta$, then $\delta - \rho = \eta_n(C) = \delta - \rho + \tau_n(C)$. Thus, $\tau_n(C) = 0$. So $C \in \mathcal{A}$ or $C \in \mu^{-1}([0, \delta])$. In both cases, $C \in N^2(\varepsilon, \mathcal{A})$. We have shown that $\mathcal{B}_n \subset N^2(\varepsilon, \mathcal{A})$. Therefore, $H^2(\mathcal{A}, \mathcal{B}_n) < \varepsilon$.

In order to prove that $SL(X)$ is not locally compact at $\mathcal{A}$, since ε is arbitrary, it is enough to show that if $\{\mathcal{B}_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{\mathcal{B}_n\}_{n=1}^{\infty}$ and $\mathcal{B}_{n_k} \to \mathcal{B}$ for some $\mathcal{B} \in C^2(X)$, then $\mathcal{B}$ is not a size level.

Since $A_n \to \{p\}$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $\mu(A_n) < \delta$. If $n \geq n_0$, since $A_n \in \mathcal{C}_n \cap \mu^{-1}([0, \delta])$, $\eta_n(A_n) = \delta - \rho$. So $A_n \in \mathcal{B}_n$. Therefore, $\{p\} \in \mathcal{B}$. Now fix a point $q \in X - \{p\}$ and suppose that $\{q\} \in \mathcal{B}$. Then there exists a sequence of subcontinua $\{B_k\}_{k=1}^{\infty}$ of X such that $B_k \in \mathcal{B}_{n_k}$ for each $k \in \mathbb{N}$ and $B_k \to \{q\}$. Thus, there exists $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $\mu(B_k) < \delta$. Thus, $\delta - \rho = \eta_{n_k}(B_k) = \max\{\rho, \delta + (\mu(B_k) - \delta)\varsigma_n(B_{n_k})\} - \rho$. So $\delta = \delta + (\mu(B_k) - \delta)\varsigma_n(B_{n_k})$ and $\varsigma_n(B_{n_k}) = 0$. By property (c), $B_k \in \mathcal{C}_{n_k}$. So $p \in A_{n_k} \subset B_k$. Thus, $p \in \lim B_k = \{q\}$ and $p = q$. This contradicts the choice of q and proves that $\mathcal{B} \cap F_1(X) = \{\{p\}\}$. Therefore, $\mathcal{B}$ is not a size level, so the proof of the theorem is complete. □

By Theorems 4 and 5, $\operatorname{cl}_{C(C(X))}(WB(X)) \not\subseteq SL(X)$. However, it is natural to ask if the following claim is true: if $\mathcal{A} \in$

$\mathrm{cl}_{C(C(X))}(WB(X))$ and either $\mathcal{A} \cap F_1(X) = \emptyset$ or $F_1(X) \subset \mathcal{A}$, then is $\mathcal{A} \in SL(X)$? Below we prove that if $X = [0,1]$, then this implication is true. After that, we present an example of a continuum X for which the implication is not true.

Proposition 6. *If $\mathcal{A} \in cl_{C(C(X))}(WB([0,1]))$ and either $\mathcal{A} \cap F_1([0,1]) = \emptyset$ or $F_1([0,1]) \subset \mathcal{A}$, then $\mathcal{A} \in SL([0,1])$.*

Proof: Let $X = [0,1]$. Suppose that $\mathcal{A} = \lim \mathcal{A}_n$, where each $\mathcal{A}_n \in WB(X)$. Then $\mathcal{A}$ is a subcontinuum of $C(X)$. According to Theorem 1, we only need to show that $\mathcal{A}$ is convex and $\mathcal{A} \cap \alpha([0,1]) \neq \emptyset$ for each large order arc α.

Let $A, B, C \in C(X)$ be such that $A \subset B \subset C$ and $A, C \in \mathcal{A}$. We need to show that $B \in \mathcal{A}$, so we may assume that $A \neq B \neq C$. Let $\varepsilon_1 > 0$ and $\varepsilon = \min\{\varepsilon_1, \frac{H(A,C)}{2}\}$. Then there exists $m_0 \in \mathbb{N}$ such that $H^2(\mathcal{A}, \mathcal{A}_n) < \varepsilon$ for each $n \geq m_0$. We need to prove that $B \in N^2(\varepsilon, \mathcal{A}_n)$ for each $n \geq m_0$.

Let $n \geq m_0$. Since $A, C \in \mathcal{A}$, there exist $D, E \in \mathcal{A}_n$ such that $H(A, D), H(C, E) < \varepsilon$. Let $\mu : C(X) \to [0,1]$ and $0 \leq s \leq t \leq 1$ be such that μ is a Whitney map, $\mu(X) = 1$ and $\mathcal{A}_n = \mu^{-1}([s,t])$. Let $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$, $D = [d_1, d_2]$ and $E = [e_1, e_2]$. Then $c_1 \leq b_1 \leq a_1 \leq a_2 \leq b_2 \leq c_2$. If $s \leq \mu(B) \leq t$, then $B \in \mathcal{A}_n \subset N^2(\varepsilon, \mathcal{A}_n)$.

Now, assume that $\mu(B) < s$. Then $D \not\subseteq B$ and $d_1 < b_1$ or $b_2 < d_2$. We analyze the case that $d_1 < b_1$, the other one is similar. Since $H(A, D) < \varepsilon$, $|a_1 - d_1|, |a_2 - d_2| < \varepsilon$. If $b_2 \leq d_2$, then $A \subset B \subset D$, whence $H(B, D) \leq H(A, D) < \varepsilon$ and $B \in N^2(\varepsilon, \mathcal{A}_n)$. So we may assume that $d_2 < b_2$. Since $\mu([b_1, b_2]) < s \leq \mu([d_1, d_2]) < \mu([d_1, b_2])$, there exists $r \in [d_1, b_1]$ such that $\mu([r, b_2]) = s$. Since $d_1 \leq r \leq b_1 \leq a_1$, $|b_1 - r| < \varepsilon$. Thus $H(B, [r, b_2]) < \varepsilon$. Hence $B \in N^2(\varepsilon, \mathcal{A}_n)$.

Finally, assume that $\mu(B) > t$. Then $B \not\subseteq E$ and $b_1 < e_1$ or $e_2 < b_2$. We analyze the case that $b_1 < e_1$, the other one is similar. Since $H(C, E) < \varepsilon$, $|c_1 - e_1|, |c_2 - e_2| < \varepsilon$. If $e_2 \leq b_2$, then $E \subset B \subset C$, whence $H(B, E) \leq H(C, E) < \varepsilon$ and $B \in N^2(\varepsilon, \mathcal{A}_n)$. So we may assume that $b_2 < e_2$. If $B \cap E \neq \emptyset$, since $\mu([b_1, b_2]) > t \geq \mu([e_1, e_2]) > \mu([e_1, b_2])$, there exists $r \in [b_1, e_1]$ such that $\mu([r, b_2]) = s$. Since $c_1 \leq b_1 \leq r \leq e_1$, $|b_1 - r| < \varepsilon$. Thus $H(B, [r, b_2]) < \varepsilon$. Hence $B \in N^2(\varepsilon, \mathcal{A}_n)$. If $B \cap E = \emptyset$, then

$B \subset [c_1, e_1]$. Thus $\text{diam}(B) < \varepsilon$. Let $F \in \mu^{-1}(t)$ such that $F \subset B$; then $H(F, B) < \varepsilon$. Thus $B \in N^2(\varepsilon, \mathcal{A}_n)$.

In any case, $B \in N^2(\varepsilon, \mathcal{A}_n)$ for each $n \geq m_0$.

Therefore, $B \in \mathcal{A}$ and $\mathcal{A}$ is convex.

Now take a large order arc $\alpha : [0, 1] \to C(X)$. Then, for each $n \in \mathbb{N}$, there exists an element $G_n \in \mathcal{A}_n \cap \alpha([0, 1])$. Let $\{G_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{G_n\}_{n=1}^{\infty}$ such that $G_{n_k} \to G$ for some $G \in C(X)$. Then $G \in \mathcal{A} \cap \alpha([0, 1])$.

Therefore, $\mathcal{A}$ is a size level and the proposition is proved. □

Example 7. *There exist a continuum X and an element $\mathcal{A} \in cl_{C(C(X))}(WB(X))$ such that $F_1(X) \subset \mathcal{A}$ and $\mathcal{A} \notin SL(X)$.*

Let X be a metric compactification of the ray $[0, \infty)$ such that the remainder of X is the simple triod $T = ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ and, if T_1 is a subtriod of T and T_1 is the limit of subcontinua of X which are contained in $[0, \infty)$, then T_1 contains one of the legs $L_1 = [-1, 0] \times \{0\}$, $L_2 = [0, 1] \times \{0\}$ or $L_3 = \{0\} \times [0, 1]$. Let $T_0 = ([-\frac{1}{2}, \frac{1}{2}] \times \{0\}) \cup (\{0\} \times [0, \frac{1}{2}])$

By Theorem 16.10 of [8], there exists a Whitney map $\mu : C(X) \to [0, 1]$ such that $\mu(L_1) = \mu(L_2) = \mu(L_3) = \frac{1}{2} < \mu(T_0) < \frac{3}{4} = \mu(T)$.

Fix sequences of numbers $0 \leq c_1 < a_1 < b_1 < c_2 < a_2 < b_2 < \ldots$ such that $A_n = [a_n, b_n] \to T$.

Let $n \in \mathbb{N}$. Using again Theorem 16.10 of [8], we conclude that there exists a Whitney map $\mu_n : C(X) \to [0, 1]$ such that $\mu_n|(C([0, c_n]) \cup C([c_{n+1}, \infty) \cup T)) = \mu|(C([0, c_n]) \cup C([c_{n+1}, \infty) \cup T))$ and $\mu_n(A_n) = \frac{1}{2}$. Let $\mathcal{A}_n = \mu_n^{-1}([0, \frac{1}{2}]) \in WB(X)$ and $\{\mathcal{A}_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\mathcal{A}_n\}_{n=1}^{\infty}$ such that $\mathcal{A}_{n_k} \to \mathcal{A}$ for some $\mathcal{A} \in C(C(X))$. Then $\mathcal{A} \in \text{cl}_{C(C(X))}(WB(X))$. Since $F_1(X) \subset \mathcal{A}_{n_k}$ for each $k \in \mathbb{N}$, $F_1(X) \subset \mathcal{A}$.

Now, we prove that $\mathcal{A} \notin SL(X)$. Suppose to the contrary that $\mathcal{A} \in SL(X)$. Since $A_{n_k} \in \mathcal{A}_{n_k}$ for each $k \in \mathbb{N}$, $T \in \mathcal{A}$. Since $\mathcal{A}$ is convex, $T_0 \in \mathcal{A}$. So, there exists a sequence of subcontinua $\{B_k\}_{k=1}^{\infty}$ of X such that $B_k \to T_0$ and $B_k \in \mathcal{A}_{n_k}$ for each $k \in \mathbb{N}$. Since $A_{n_k} \to T$, we may assume that $A_{n_k} \not\subseteq B_k$ and $T \not\subseteq B_k$ for each $k \in \mathbb{N}$. If $B_{k_l} \subset T$ for a sequence $k_1 < k_2 < k_3 < \ldots$, then $\mu_{n_{k_l}}(B_{k_l}) = \mu(B_{k_l}) \to \mu(T_0) = \frac{3}{4}$. This is absurd since $B_{k_l} \in \mathcal{A}_{n_{k_l}}$. Thus we may assume that $B_k \not\subseteq T$ for each $k \in \mathbb{N}$. Since T is terminal in X, we have that $B_k \cap T = \emptyset$ for each $k \in \mathbb{N}$. Then

$B_k \subset [0,\infty)$ for each $k \in \mathbb{N}$. This is a contradiction since T_0 is not the limit of subcontinua of X contained in $[0,\infty)$. Therefore, $\mathcal{A} \notin SL(X)$.

Remark. Consider the following three conditions that a subcontinuum $\mathcal{A}$ of $C(X)$ may satisfy:
(a) $\mathcal{A} \in \mathrm{cl}_{C(C(X))}(WB(X))$;
(b) either $\mathcal{A} \cap F_1(X) = \emptyset$ or $F_1(X) \subset \mathcal{A}$;
(c) $\mathcal{A} \in SL(X)$.

Proposition 6 shows that if $X = [0,1]$, then each subcontinuum $\mathcal{A}$ of $C(X)$ satisfying (a) and (b) satisfies (c). On the other hand, for the continuum X of Example 7, there exists a subcontinuum $\mathcal{A}$ of $C(X)$ satisfying (a) and (b) but not satisfying (c). Thus the following problem arises naturally.

Problem. Give necessary and/or sufficient conditions on a continuum X in such a way that every subcontinuum $\mathcal{A}$ of $C(X)$ satisfying (a) and (b) satisfies (c). We do not even know if local connectedness of X is a sufficient condition.

4. Applications

In [6] and [5, Theorem 5.6], it was proved that, for each continuum X, the spaces $WL(X)$ and $W(X)$ are homemorphic to the Hilbert space l_2, for each continuum X. This motivated the questions of whether the spaces $SL(X)$ and $S(X)$ are homeomorphic to the Hilbert space l_2 ([8, Question 83.15]). We are ready to show that the space $SL([0,1])$ is not homeomorphic to l_2. The respective question for $S(X)$ is still open.

Theorem 8. *The space $SL([0,1])$ is not homeomorphic to the Hilbert space l_2.*

Proof: Let $X = [0,1]$. Let $\mathfrak{A} = \{\mathcal{A} \in C(C(X)) : F_1(X) \cap \mathcal{A} \neq \emptyset\}$. It is easy to show that $\mathfrak{A}$ is a closed subset of $C(C(X))$. Let $\varepsilon > 0$ be such that $\mathrm{cl}_{C(C(X))}(B^2(\varepsilon, \{X\})) \cap \mathfrak{A} = \emptyset$. So $\mathfrak{B} = \mathrm{cl}_{C(C(X))}(B^2(\varepsilon, \{X\})) \cap \mathrm{cl}_{C(C(X))}(WB([0,1]))$ is compact. By Proposition 6, $\mathfrak{B} \subset SL(X)$. By Theorem 4, $\mathrm{cl}_{C(C(X))}(B^2(\varepsilon, \{X\})) \cap SL(X) \subset \mathfrak{B}$. Therefore, $\mathfrak{B}$ is a compact neighborhood of $\{X\}$ in $SL(X)$. This proves that $SL(X)$ is locally compact at $\{X\}$. Since

l_2 is not locally compact at any of its elements, we conclude that $SL([0,1])$ is not homeomorphic to l_2. □

Question 9. Is there a continuum X such that $SL(X)$ is homeomorphic to l_2? Is there a continuum X such that $SL(X)$ is not locally compact at any of its elements? Is there a continuum X such that $SL(X)$ is not locally compact at $\{X\}$?

We finish this paper by giving topological characterizations of the size levels of $C([0,1])$ and $C(S^1)$, where S^1 is the unit circle in the Euclidean plane $\mathbb{R}^2$. Recall that C_0 and A_0 denote the unit square and the annulus as defined in the introduction.

Lemma 10. *Let X be $[0,1]$ or S^1. Let $C \in 2^X$; if $X = S^1$ we ask that C is nondegenerate. Let $\mathcal{A} = F_1(X) \cup$ $cl_{C(X)}\{A \in C(X) : A \subset X - C\}$. Then $\mathcal{A}$ is a size level of $C(X)$ and, if $X = [0,1]$ (respectively, $X = S^1$), then $\mathcal{A}$ is the pinched square (respectively, annulus) that can be obtained from C_0 (respectively, A_0) by taking the closed subset C of $[0,1]$ (respectively, of S^1) and identifying each one of the fibers of $\pi_1^{-1}(C)$ (respectively, $\pi_2^{-1}(C)$) to a point.*

Proof: If $A \subset X - C$, taking an order arc from a point of A to A it is possible to connect A to an element in $F_1(X)$ by subcontinua of $X - C$. Thus $\mathcal{A}$ is a subcontinuum of X. Since $F_1(X) \subset \mathcal{A}$, according to Theorem 1, we only need to prove that $\mathcal{A}$ is convex. Let B be a nondegenerate element of $\mathrm{cl}_{C(X)}\{A \in C(X) : A \subset X - C\}$. Then B is an arc. Let a and b be the end points of B. Then $B - \{a,b\} \subset X - C$. Thus each subcontinuum of B can be approximated by continua contained in $B - \{a,b\}$. That is, $C(B) \subset \mathcal{A}$. Therefore, $\mathcal{A}$ is convex and, in consequence, $\mathcal{A}$ is a size level.

If $C = X$, then $\mathcal{A} = F_1(X)$ which is a pinched square if $X = [0,1]$, and it is a pinched annulus if $X = S^1$.

For each component D of $X - C$, $\mathrm{cl}_X(D)$ is an arc and, by Example 5.1 of [8], $C(\mathrm{cl}_X(D))$ is a solid triangle with vertices $\{a\}$, $\{b\}$ and $\mathrm{cl}_X(D)$, where a and b are the end points of $\mathrm{cl}_X(D)$, and basis $F_1(\mathrm{cl}_X(D))$. Notice that $\mathcal{A} = F_1(X) \cup (\bigcup\{C(\mathrm{cl}_X(D)) : D$ is a component of $X - C\})$. Therefore, $\mathcal{A}$ is $F_1(X)$ with solid triangles glued at each of the sets of the form $F_1(\mathrm{cl}_X(D))$. This proves the second part of the lemma. □

The proof of the following result is similar to the proof of the previous lemma.

Lemma 11. *Let* $p = (1,0) \in S^1$ *and* $X = S^1$*. Let* $\mathcal{A} = \{p\} \cup cl_{C(X)}\{A \in C(X) : A \subset X - \{p\}$ *and length*$(A) \leq \pi\}$*. Then* $\mathcal{A}$ *is a size level of* $C(X)$*.*

Corollary 12. *Each pinched square (respectively, annulus) can be obtained as a size level of* $C([0,1])$ *(respectively,* $C(S^1)$*).*

Let X be $[0,1]$ or S^1. Given a subarc A of X, let $m(A)$ be the middle point of A and, for each subcontinuum A of X, let $L(A)$ be the length of A. Given a point $p \in X$, let $\mathcal{B}'_p = \{A \in C(X) : m(A) = p\}$ and

$$\mathcal{B}_p = \begin{cases} \mathcal{B}'_p, & \text{if } X = [0,1], \\ \mathcal{B}'_p \cup \{X\}, & \text{if } X = S^1. \end{cases}$$

Clearly, $\mathcal{B}_p$ is (the image of) an order arc or a one-point set (in the case that $X = [0,1]$ and $p = 0$ or $p = 1$). If $X = S^1$, then $\mathcal{B}_p$ is (the image of) a large order arc.

Let $\mathcal{A}$ be a size level of $C(X)$ and $A_{\mathcal{A}} = \{p \in X : \mathcal{B}_p \cap \mathcal{A} \neq \emptyset\}$. By Theorem 1, in the case that $X = S^1$, $A_{\mathcal{A}} = S^1$. In any case, by the convexity of $\mathcal{A}$, $\mathcal{B}_p \cap \mathcal{A}$ is always a (possibly degenerate) subarc of $\mathcal{B}_p$. For each $p \in A_{\mathcal{A}}$, let $h(p)$ (respectively, $f(p)$) be the minimal (respectively, maximal), with respect to inclusion, element of $\mathcal{B}_p \cap \mathcal{A}$.

Lemma 13. *The functions* $h, f : A_{\mathcal{A}} \to C(X)$ *are continuous and, if* $X = [0,1]$*, then* $A_{\mathcal{A}}$ *is a (possibly degenerate) subarc of* $[0,1]$*.*

Proof: We only prove that h is continuous; the proof for f is similar. It is easy to show that $A_{\mathcal{A}}$ is closed. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $A_{\mathcal{A}}$ converging to a point $p \in A_{\mathcal{A}}$. We may assume that $h(p_n) \to A$ for some $A \in C(X)$. Since $h(p_n) \in \mathcal{A}$ for each $n \in \mathbb{N}$, $A \in \mathcal{A}$. If A is an arc or A is degenerate, $m(A) = p$, so $A \in \mathcal{B}_p \cap \mathcal{A}$ and $h(p) \subset A$. If $A = S^1$, then $h(p) \subset A$. In any case, $h(p) \subset A$. Suppose that $h(p) \neq A$. Then $h(p)$ is a (possibly degenerate) arc. For each $n \in \mathbb{N}$, let s_n be the length of the minimal arc containing p and p_n. Then $s_n \to 0$. Fix a number s_0 such that $L(h(p)) < s_0 < L(A) = \lim L(h(p_n))$. Then there exists $n \in \mathbb{N}$ such that $L(h(p)) + 2s_n < s_0 < L(h(p_n))$. Therefore there exists a subarc B of X such that $p_n = m(B)$, $h(p) \subset B$ and $L(B) = L(h(p)) + 2s_n$.

Then B is a proper subset of $h(p_n)$. By the convexity of $\mathcal{A}$, $B \in \mathcal{A}$. This is a contradiction since $B \in \mathcal{B}_{p_n}$ and B is smaller than $h(p_n)$. Hence, $h(p) = A$. This completes the proof that h is continuous.

In order to prove the second part of the lemma, assume that $X = [0,1]$ and $\mathcal{A} = \sigma^{-1}(t)$, where $\sigma : C(X) \to [0,\infty)$ is a size level and $t \in [0, \sigma(X)]$. Let $0 \leq a < b < c \leq 1$ be such that $a, c \in A_{\mathcal{A}}$. Take $D \in \mathcal{B}_a \cap \mathcal{A}$ and $F \in \mathcal{B}_c \cap \mathcal{A}$ and consider the maximal subarc E of $[0,1]$ such that $b = m(E)$. Then $0 \in E$ or $1 \in E$, so $D \subset E$ or $F \subset E$. Since $\mathcal{B}_b$ is connected, $\{b\}, E \in \mathcal{B}_b$ and $\sigma(\{b\}) = 0 \leq t = \sigma(D) = \sigma(F) \leq \sigma(E)$. Hence there exists $G \in \mathcal{B}_b$ such that $\sigma(G) = t$. Thus $G \in \mathcal{B}_b \cap \mathcal{A}$ and $b \in A_{\mathcal{A}}$. This proves that $A_{\mathcal{A}}$ is a subinterval of $[0,1]$ and completes the proof of the lemma. $\square$

Theorem 14. *Let X be $[0,1]$ or S^1 and $\mathcal{A}$ be a size level of $C(X)$. Then*:
(a) *if $X = [0,1]$, then $\mathcal{A}$ is degenerate or $\mathcal{A}$ is a pinched square*,
(b) *if $X = S^1$ and $X \notin \mathcal{A}$, then $\mathcal{A}$ is a pinched annulus*,
(c) *if $X = S^1$ and $X \in \mathcal{A}$, then $\mathcal{A}$ is degenerate or $\mathcal{A}$ is a bunch of 2-cells.*

Proof: (a) If $A_{\mathcal{A}}$ is a one-point set $\{p\}$ and $\mathcal{A}$ is nondegenerate, there exist two subarcs A and B of $[0,1]$ such that $m(A) = m(B) = p$ and $A \subset B$. It is easy to construct a subarc C of B such that $A \subset C$ and $m(C) \neq p$. By the convexity of $\mathcal{A}$, $C \in \mathcal{A}$. Thus, $m(C) \in A_{\mathcal{A}} - \{p\}$. This contradiction proves that if $A_{\mathcal{A}}$ is a one-point set, then $\mathcal{A}$ is degenerate. In fact, each size level contains an element containing 0 and an element containing 1, so if $\mathcal{A}$ is degenerate, $\mathcal{A} = \{[0,1]\}$. Therefore, if $A_{\mathcal{A}}$ is a one-point set, then $\mathcal{A} = \{[0,1]\}$ and $A_{\mathcal{A}} = \{\frac{1}{2}\}$.

Now, suppose that $A_{\mathcal{A}}$ is nondegenerate. Then $A_{\mathcal{A}}$ is an arc. Let $g : A_{\mathcal{A}} \times [0,1] \to \mathcal{A}$ be given by

$$g(p,t) = [p - \tfrac{1}{2}(tL(f(p)) + (1-t)L(h(p))), p + \tfrac{1}{2}(tL(f(p)) + (1-t)L(h(p)))].$$

Since $h(p) \subset g(p,t) \subset f(p)$ and $\mathcal{A}$ is convex, $g(p,t) \in \mathcal{A}$ for each $(p,t) \in A_{\mathcal{A}} \times [0,1]$. Clearly, g is continuous and onto.

Let $D_{\mathcal{A}} = \{p \in A_{\mathcal{A}} : h(p) = f(p)\}$. Since h and f are continuous, $D_{\mathcal{A}}$ is a closed subset of $A_{\mathcal{A}}$. Let $P_{\mathcal{A}}$ be the pinched square obtained from $A_{\mathcal{A}} \times [0,1]$ by identifying each one of the sets of the form

$\{p\} \times [0,1]$ to a point, where p is taken in $D_{\mathcal{A}}$. Let $\varphi : A_{\mathcal{A}} \times [0,1] \to P_{\mathcal{A}}$ be the identification map. Observe that the fibers of g are exactly the same as the fibers of φ. Therefore, $\mathcal{A}$ is homeomorphic to $P_{\mathcal{A}}$. This completes the proof of (a).

(b) The proof is similar to the proof of the second part of (a).

(c) Suppose that $X = S^1$ and $X \in \mathcal{A}$. Let $Z = \{p \in S^1 : h(p) = X\}$. Then Z is a closed subset of S^1. Let $\psi : S^1 \times [0,1] \to \mathcal{A}$ be given by $\psi(p,t) = X$, if $p \in Z$ or $t = 1$, and $\psi(p,t)$ is the subarc A of S^1 such that $m(A) = p$ and $L(A) = t2\pi + (1-t)L(h(p))$, if $p \notin Z$ and $t < 1$.

Then ψ is continuous. Since $\mathcal{A}$ is convex, $\psi(S^1 \times [0,1]) = \mathcal{A}$. If $p \notin Z$ and $t < 1$, then $\psi^{-1}(\psi(p,t)) = \{(p,t)\}$, and $\psi^{-1}(X) = (Z \times [0,1]) \cup (S^1 \times \{1\})$. Therefore, $\mathcal{A}$ is homeomorphic to the continuum $(S^1 \times [0,1])/(Z \times [0,1]) \cup (S^1 \times \{1\})$, which is the space obtained from the continuum $S^1 \times [0,1]$ by identifying the closed subset $(Z \times [0,1]) \cup (S^1 \times \{1\})$ to a one-point set.

If $Z = \emptyset$, then $\mathcal{A}$ is homeomorphic to $(S^1 \times [0,1])/(S^1 \times \{1\})$, which is a 2-cell.

If $Z = \{p\}$ for some $p \in S^1$, then $\mathcal{A}$ is homeomorphic to $(S^1 \times [0,1])/(\{p\} \times [0,1]) \cup (S^1 \times \{1\})$ which again is a 2-cell.

If $Z = X$, then $A = \{X\}$ is degenerate.

So, we may assume that Z is a nonempty and nondegenerate proper subset of X.

Let Y be the pinched annulus that can be obtained from $S^1 \times [0,1]$ by identifying each one of the sets of the form $\{z\} \times [0,1]$ to a one-point set. Then there is a homeomorphism from Y to the continuum $Y_0 = \{(p,t) \in S^1 \times [0,1] : 0 \leq t \leq \frac{1}{2\pi} d(p,Z)\}$ which is the identity on the set $S^1 \times \{0\}$. (Here, $d(p,Z)$ is the length of the minimum subarc of S^1 joining p to some point of Z.) Notice that Y_0 is the union of the simple closed curve $S^1 \times \{0\}$ and an at most countable family of solid triangles $\mathcal{G}$ such that the intersection of each element of $\mathcal{G}$ with $S^1 \times \{0\}$ is a side of the triangle and a subarc of $S^1 \times \{0\}$; each two different triangles in $\mathcal{G}$ touches at most one or two of their vertices. Furthermore, each countable convergent family of elements of $\mathcal{G}$ tends to one point in $S^1 \times \{0\}$.

Observe that $(S^1 \times [0,1])/(Z \times [0,1]) \cup (S^1 \times \{1\})$ is homeomorphic to $Y/(S^1 \times \{0\})$. Therefore, $\mathcal{A}$ is homeomorphic to $Y/(S^1 \times \{0\})$, which is a union of a nonempty and at most countable family $\mathcal{F}$ of

2-cells such that the intersection of any two different elements of $\mathcal{F}$ is exactly the point q_0 that represents to $S^1 \times \{0\}$ and, in the case that the family is countable, the 2-cells tend to q_0.

This completes the proof of (c) and the proof of the theorem. □

Theorem 15. *Let $\mathcal{B}$ be a bunch of 2-cells. Then there exists a size level $\mathcal{A}$ of $C(S^1)$ such that $\mathcal{A}$ is homeomorphic to $\mathcal{B}$.*

Proof: We only consider the case that $\mathcal{B}$ is a countable bunch of 2-cells. The other case (finite bunch of 2-cells) is similar.

Let $X = S^1$. Consider a sequence of subarcs $\{A_n\}_{n=1}^{\infty}$ of S^1 such that $S^1 - A_1, S^1 - A_2, ...$ are pairwise disjoint and the lengths of the intervals $S^1 - A_n$ tend to 0.

For each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{A \in C(S^1) : A_n \subset A\}$ and let $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in \mathbb{N}\}$. Notice that $\mathcal{A}$ is a subcontinuum of $C(X)$, $\mathcal{A} \cap F_1(X) = \emptyset$, $\mathcal{A}_n$ is a 2-cell, X belongs to the manifold boundary of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\mathcal{A}_n \to X$. Therefore, $\mathcal{A}$ is a countable bunch of 2-cells.

According to Theorem 1, in order to prove that $\mathcal{A}$ is a size level of $C(X)$, we only need to show that if $A \subset B \subset C$ and $A, C \in \mathcal{A}$, then $B \in \mathcal{A}$. Since $A \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $A_n \subset A \subset C$. Thus $C \in \mathcal{A}_n \subset \mathcal{A}$. Therefore, $\mathcal{A}$ is a size level of $C(X)$. This ends the proof of the theorem. □

Combining Corollary 12, Theorem 14 and Theorem 15, we resume the topological characterizations of size levels of $[0, 1]$ and S^1 in the following corollary.

Corollary 16. *The nondegenerate size levels of $C([0, 1])$ are exactly the pinched squares. The nondegenerate size levels of $C(S^1)$ are exactly the pinched annulus and the bunches of 2-cells.*

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Added in proof. In a private communication, A. Samulewicz has shown to the author a very short proof of Theorem 1, by using an extension theorem for size levels.

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