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ANOTHER CHARACTERIZATION OF NON-SEPARATING PLANAR CONTINUA

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ABSTRACT. It is proved that a planar continuum is nonseparating if and only if it is H-connected. A fixed-point theorem for spaces that can be written in terms of sequences of H-connected continua follows as a consequence.

1. INTRODUCTION

One reason for interest in nonseparating planar continua is the classical unsolved problem of whether or not every nonseparating planar continuum has the fixed-point property. For a recent summary of results and problems relating to this question the reader is referred to Hagopian [4]. Hagopian's main result [4, Corollary 3], or [5, 9.1], is that every simply connected plane continuum has the fixed point property. On the other hand, Lau [10] noted that as a consequence of [10, Corollary 3], every local homeomorphism from a continuum onto a nonseparating continuum is a homeomorphism. So a nonseparating simply connected plane continuum M has the fixed-point property, and it also has the property that every proper local homeomorphism from a continuum onto M must to be

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a homeomorphism. This observation leads naturally to the question of whether there is a more intimate connection between these two properties and to a concept introduced by the first author [6].

Definition. [6, 3.1]. A connected Hausdorff space Y is H-connected if every proper local homeomorphism of a connected Hausdorff space X onto Y is a homeomorphism.

Thus, Lau's observation has a corollary that every nonseparating planar continuum is H-connected. As a first step in analyzing the relationship between H-connectedness and the fixed-point property, we prove that every H-connected planar continuum is nonseparating. This enables us to conclude,

Theorem 1.1. *A planar continuum is nonseparating if and only if it is H-connected.*

Combining this with a result of Hagopian [4] we have as a corollary that every path connected planar H-connected continuum has the fixed-point property. Properties of H-connected spaces are developed in [6], [7], and [8]. Of particular relevance to this paper is the development of the "algebra" of H-connected spaces begun by the first author in [6, Ch. 3]. These results give methods for recognizing H-connected spaces. Specifically, they give methods for combining H-connected spaces in such a way that H-connected spaces result. Two of the results from this algebra of H-connectedness, [6, 3.8] and [6, 3.10], are of particular interest in the context of this paper. They provide a model for generating examples of interest and are instrumental in obtaining fixed-point results, Theorem 2.3 and its corollaries, for certain types of planar continua. Thus, above and beyond its generic value, Theorem 1.1 might prove useful in obtaining a solution to the question as to whether or not a nonseparating planar continuum has the fixed-point property.

The paper consists of four sections. In the remainder of the first section we confirm terminology and provide statements of known results needed to prove Theorem 1.1. Section 2 contains the proof of the theorem and its corollaries. Section 3 contains examples. Section 4 contains a question.

We let N denote the set of positive integers. A continuum is a compact connected subset of a metric space. A subset M of a metric space (X, d) has the fixed-point property if every continuous map $f : M \to M$ has a fixed point−a point $p \in$M such that $f(p) = p$. A map $f : X \to Y$ is proper if for every compact $K \subset Y$, $f^{-1}(K)$ is compact. We shall appeal to the following propositions.

Proposition 1.2. (Knaster & Kuratowski [9]) *If neither of the point sets A and B separates the point a from the point b, each of the sets is closed in their sum, one of the sets is compact and their common part is a continuum or does not exist, then the sum of A and B does not separate a from b. (See also* Bing [1]*.)*

Proposition 1.3. (Christenson & Voxman [3, Ex. 5. C. 28, p. 144]) *Suppose that K is a compact subset of an open connected subset $U \subset E^2$. Then there is a rectilinear simple closed curve R such that $K \subset I(R)$ and $R \subset U$. ($I(R)$ is the interior of R, i.e., the bounded component of $(E^2 - R)$).*

Proposition 1.4. [6, 3.8] *Let $Y = A \cup C$ be a continuum in which A and C are disjoint H-connected subsets of Y. Suppose that C is closed, that A is open, and that for each $i \in N$, A_i is a connected subset of A such that $A - A_i$ is closed and connected. If $\emptyset \neq \limsup_i(A_i) \subset C$, then Y is H-connected.*

Proposition 1.5. [6, 3.10] *Let Y be a first countable Hausdorff space which can be expressed as a union of countably many sets A_i, such that $A_i \subset Int(A_{i+1})$, $i \in N$. If each A_i is H-connected, then Y is H-connected.*

2. Main Results

Theorem 2.1. *Suppose M is a planar continuum. If M separates the plane, then M is not H-connected.*

Proof: By hypothesis, we may assume that M is a subset of an annulus, say X, and that M separates the inner and outer circles C_i and C_o, respectively, of X. Coordinatize the plane with polar coordinates (r, θ) so that the center of X is at the origin, and the functions $r = r_i$ and $r = r_o$ trace C_i and C_o, respectively.

Define the function $p : X \to X$ by $p\ (r, \theta) = (r, 2\theta)$ for $r_i \leq r \leq r_o$. Then (X, p) is a 2-fold self covering of X and p is a proper local homeomorphism. We also let $M^*= p^{-1}(M)$, ${C_k}^* = p^{-1}(C_k)$ for $k = i, o$, and $p^* = p|_{M^*}$, the restriction of p to M^*. Thus, p^* is a proper local homeomorphism of M^* onto M. Observe that ${C_k}^* = C_k$.

If M^* is connected, M could not be H-connected since p^* is a 2-fold proper local homeomorphism. So suppose that M^* is not connected. Then, as a consequence of Jungck and Timm [7, 1.4], the following is obtained.

Lemma. *$M^* = A \cup B$, where A and B are closed disjoint connected subsets of X, and $p|_{A(B)}$ is a homeomorphism of $A(B)$ onto M. In fact, there exist disjoint connected open sets U_A and U_B containing A and B, respectively, such that p is a homeomorphism on each and $p(U_A) = p(U_B)$.*

Note that M^* separates ${C_i}^*$ and ${C_o}^*$ since $M = p(M^*)$ separates C_i and C_o. We now appeal to Proposition 1.2, which−by the above lemma−says that at least one of A or B separates ${C_i}^*$ and ${C_o}^*$. (To apply the proposition, pick points a and b on ${C_i}^*$ and ${C_o}^*$, respectively, and use the fact that these sets are connected.) So we assume that A does. Consequently, U_A can be chosen so as to separate ${C_i}^*$ and ${C_o}^*$. Now, Proposition 1.3 yields a simple closed curve $R^* \subset U_A$ such that $A \subset I(R^*)$. Clearly, ${C_i}^* \subset I(R^*)$, since otherwise A does not separate the indicated sets. Thus, R^* is a simple closed curve which separates ${C_i}^*$ and ${C_o}^*$.

Now since $p|_{U_A}$ is a homeomorphism, $p(R^*)$ is a simple closed curve. But $p|_{U_B}$ is also a homeomorphism and $p(\mathrm{R}^*) \subset p(U_A) = p(U_B)$. So if $R^{**} = p_{U_B}^{-1}(p(R^*))$, then R^{**} is a simple closed curve in U_B. Since $R^* \subset U_A$ and $U_A \cap U_B = \emptyset$, R^{**} is interior to R^* or exterior to R^*. Suppose that R^{**} is exterior to R^*.

Let $r_1 = imf\{r : (r, \theta) \in R^*\}$. Since R^* is compact, there is a θ_1 such that $(r_1, \theta_1) \in R^*$. Then $(r_1, \theta_1 + \pi) = (p|_{U_B})^{-1}(p(r_1, \theta_1)) \in R^{**}$. But $R^* \cap R^{**} = \emptyset$, so the choice of r_1 implies that $(r_1, \theta_1 + \pi) \in I(R^*)$, since R^* separates C_i and C_o. This contradicts the above assumption that R^{**} is exterior to R^*. In like manner, the assumption that R^{**} is interior to R^* leads to a contradiction.

Thus, in any event, the assumption that M^* is not connected is unacceptable, and we conclude that M is not H-connected, as desired.

Corollary 2.2. *Every path-connected H-connected planar continuum has the fixed-point property.*

Proof: By [4], every path-connected nonseparating planar continuum has the fixed-point property. □

The following theorems provide applications of the above results and information relating H-connectedness to the fixed-point property.

Theorem 2.3. *Let C and B_i be path connected continua in a metric space X such that $B_i \subset B_{i+1}$ and $C \cap B_i = \emptyset$ for $i \in N$. Suppose that C and all B_i have the fixed-point property, $B = \cup\{B_i : i \in N\}$, and $\emptyset \neq \limsup_i(B - B_i) \subset C$. Then $Y = B \cup C$ is a continuum, and if Y is not path connected, Y has the fixed-point property.*

Proof: Clearly B is connected. Since C is connected, $\emptyset \neq \limsup$ $(B_i - B)$ implies that Y is connected. To see that Y is compact, let $\{x_n\}$ be a sequence in Y. If $x_n \in C$ for infinitely many n, then some subsequence of $\{x_n\}$ converges to some $c \in C \subset Y$, since C is compact. Similarly, if there is a $j \in N$ such that $x_n \in B_j$ for infinitely many n, then some subsequence of $\{x_n\}$ converges to a point $b \in B_j \subset Y$, since B_j is compact. So assume that the only other possibility occurs; i.e., for each $k \in N$ there is an $n(k) \geq k$ such that $x_i \notin C \cup B_k$ for $i > n(k)$; i.e., $x_i \in B - B_k$ for $i > n(k)$. But then $\emptyset \neq \limsup(B - B_i) \subset C$ implies that some subsequence of $\{x_n\}$ converges to a point $c \in C \subset Y$, since C is compact. Thus, in any event, $\{x_n\}$ has a subsequence which converges to a point of Y, so that Y is compact.

Now assume that Y is not path connected and let $f : Y \to Y$ be a continuous map. Now B and C are each path connected. So since Y is not path connected, and each of $f(B)$ and $f(C)$ is path connected , it follows that (1) $f(C) \subset C$ or (2) $f(C) \subset B$. Similarly, (3) $f(B) \subset C$ or (4) $f(B) \subset B$.

If (1) holds, $f(c) = c$ for some $c \in C \subset Y$ since C has the fixed point property. So suppose that (2) is true. If (3) also holds, we have

$$f(Y) = f(B) \cup f(C) = (f(B) \cap C) \cup f(C).$$

Since C and $f(C)$ are disjoint and each set is closed (f is continuous) the preceding equalities say that $f(Y)$ is separated. This is a contradiction, because the connectedness of Y and the continuity of f require that $f(Y)$ be connected.

Consider the remaining case in which (2) and (4) hold simultaneously. We assert that there is an $i \in N$ such that $f(B_i) \subset B_i$. If not, for each i there is a $p_i \in B_i$ such that $f(p_i) \in (B - B_i)$. As above, the fact that $\emptyset \neq \limsup_i(B - B_i) \subset C$ and the compactness of C demand that some subsequence of $\{f(p_i)\}$ converges to a point $p \in C$. But Y is compact, so some subsequence of $\{p_i\}$ converges to a point $q \in Y$. Then continuity implies that $f(q) = p \in C$. This is impossible since $f(Y) = f(C) \cup f(B)$ and (2) and (4) demand that $f(q) \in B$. (Remember, $B \cap C = \emptyset$.) Thus, $f(B_i) \subset B_i$ for some i, and since B_i has the fixed-point property, there is a $p \in B_i \subset Y$ such that $f(p) = p$. So f has a fixed point; i.e., Y has the fixed-point property. □

Definition 2.4. Let C and B_i be continua such that $B_i \cap C = \emptyset$ for $i \in N$. Let $B = \cup\{B_i : i \in N\}$ and $Y = B \cup C$. If $\emptyset \neq \limsup_i(B - B_i) \subset C$ we say that Y is a B-spiral to C or a $\{B_i\}$-spiral to C. If $B_i \subset B_{i+1}$ for all $i \in N$, we say that the $\{B_i\}$-spiral is a nested spiral.

Note that if $Y = B \cup C$ is a nested B-spiral to C in which each B_i is path connected, then Theorem 2.3 implies that Y is a continuum. The traditional spiral to the triod mentioned in Hagopian [4] and the "crooked spirals" defined in [2] are nested $\{B_i\}$-spirals to continua C in which each B_i is a closed interval and B is an embedding of the ray $[0, +\infty)$ in Y in such a way that B is an open dense subset of Y and C is the remainder (see [2]) of B in Y. Also note that given any continuum C there is a continuum Y that is a nested $\{B_i\}$-spiral to C in which the B_i are closed intervals.

Corollary 2.5. *Let $Y = B \cup C$ be a nested $\{B_i\}$-spiral to C in which C and each B_i are H-connected, path connected continua in E^2. Then Y has the fixed-point property provided that one of the following holds:*

(a) *Y is not path connected.*
(b) *Y is planar, $B_i \subset Int\,(B_{i+1})$, and $(B - B_i)$ is connected for $i \in N$.*

Proof: Suppose (a) holds. By Corollary 2.2, all B_i and C have the fixed-point property, so Theorem 2.3 tells us that Y has the fixed-point property.

Suppose (b) holds. If Y is not path connected, Y has the fixed-point property by (a). So assume that Y is path connected. Since B_i is H-connected and $B_i \subset Int(B_{i+1})$ for all $i \in N$, B is H-connected by Proposition 1.5. Since C is H-connected and $B \cap C = \emptyset$, we appeal to Proposition 1.4 and set $A = B$ and $A_i = B - B_i$ to conclude that Y is H-connected. Since Y is path connected and planar, Corollary 2.2 implies that Y has the fixed-point property. □

Corollary 2.6. *Let C be a simply connected planar continuum and $Y = B \cup C$ a $\{B_i\}$-spiral to C in which B is homeomorphic to the ray $[0, +\infty)$, B_i the image of $[1, i+1]$, and $C = \limsup_i (B - B_i)$. Then Y has the fixed-point property.*

3. Examples

We begin by remarking that the unit circle S^1 is an elementary planar example that shows that either the assumption that the continuum Y of Theorem 2.3 and Corollary 2.5 not be path-connected or the assumption that $B - B_i$ be connected is necessary: S^1 does not have the fixed-point property and it can be written as a point p union of a disjoint open arc B, and B can be written as a countable union of closed arcs $B_i \subset Int(B_{i+1})$ such that $\limsup_i (B - B_i) = \{p\}$.

Example 3.1. We construct a non-simply connected example that can be shown to have the fixed-point property using our results. Let $S_1 = \{(\frac{1}{x}, \sin(x\pi)) : x \in [\frac{1}{2}, +\infty)\}$,

$S_2 = \{(\frac{1}{x}, -\sin(x\pi)) : x \in (-\infty, -\frac{1}{2}]\}$, $L = \{(0, y) : y \in [-1, 1]\}$

$A_1 = \{(2, y) : y \in [-2, 0]\}, A_3 = \{(x, -2) : x \in [-2, 2]\}$,

$A_2 = \{(-2, y) : y \in [-2, 0)\}$.

Consider the plane continuum $M = \mathrm{S}_1 \cup L \cup S_2 \cup A_1 \cup A_2 \cup A_3$. For $i = 1, 2$ let $S_i(n) = \{(\frac{1}{x}, (-1)^i \sin(x\pi)) : x \in [(-1)^i \frac{1}{2}, (-1)^i n)\}$.

Let $C = L$. For $n \geq 2$, let $B_n = (\bigcup_{i=1}^{2} S_i(n)) \cup A_1 \cup A_2 \cup A_3$. Then L and each B_n are closed intervals and so have the fixed-point property. Also $B = M - L = \bigcup_{n=2}^{\infty} B_n$ is path connected and written as an increasing union of spaces with the fixed-point property, $\limsup_n (M - B_n) = C$, and M is not path connected. So, by Theorem 2.3, M has the fixed-point property. We remark that by Jungck [6, 7.6], it is easy to see that M has nontrivial finite sheeted covering spaces and so M is not H-connected.

Example 3.2. This example is a variation on Example 3.1 and is used to show that some assumptions about the path connectedness of $\limsup(B - B_i)$ or the connected subset $B = \bigcup_n^\infty B_n$, such as those in Theorem 2.3 and Corollary 2.5, are necessary. Let $M_+ = \{(x, y+2) : (x, y) \in M - A_3$. Let M_- denote the reflection of M_+ across the x-axis. Let $M' = M_+ \cup M_- \cup \{(2,0), (-2,0)\}$. By construction, M' is set-wise invariant under a rotation ρ of the plane by 180° and $(\rho|M') : M' \to M'$ is fixed-point free. (a) To see that some hypothesis on the path connectedness of the $\limsup_i(B - B_i)$ is necessary, e.g. the subcontinuum C of Theorem 2.3 and Corollary 2.5(a) is assumed to be path connected, we take $C = M' \cap \{(x, y) : x \leq 0\}$ and write $B = M' \cap \{(x, y) : x > 0\}$ as a countable union of nested closed intervals B_n with the nth interval contained in the interior of the $(n+1)$ interval. Note that both C and each B_n have the fixed-point property. (b) To see that the set B of the theorem must be path connected, take the closed interval $C = \{(0, y) : y \in [1, 3]\}$. Let $B_1 = M_- \cup \{(2,0), (-2,0)\}$. Define B_{n+1} recursively: B_{n+1} is obtained from B_n by attaching two closed intervals to the two endpoints of B_n in such a way that $B = \bigcup_n B_n = M' - C$. In this case, also note that C and each B_n have the fixed-point property.

Example 3.3. There are examples of continua that are $\{B_i\}$-spirals to path connected sets C that do not have the fixed-point property. Let

$N_+ =$
$(M_+ \cap \{(x, y) : x \geq 0, y \geq 0\}) \cup \{(2, y) : -3 \leq y \leq 2\} \cup \{(x, -3) : 0 \leq x \leq 2\}$.

Let N_- be the rotation of N_+ by 180° about the origin and let $N = N_+ \cup N_-$. Now take $C = (N_+ \cap \{(x, y) : x > 0\}) \cup \{(0, y) : -3 \leq y \leq -1\}$. Take $B = N - C$ and write B as an ascending union of nested closed intervals where $B_1 = \{(0, y) : 1 \leq y \leq 3\}$. By construction, N is set-wise invariant under a 180° rotation of the plane, and this map clearly has no fixed point.

Problem 3.4. In the context of Theorem 2.3, there are two possible reasons why Example 3.3 may fail to have the fixed-point property: either (1) C is not a continuum or (2) C does not have the fixed-point property. We wonder if there exists a similar example of a planar continuum N that does not have the fixed-point property, that is a nested $\{B_i\}$-spiral to a path connected set C, with each B_i a path connected continuum with the fixed-point property, but in which the path connected set C does have the fixed-point property.

In the remainder of this section we will use our techniques to analyze the cone on the traditional spiral to the triod. This example was brought to the attention of the authors by Hagopian [4].

Definition 3.5. Let M denote the traditional spiral to a triod and $C(M)$ be the cone on M. Decompose M into the spiral, which we will denote by either S or, by abuse of notation, as $[1, +\infty)$, and the triod which we denote by Y. Let $\mathbf{0}$ denote the cone point of $C(M)$. We will write $C(M) = [0, 1] \times M/\sim$ where $(t, x) \sim (s, y)$ if and only if (1) $t = s = 0$ or (2) $t = s \neq 0$ and $x = y$. With this identification, $\mathbf{0} = \{(0, x) : x \in M\}$. Let $f : C(M) \to C(M)$ be a continuous function. We say f has a radially fixed point if there is a point $(t, x) \in C(M)$ such that $f(t, x) = (s, x)$. We say f has a tangentially fixed point if there is a point $(t, x) \in C(M)$ such that $f(t, x) = (t, y)$. Recall that it is not known if $C(M)$ has the fixed-point property.

Corollary 3.6. *Let M be the spiral to a triod and $C(M)$ the cone on M. Then every self-map of M has a radially fixed point and a tangentially fixed point.*

Proof: Let $f : C(M) \to C(M)$ be a continuous function. We want to show that f has a radially fixed point. □

Claim 1. *M has the fixed-point property.*

Proof: The triod Y is path connected and, by Hagopian [4, Theorem 2], has the fixed-point property. We again abuse notation and write the spiral as $S = [1, +\infty) = \cup\{[1, n]\colon n \in N\}$. So setting $B_n = [1, n]$ and $C = Y$ we may apply our Corollary 2.6 and conclude that M has the fixed-point property. □

Claim 2. *If I is a finite closed interval, then $I \times M$ has the fixed-point property.*

Proof: We may write $I \times M = (I \times [0, +\infty)) \cup (I \times Y))$. Set $B_n = I \times [0, n]$ and $C = I \times Y$. By the Lefshetz Fixed-point Theorem, C and each B_n have the fixed-point property. Now applying our Theorem 2.3, it follows that $I \times M$ has the fixed-point property. □

Claim 3. *If $\mathbf{0} \notin f(C(M))$, then f has a fixed point.*

Proof: Suppose that $\mathbf{0} \notin f(C(M))$. Then $f(C(M))$ compact implies there is an open ϵ-cone $C_\epsilon(M) = [0,\epsilon) \times M/\sim$ such that $C_\epsilon(M) \cap f(C(M)) = \emptyset$. Therefore, $f([\epsilon,1] \times M) \subset [\epsilon,1] \times M$. Hence, by Claim 2, the restriction $(f|([\epsilon,1] \times M))$ has a fixed point, and Claim 3 is proved. □

We now show that f has a radially fixed point. We observe that if $\mathbf{0} \in f^{-1}(\mathbf{0})$, then the cone point is fixed and we are done. So, we may assume that $\mathbf{0} \notin f^{-1}(\mathbf{0})$. Now $f^{-1}(\mathbf{0})$ is closed, and therefore compact. So there is an open ϵ_0-cone such that $C_{\epsilon_0}(M) \cap f^{-1}(\mathbf{0}) = \emptyset$. We may without loss of generality assume that ϵ_0 is the largest ϵ for which $C_\epsilon(M) \cap f^{-1}(\mathbf{0})$ is empty. Consider the closed $\frac{\epsilon_0}{2}$-cone $\overline{C}_{\epsilon_0/2}(M) = [0, \frac{\epsilon_0}{2}] \times M/\sim$. Let $\pi : C(M) \to \overline{C}_{\epsilon_0/2}(M)$ be the "radial" retract defined by $\pi(t,x) = \begin{cases} (t,x), \text{if } x \in \overline{C}_{\epsilon_0/2}(M), \\ (\frac{\epsilon_0}{2}, x), \text{otherwise.} \end{cases}$

Then the map $\pi \circ f \circ \pi : C(M) \to C(M)$ is a map such that $\mathbf{0} \notin (\pi \circ f \circ \pi)^{-1}(\mathbf{0})$. So, by Claim 3, $\pi \circ f \circ \pi$ has a fixed point. If this fixed point is in the open cone $C_{\frac{\epsilon_0}{2}}(M)$ then, in fact, it follows that f has a fixed point. Otherwise, the definition of π implies that there is a (t_0, x_0) that is radially fixed, i.e., (t_0, x_0) is such that $f(t_0, x_0) = (t_1, x_0)$ and $t_0, t_1 \geq \frac{\epsilon_0}{2}$.

We now use the Intermediate Value Theorem to show that the map f has a tangentially fixed point. First, note that we may think of M as sitting on the surface of the unit sphere in 3-space and the cone point $\mathbf{0}$ as the origin of R^3. Let d denote the standard distance function for 3-space. Define $g : C(M) \to C(M)$ by $g(s,x) = d(\mathbf{0},(s,x)) - d(\mathbf{0}, f(s,x))$. If the cone point $\mathbf{0}$ is fixed by f we are done. If not, an application of the Intermediate Value Theorem to g yields a tangentially fixed point. □

Example 3.7. We use the notation of the previous corollary in this example. We show that for each $\epsilon > 0$, there is subcontinuum X_ϵ of $C(M)$ such that (1) X_ϵ contains the entire cone on the triod $C(Y)$, including the cone point, (2) $C(M) - X_\epsilon \subset C_\epsilon(M)$, and (3) X_ϵ has the fixed-point property.

We will think of M as $M = [1, +\infty) \cup Y$ and we will assume the spiral $S = [1, +\infty)$ is parametrized in such a way that for each n, the closed interval $[n, n+1]$ wraps around the triod Y exactly once. Fix $1 > \epsilon > 0$. We define for each m a path-connected subset of the cone on the spiral $S = [1, +\infty)$. For $m = 1$, define $B_{\epsilon,1} = ([\epsilon, 1] \times [1, 2]) \cup \{(t, 2) \in [0, 1] \times S/\sim: \frac{\epsilon}{2} \leq t \leq \epsilon\}$.

For each $m \geq 2$, define $B_{\epsilon,m} = B_{\epsilon,(m-1)} \cup ([\epsilon, 1] \times [m, m+1]) \cup \{(t, m + k(\frac{1}{m+1})) \in [0, 1] \times S/\sim: \frac{\epsilon}{m+1} \leq t \leq \epsilon, k \in N, 1 \leq k \leq m+1\}$.

Each $B_{\epsilon,m}$ has the fixed-point property since it is a closed 2-disk with $\sum_{k=1}^{m-1} k$ closed intervals sticking off its boundary toward the cone point $\mathbf{0}$.

Now let $B_\epsilon = \bigcup_m B_{\epsilon,m}$, let $C = C(Y)$, and let $X_\epsilon = B_\epsilon \cup C$. By construction, properties (1)-(3) above are true of X_ϵ. Also, by construction, Theorem 2.3 applies to X_ϵ, and so it has the fixed-point property.

4. A question

The results reported in this paper were motivated by an attempt to understand the relationship between H-connectedness and the fixed-point property for nonseparating planar continua. Theorem 2.3 certainly suggests this may be a possibility. With Theorem 2.3 in mind, suppose we decide to prove that every planar H-connected continuum has the fixed-point property by using a proof by contradiction. We assume Y is an H-connected planar continuum, and $f : Y \to Y$ is a continuous map that does not have a fixed point. We now want to show that there is a space X and a surjective proper local homeomorphism $p : X \to Y$ that is not a global homeomorphism. That is, we want to show that Y has a nontrivial finite sheeted connected covering space. This observation prompts our last question.

Question 4.1. Given a continuous self-map $f : Y \to Y$, how can f be used to construct a nontrivial connected covering space for Y? As a warm-up exercise, consider S^1, the circle, and $f : S^1 \to S^1$ is a rotation of S^1 by $\epsilon > 0$ radians. How can f be used to construct a finite-sheeted connected cover of S^1?

References

[1] R. H. Bing, *Generalizations of two theorems of Janiszewski,* Bulletin of the American Math. Soc., **51** (1945), 954–960.

[2] J. J. Charatonik and P. Pyrih, *Atomic mappings can spoil lightness of open maps,* Tsukuba Journal of Mathematics, **24** (2000), 157–169.

[3] C. Christenson and W. Voxman, *Aspects of Topology*, Marcel Dekker Inc., 1997

[4] C. L. Hagopian, *Simply connected plane continua have the fixed point property,* Topology Proc., **19** (1994), 111–119.

[5] C. L. Hagopian, *The fixed-point property for simply connected plane continua,* Trans. AMS, **348** (1996), 4525–4548.

[6] G. Jungck, *Local Homeomorphisms,* Dissertationes Mathematicae, **209** (1983), 1–51.

[7] G. Jungck and M. Timm, *Domains of proper local homeomorphisms,* Topology Proc., **17** (1992), 145–149.

[8] G. Jungck and M. Timm, *H-connected inverse limit spaces,* Topology Proc., **24** (1999), 173–187.

[9] B. Knaster and C. Kuratowski, *Sur les continua non-bornes,* Fund. Math., **5** (1924), 23–38.

[10] A. Y. W. Lau, *Certain local homeomorphisms of continua are homeomorphisms,* Bulletin de L'Academie Polonaise des Sciences, **26** (1973), 315–317.

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