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ON THE COMPARISON OF HEREDITY OF GENERALIZED METRIC PROPERTIES TO MAPPING SPACES AND HYPERSPACES

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Abstract. We compare the heredity of some generalized metric properties to mapping spaces with compact open topology and hyperspaces of compact subsets with finite topology. Especially, we treat here spaces with a G_δ-diagonal, G^*_δ-diagonal, regular G_δ-diagonal, paracompact M-space, M-space, topologically complete spaces and Moore spaces.

1. Introduction

Throughout this paper, all spaces are assumed to be regular T_1 unless the contrary is stated explicitly. For a space X, we denote the topology of X by $\tau(X)$.

In this paper, we compare the heredity of generalized metric properties of a space Z to the hyperspace $\mathcal{K}(Z)$ of non-empty compact subsets of Z and to the mapping space $C(X, Z)$ with the compact domain. That is, we study two topological operations under classes of topological spaces in terms of generalized metric properties as follows: Let $\mathcal{P}$ be a class of spaces with some generalized metric property;

(I) if $Z \in \mathcal{P}$, then does $\mathcal{K}(Z) \in \mathcal{P}$?
(II) if X is a compact space and $Z \in \mathcal{P}$, then does $C(X, Z) \in \mathcal{P}$?

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Here, $\mathcal{K}(Z)$ is the space of non-empty compact subsets of Z with a finite topology in the sense of [4], called frequently the "Vietoris topology" also, which has a base

$$\{\langle U_1, \cdots, U_k\rangle | U_1, \cdots, U_k \in \tau(X) \text{ and } k = 1, 2, \cdots\},$$

where $\langle U_1, \cdots, U_k\rangle =$

$$\left\{ K \in \mathcal{K}(Z) \;\middle|\; K \subset \bigcup_i U_i \text{ and } K \cap U_i \neq \emptyset \text{ for each } i \right\},$$

and $C(X, Z)$ has the compact-open topology which has a base

$$\{W(K_1, \cdots, K_n; U_1, \cdots, U_n) | K_1, \cdots, K_n \in \mathcal{K}(X),$$

$$U_1, \cdots, U_n \in \tau(Z),\ n = 1, 2, \cdots\},$$

where $W(K_1, \cdots, K_n; U_1, \cdots, U_n) =$

$$\{f \in C(X, Z) | f(K_i) \subset U_i \text{ for each } i\}.$$

For brevity, we write $\langle \mathcal{U}\rangle$ in place of $\langle U_1, \cdots, U_k\rangle$ when $\mathcal{U} = \{U_1, \cdots, U_k\}$.

Both $\mathcal{K}(Z)$ and $C(X, Z)$ are regular T_1 for a regular T_1-space Z ([4, Theorem 4.9.10] and [2, Theorem 3.4.13], respectively).

As for undefined terms in generalized metric spaces treated here, refer to [3].

2. The case of M-spaces

Čoban showed in [1, Proposition 2] that if f is a perfect mapping of a completely regular space X onto a completely regular space Y, then the mapping $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by

$$f^*(K) = f(K),\ K \in \mathcal{K}(X),$$

is also perfect. Using this, he showed that paracompact M-spaces are closed under the operation I. But we can show the perfectness of f^* for more general spaces, the proof of which is more complicated than his. We note that his proof indeed proceeds via the Stone-Čeck compactifications βX and βY and we do not use them.

Proposition 2.1. *Let X, Y be T_2-spaces and let f be a perfect mapping of X onto Y. Then $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ is also perfect.*

Proof: f^* is continuous [4, Theorem 5.10.1] and obviously onto. Since for each $K \in \mathcal{K}(Y)$, $\mathcal{K}(f^{-1}(K))$ is compact, $(f^*)^{-1}(K)$ is compact. Thus it remains to show that f^* is closed. To this end, let $\hat{O}$ be any open subset of $\mathcal{K}(X)$ such that $(f^*)^{-1}(K) \subset \hat{O}$. Take finite collections $\mathcal{U}_i$, $i = 1, \cdots, k$ of open subsets of X such that

$$(1) \qquad (f^*)^{-1}(K) \subset \bigcup_i \langle \mathcal{U}_i \rangle \subset \hat{O},$$

$$\langle \mathcal{U}_i \rangle \cap (f^*)^{-1}(K) \neq \emptyset, \ i = 1, \cdots, k.$$

Let $\{\mathcal{U}(\delta) | \delta \in \Delta\}$ be the totality of finite covers of $f^{-1}(K)$ by members of $\bigcup_i \mathcal{U}_i$. Obviously Δ is finite. Let $\delta \in \Delta$ and let $\mathcal{V}(1), \cdots, \mathcal{V}(s(\delta))$ be the totality of subcollections of $\mathcal{U}(\delta)$ such that $(Y \setminus f(X \setminus \bigcup \mathcal{V}(i))) \cap K \neq \emptyset$ for each i. Let

$$\hat{V}(\delta) = \left\langle \left\{ Y \setminus f\left(X \setminus \bigcup \mathcal{V}(i) \right) \,\middle|\, i = 1, \cdots, s(\delta) \right\} \right\rangle,$$

which is an open neighborhood of K in $\mathcal{K}(Y)$. Thus, if we define

$$(2) \qquad \hat{V} = \bigcap \{\hat{V}(\delta) | \delta \in \Delta\},$$

then $\hat{V}$ is an open neighborhood of K in $\mathcal{K}(Y)$. To show the closedness of f^*, it suffices to show that $(f^*)^{-1}(\hat{V}) \subset \bigcup_i \langle \mathcal{U}_i \rangle$. On the contrary, assume that there exists

$$(3) \qquad L \in (f^*)^{-1}(\hat{V}) \setminus \bigcup_i \langle \mathcal{U}_i \rangle.$$

By (1), there exists $i(0)$ such that $f^{-1}(K) \in \langle \mathcal{U}_{i(0)} \rangle$. Let $\mathcal{U}_{i(0)} = \mathcal{U}(\delta_0)$, $\delta_0 \in \Delta$. If we set

$$\mathcal{U}'_{i(0)} = \{U \in \mathcal{U}_{i(0)} | U \cap L = \emptyset\},$$

then $\mathcal{U}'_{i(0)} \neq \emptyset$ because $f(L) \in \hat{V}(\delta_0)$ by (2) and $L \notin \langle \mathcal{U}_{i(0)} \rangle$. We show the validity of

$$(4) \qquad f\left(f^{-1}(K) \setminus \bigcup \mathcal{U}'_{i(0)} \right) = K.$$

Otherwise, there exists

$$p \in K \setminus f\left(f^{-1}(K) \setminus \bigcup \mathcal{U}'_{i(0)} \right),$$

which implies $f^{-1}(p) \subset \bigcup \mathcal{U}'_{i(0)}$. But this is a contradiction because $f(L) \cap (Y \backslash f(X \setminus \bigcup \mathcal{U}'_{i(0)})) \neq \emptyset$ by (3) and by the construction of $V(\delta)$. Again, by (1) there exists $i(1)$ such that

$$f^{-1}(K) \setminus \bigcup \mathcal{U}'_{i(0)} \in \langle \mathcal{U}_{i(1)} \rangle.$$

Note that $i(0)$ and $i(1)$ are distinct. Choose $\delta_1 \in \Delta$ such that $\mathcal{U}'_{i(0)} \cup \mathcal{U}_{i(1)} = \mathcal{U}(\delta_1)$. Set

$$\mathcal{U}'_{i(1)} = \{U \in \mathcal{U}_{i(1)} | L \cap U = \emptyset\},$$

then $\mathcal{U}'_{i(1)} \neq \emptyset$, for otherwise, $L \in \langle \mathcal{U}_{i(1)} \rangle$ follows and this is a contradiction to (1). By the same argument as (4), we have

$$f\left(f^{-1}(K) \setminus \bigcup (\mathcal{U}'_{i(0)} \cup \mathcal{U}'_{i(1)})\right) = K.$$

Then there exists $i(2)$ with $i(2) \neq i(0), i(1)$ such that

$$f^{-1}(K) \setminus \bigcup (\mathcal{U}'_{i(0)} \cup \mathcal{U}'_{i(1)}) \in \langle \mathcal{U}_{i(2)} \rangle.$$

Repeating this process as many times as possible, we should come to a contradiction because $\bigcup_i \langle \mathcal{U}_i \rangle$ covers $(f^*)^{-1}(K)$. □

Quasi-perfect mappings do not have this property, even if both X and Y are completely regular. To see it, it suffices to recall that the existence of a countably compact completely regular space X such that $\mathcal{K}(X)$ is not M [6, Example 2] and that M-spaces are characterized as a quasi-perfect preimage of a metric space.

The next two examples show that paracompact M-spaces and M-spaces are not closed under the operation II.

Example 2.2. *There exists compact spaces X and Y such that $C(X,Y)$ is not paracompact* M.

Construction: Let X be the unit interval $[0,1]$ with the usual topology. It is well known that $C(X,X)$ is not countably compact. Let Y be the long segment due to Michael, i.e., Y consists of ω_1 together (α, t) with $\alpha < \omega_1$ and $0 \leq t < 1$ and has the interval topology [2, Problem 3.12.19]. We show that $C(X,Y)$ is not paracompact M. Here, we identify $C(X,Y)$ with the subspace

$$\left\{\{(x, f(x)) | x \in X\} \;\middle|\; f \in C(X,Y)\right\}$$

of $\mathcal{K}(X \times Y)$, [2, Problem 3.12.27(j)]. Assume that $C(X,Y)$ is paracompact M. Let $(\hat{\mathcal{U}}_n)$ be an M-sequence for $C(X,Y)$. Since

$C(X,Y)$ is paracompact, we can assume that each $\hat{\mathcal{U}}_n$ is locally finite in $C(X,Y)$. Let $\hat{\mathcal{F}}_n$ be a locally finite closed cover of $C(X,Y)$ shrinking $\hat{\mathcal{U}}_n$. For each n, there exists an open neighborhood $\hat{O}(n)$ of

$$f = \{(x, \omega_1) | x \in X\}$$

such that

$$\hat{O}(n) \cap \left(\bigcup \{\hat{F} \in \hat{\mathcal{F}}_n | f \notin \hat{F}\} \right) = \emptyset,$$

$$\hat{O}(n) = \left\langle \left\{ U_i \times ((\alpha_n, 0), \omega_1] \;\middle|\; i = 1, \cdots, k_n \right\} \right\rangle \cap C(X,Y),$$

where $\alpha_n < \omega_1$ and $U_i \in \tau(X)$ for $i = 1, \cdots, k_n$. Let $\alpha = \sup_n \alpha_n$ and let

$$\hat{G}(n) = \left\langle \left\{ \overline{U_i} \times [(\alpha, 0), \omega_1] \;\middle|\; i = 1, \cdots, k_n \right\} \right\rangle \cap C(X,Y), \; n \in \omega.$$

Then $\hat{G}(n) \subset \mathrm{S}(f, \hat{\mathcal{F}}_n) \subset \mathrm{S}(f, \hat{\mathcal{U}}_n)$. Since $(\hat{\mathcal{U}}_n)$ is an M-sequence for $C(X,Y)$, $\bigcap_n \hat{G}(n)$ is compact. But this is a contradiction. For, we can easily notice that $\bigcap_n \hat{G}(n)$ contains the closed subspace homeomorphic to $C(X,X)$ which is not countably compact.

Example 2.3. *There exists a compact space X and a countably compact space Y such that $C(X,Y)$ is not a* wΔ-space.

Construction: Let Z_1, Z_2 be countably compact spaces such that $Z_1 \times Z_2$ is not countably compact [2, Example 3.10.19]. We separate the non-limit ordinals in ω_1 to L_1 and L_2 as

$$\begin{aligned} L_1 &= \{\alpha + 2n + 1 | n \in \omega \text{ and } \alpha \text{ is a limit ordinal}\}, \\ L_2 &= \{\alpha + 2n + 2 | n \in \omega \text{ and } \alpha \text{ is a limit ordinal}\}. \end{aligned}$$

For each $\alpha \in \omega_1 + 1$, let

$$\begin{aligned} \alpha^* &= \{\alpha\} \times Z_i \quad && \text{if } \alpha \in L_i \ (i = 1, 2) \\ \alpha^* &= \{\alpha\} && \text{otherwise} \end{aligned}$$

and let $Y = \bigcup \{\alpha^* | \alpha \in \omega_1 + 1\}$. We topologize Y as follows: For each $\alpha \in L_i \, (i = 1, 2)$ we make α^* a clopen subspace homeomorphic to Z_i. For each limit ordinal α in $\omega_1 + 1$, let $\{(\beta^*, \alpha^*] | \beta < \alpha\}$ be a neighborhood base of α in Y, where

$$(\beta^*, \alpha^*] = \bigcup \{\gamma^* | \beta < \gamma \leq \alpha\}.$$

Then obviously Y is countably compact. Let $X = \omega_1 + 1$ with order topology. Now, we show that $C(X,Y)$ is not a wΔ-space. Assume that there exists a wΔ-sequence $(\hat{\mathcal{U}}_n)$ for $C(X,Y)$. As in the previous example, we identify $C(X,Y)$ with the subspace of $\mathcal{K}(X \times Y)$. Let

$$f = \{(\alpha, \omega_1) | \alpha \in \omega_1 + 1\}.$$

For each n, there exists a closed neighborhood $\hat{O}(n)$ of f in $C(X,Y)$ such that

$$\hat{O}(n) = \left\langle \left\{ \overline{U_i} \times [\alpha_n^*, \omega_1^*] \;\middle|\; i = 1, \cdots, k_n \right\} \right\rangle \cap C(X,Y) \subset \mathrm{S}(f, \hat{\mathcal{U}}_n),$$

where α_n is a limit ordinal and $\{U_i | i = 1, \cdots, k_n\}$ is an open cover of $\omega_1 + 1$. ($[\alpha_n^*, \omega_1^*]$ is defined similarly to the above.) Let $\alpha = \sup_n \alpha_n$. Then

$$C = \bigcap_n \left\langle \left\{ \overline{U_i} \times [\alpha^*, \omega_1^*] \;\middle|\; i = 1, \cdots, k_n \right\} \right\rangle \cap C(X,Y)$$

is a countably compact closed subset of $C(X,Y)$. Since as easily seen, $Z_1 \times Z_2$ is embedded in C as a closed subset, C cannot be countably compact, a contradiction.

3. The case of topologically complete spaces

A space X is called *topologically complete* if there exists a uniformity μ compatible with $\tau(X)$ such that (X, μ) is complete. Such spaces coincide with inverse limits of metric spaces [8]. Using this, Zenor proved that topologically complete spaces are closed under the operation I [9]. Similarly, we show that they are closed under the operation II when the domain X is a k-space. For it, we prepare the following notation. For a space X, let $\mathcal{K}(X) = \{K_\alpha | \alpha \in A\}$ and we introduce the order $\leq$ in A as follows: For $\alpha, \beta \in A$, $\alpha \leq \beta$ if and only if $K_\alpha \subset K_\beta$. For a directed set B, we introduce the order $\leq$ in $A \times B$ as follows: For $\lambda_1 = (\alpha_1, \beta_1), \lambda_2 = (\alpha_2, \beta_2) \in A \times B$, $\lambda_1 \leq \lambda_2$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. For $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \leq \alpha_2$, let $i_{\alpha_1 \alpha_2}$ be the inclusion mapping of K_{α_1} into K_{α_2}.

Proposition 3.1. *Let X be a k-space and $Y = \varprojlim \{Y_\beta, \pi_{\beta_1 \beta_2} | \beta_1, \beta_2 \in B, \beta_1 > \beta_2\}$. Then $C(X,Y)$ is the inverse limit of $\{C(K_\alpha, Y_\beta) | (\alpha, \beta) \in A \times B\}$.*

Proof: As the bonding mappings, we define

$$\Pi_{\lambda_1 \lambda_2} : C(K_{\alpha_1}, Y_{\beta_1}) \to C(K_{\alpha_2}, Y_{\beta_2}),$$

for $\lambda_1(\alpha_1, \beta_1), \lambda_2(\alpha_2, \beta_2) \in A \times B$ and $\lambda_2 < \lambda_1$ as follows:

$$\Pi_{\lambda_1 \lambda_2}(f) = \pi_{\beta_1 \beta_2} \circ f \circ i_{\alpha_2 \alpha_1}, \; f \in C(K_{\alpha_1}, Y_{\beta_1}).$$

Since it is easy to check for $\lambda_i = (\alpha_i, \beta_i) \in A \times B\,(i = 1, 2, 3)$ with $\lambda_3 < \lambda_2 < \lambda_1$

$$\begin{aligned}\Pi_{\lambda_2\lambda_3} \circ \Pi_{\lambda_1\lambda_2}(f) &= \Pi_{\lambda_2\lambda_3} \circ (\pi_{\beta_1\beta_2} \circ f \circ i_{\alpha_2\alpha_1})\\ &= \pi_{\beta_2\beta_3} \circ (\pi_{\beta_1\beta_2} \circ f \circ i_{\alpha_2\alpha_1}) \circ i_{\alpha_3\alpha_2}\\ &= \pi_{\beta_1\beta_3} \circ f \circ i_{\alpha_3\alpha_1}\\ &= \Pi_{\lambda_1\lambda_3}(f),\end{aligned}$$

$\{C(K_\alpha, Y_\beta), \Pi_{\lambda_1\lambda_2} | \lambda_1, \lambda_2 \in A \times B, \lambda_2 < \lambda_1\}$ forms an inverse limit system. Let

$$P = \varprojlim C(K_\alpha, Y_\beta)$$

be its inverse limit. Then we show that $C(X, Y) \cong P$. To this end, we define a homeomorphism Φ of $C(X, Y)$ onto P as follows: Let $\pi_\beta : Y \to Y_\beta$ be the projection for each $\beta \in B$. Let f be an arbitrary element of $C(X, Y)$. Then we let

$$\Phi(f) = (f_\lambda)_{\lambda \in A \times B},$$

where for each $\lambda = (\alpha, \beta) \in A \times B$,

$$f_\lambda = (\pi_\beta \circ f)|K_\alpha.$$

Then clealy $(f_\lambda) \in P$. To show that Φ is one-to-one, let $f \neq g$, $f, g \in C(X, Y)$. There exist $x \in X$ and $\beta \in B$ such that $(\pi_\beta \circ f)(x) \neq (\pi_\beta \circ g)(x)$. Pick $\alpha \in A$ with $K_\alpha = \{x\}$. For this $\lambda = (\alpha, \beta)$, clearly $f_\lambda \neq g_\lambda$, which means $\Phi(f) \neq \Phi(g)$. To see that Φ is onto, let $(f_\lambda) \in P$. Let x be an arbitrary point of X and pick $\alpha(x) \in A$ with $K_{\alpha(x)} = \{x\}$. For each $\beta \in B$, $f_{(\alpha(x),\beta)}(x) \in Y_\beta$. Then we define a correspondence $f : X \to Y$ such that

$$f(x) = \left(f_{(\alpha(x),\beta)}(x)\right)_{\beta \in B}.$$

Note that $f|K_\alpha = (f_{(\alpha,\beta)})_{\beta \in B}$ for each $\alpha \in A$. Since X is a k-space, $f \in C(X, Y)$ and obviously $\Phi(f) = (f_\lambda)$. To see the continuity, let $\lambda = (\alpha, \beta) \in A \times B$. Let C_α be a compact subset of K_α and U_β an open subset of Y_β. Then we have the equality

$$\Phi^{-1}\left(\Pi_\lambda^{-1}(W(C_\alpha; U_\beta))\right) = W\left(C_\alpha; \pi_\beta^{-1}(U_\beta)\right),$$

where $\Pi_\lambda : P \to C(K_\alpha, Y_\beta)$ is the projection, which proves the continuity. Conversely, to see the continuity of Φ^{-1}, let $\alpha \in A$ and let U_β be an open subset of Y_β. Then it is easy to see

$$\Phi\left(W\left(K_\alpha;\pi_\beta^{-1}(U_\beta)\right)\right)=\Pi_{(\alpha,\beta)}^{-1}(W(K_\alpha;U_\beta)),$$

which proves the continuity. Hence Φ is a homeomorphism. □

Theorem 3.2. *Let X be a k-space. If Y is a topologically complete space, then so is $C(X,Y)$.*

Proof: Let $Y=\varprojlim Y_\alpha$, where Y_α's are metric spaces. By the above proposition, $C(X,Y)=\varprojlim C(K_\alpha,Y_\beta)$. Since $C(K_\alpha,Y_\beta)$ is metrizable [2, Exercise 4.2.H], $C(X,Y)$ is topologically complete. □

4. The case of G_δ-diagonals.

From the definitions [3, Definitions 2.1, 2.10], the following implication holds true:
Regular G_δ-diagonal → G_δ^*-diagonal → G_δ-diagonal.
It is shown in [6, Example 3] that spaces with a G_δ-diagonal are not closed under the operation I, but this example is not a T_2-space. So, we give here another example.

Example 4.1. *There exists a T_2-space Z with a G_δ-diagonal such that $\mathcal{K}(Z)$ has no G_δ-diagonal.*

Construction: Let $X=X(1)\cup X(2)$, where

$$\begin{aligned} X(1) &= \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; y=0,1,\frac{1}{2},\frac{1}{3},\cdots\right\},\\ X(2) &= \mathbb{R}\times\{-1\}.\end{aligned}$$

Topologize X as follows: Each $\mathbb{R}\times\{\frac{1}{n}\}$ is an open Michael line. For each $p=(x,0)$ or $(x,-1)$ has a neighborhood base $\{N(p;\varepsilon)|\varepsilon>0\}$, where $N(p;\varepsilon)$'s are defined by;

(1) if $p=(x,0)$ and $x\in\mathbb{Q}$ (the rationals), then $N(p;\varepsilon)=$

$$\{p\}\cup\left\{(x',y')\in X(1)\;\middle|\;0\le y'\le\frac{1}{\sqrt{2}}|x'-x| \text{ and } 0\ne|x'-x|<\varepsilon\right\};$$

(2) if $p=(x,-1)$ and $x\in\mathbb{Q}$, then $N(p;\varepsilon)=$

$$\{p\}\cup\left\{(x',y')\in X(1)\;\middle|\;\frac{1}{\sqrt{2}}|x'-x|<y'<\varepsilon\right\};$$

(3) if $p=(x,0)$ and $x\in\mathbb{R}\setminus\mathbb{Q}$, then $N(p;\varepsilon)=$

$$\{p\}\cup\left\{(x',y')\in X(1)\;\middle|\;0\le y'\le|x'-x| \text{ and } 0\ne|x'-x|<\varepsilon\right\};$$

(4) if $p=(x,-1)$ and $x\in\mathbb{R}\setminus\mathbb{Q}$, then $N(p;\varepsilon)=$

$$\{p\}\cup\left\{(x',y')\;\middle|\;|x'-x|<y'<\varepsilon\right\}.$$

Then X has a G_δ-diagonal. In fact, since $X(1)$ is submetrizable, there exists a G_δ-diagonal sequence $\{\mathcal{U}(n)|n=1,2,\cdots\}$ for $X(1)$. For each n, let

$$\mathcal{V}(n)=\mathcal{U}(n)\cup\left\{N\left(p;\frac{1}{n}\right)\;\middle|\;p\in X(2)\right\}.$$

Then it is easy to see that $(\mathcal{V}(n))$ is a G_δ-diagonal sequence for X. Let Y be the Michael line and let $Z=X\cup_f Y$ be the adjunction space of X and Y with respect to $f: X(2)\to Y$ such that

$$f((x,-1))=x,\ x\in\mathbb{R}.$$

By [5, Lemma] Z has a G_δ-diagonal. We show that $\mathcal{K}(Z)$ has no G_δ-diagonal. For a contradiction, assume that $\mathcal{K}(Z)$ has a G_δ-diagonal sequence $(\hat{\mathcal{U}}(n))$. For each $s\in\mathbb{R}$, let

$$\begin{aligned} K(s) &= \left\{(s,y)\;\middle|\;y=-1,1,\frac{1}{2},\frac{1}{3},\cdots\right\},\\ L(s) &= K(s)\cup\{(s,0)\}.\end{aligned}$$

Then $K(s),L(s)\in\mathcal{K}(Z)$ and $K(s)\ne L(s)$. Since $(\hat{\mathcal{U}}(n))$ is a G_δ-diagonal sequence, there exists $n(s)$ such that
$L(s)\notin \mathrm{S}(K(s),\hat{\mathcal{U}}(n(s)))$. By the second category theorem, there exists n such that

$$A=\mathrm{Int}_{\mathbb{R}}(\mathrm{Cl}_{\mathbb{R}}(\{s\in\mathbb{R}|n(s)=n\}))\ne\emptyset,$$

where $\mathrm{Int}_{\mathbb{R}}$, $\mathrm{Cl}_{\mathbb{R}}$ means the operator in the usual sense. Take $r\in A\cap\mathbb{Q}$. Since $\hat{\mathcal{U}}(n)$ covers $\mathcal{K}(Z)$, there exists $\hat{U}\in\hat{\mathcal{U}}(n)$ such that $L(r)\in\hat{U}$. Then it is easily observed that there exists $s\in\mathbb{R}$ with $n(s)=n$ such that both $K(s)$ and $L(s)$ belong to $\hat{U}$, a contradiction.

Note that this example is not regular. As for G^*_δ-diagonals, we do not know the following:

Question 4.2. If a space X has a G^*_δ-diagonal, then does $\mathcal{K}(X)$ have a G^*_δ-diagonal?

As a partial answer, we can easily show the following: (1) If a space X has a sequence $(\mathcal{U}(n))$ of open covers of X such that $K = \bigcap_n \overline{\mathrm{S}(K,\mathcal{U}(n))}$ for each $K \in \mathcal{K}(X)$, then $\mathcal{K}(X)$ has a G^*_δ-diagonal. (2) If a space X has a sequence $(\mathcal{U}(n))$ of open covers of X such that $K = \bigcap_n \mathrm{S}(K,\mathcal{U}(n))$ for each $K \in \mathcal{K}(X)$, then $\mathcal{K}(X)$ has a G_δ-diagonal.

We give the positive results to I and II.

Theorem 4.3. *If a space X has a regular G_δ-diagonal, then so does $\mathcal{K}(X)$.*

Proof: Recall the characterization of a space X having a regular G_δ-diagonal [10, Theorem 1] that there exists a sequence $(\mathcal{U}(n))$ of open covers of X such that if $x, y \in X$ with $x \neq y$, then there exist $n \in \omega$ and neighborhoods U, V of x, y, respectively, such that $U \cap \mathrm{S}(V,\mathcal{U}(n)) = \emptyset$. Suppose that X has such a sequence $(\mathcal{U}(n))$ with $\mathcal{U}(n+1) < \mathcal{U}(n)$, $n \in \omega$. For each n, let

$$\hat{\mathcal{U}}(n) = \{\langle \mathcal{U}_0 \rangle | \mathcal{U}_0 \subset \mathcal{U}(n) \text{ is finite}\}.$$

Then $\hat{\mathcal{U}}(n)$ is an open cover of $\mathcal{K}(X)$. We show that $(\hat{\mathcal{U}}(n))$ has the required property. Let $K, L \in \mathcal{K}(X)$ and $K \neq L$. Suppose that there exists a point $p \in K \setminus L$. Then there exist neighborhoods U, V of p, L in X, respectively, and $n \in \omega$ such that $U \cap \mathrm{S}(V,\mathcal{U}(n)) = \emptyset$. Then it is easy to see that

$$\langle X, U \rangle \cap \mathrm{S}(\langle V \rangle, \hat{\mathcal{U}}(n)) = \emptyset.$$

Since $\langle X, U \rangle$, $\langle V \rangle$ are neighborhoods of K, L in $\mathcal{K}(X)$, respectively, we can say that $(\hat{\mathcal{U}}(n))$ has the required property. $\square$

Theorem 4.4. *Let X be such a space that has a σ-compact dense subset. If Y has a G_δ-diagonal, G^*_δ-diagonal, regular G_δ-diagonal, then so does $C(X,Y)$, respectively.*

Proof: We show only the case of a regular G_δ-diagonal, and the others are similar. By the assumption on X, there is no loss if we assume that $X = \bigcup_n X_n$, where $X_n \in \mathcal{K}(X)$, $n \in \omega$. By the characterization stated above, there exists a sequence $(\mathcal{U}(n))$ of open covers satisfying the same condition as there. For $m, n \in \omega$, let $\Delta(m,n)$ be the totality of pairs $\delta = (\mathcal{K}(\delta), \mathcal{U}(\delta))$ of subsets $\mathcal{K}(\delta)$, $\mathcal{U}(\delta)$ of $\mathcal{K}(X)$, $\mathcal{U}(n)$, respectively, such that $\mathcal{K}(\delta) = \{K_1, \cdots, K_t\}$ is a finite cover of X_m and $\mathcal{U}(\delta) = \{U_1, \cdots, U_t\}$. For each $\delta \in \Delta(m,n)$, let

$$W(\delta) = W(K_1, \cdots, K_t; U_1, \cdots, U_t)$$

and $\mathcal{W}(m,n) = \{W(\delta) | \delta \in \Delta(m,n)\}$. Since each X_m is compact, it is easy to see that $\mathcal{W}(m,n)$ covers $C(X,Y)$. To see that $(\mathcal{W}(m,n))$ has the required property, let $f \neq g$, $f, g \in C(X,Y)$. Then there exist $m \in \omega$ and x_0 such that $x_0 \in X_m$ and $f(x_0) \neq g(x_0)$. There exist neighborhoods O and O' of $f(x_0)$ and $g(x_0)$ in Y, respectively, and n, such that $\mathrm{S}(O, \mathcal{U}(n)) \cap O' = \emptyset$. It is easily checked that

$$\mathrm{S}(W(\{x_0\}; O), \mathcal{W}(m,n)) \cap W(\{x_0\}; O') = \emptyset.$$

Hence, $C(X,Y)$ has a regular G_δ-diagonal. □

5. The case of Moore spaces and the conclusion

It is known that Moore spaces are closed under the operation I, [5]. Moreover, it is shown in [7] that Moore spaces with a regular G_δ-diagonal are closed under the operation I and II. But we do not know whether Moore spaces are closed under the operation II.

Question 5.1. Let X be a compact space and let Y be a Moore space. Then is $C(X,Y)$ Moore?

As the conclusion, the results in this are shown by the following figure, where $+$, $-$ means the operation I and II holds true or not, respectively.

spaces	operation I	operation II
M-space	$-$	$-$
paracompact M-space	$+$	$-$
topologically complete space	$+$	$+$
G_δ-diagonal	$-$ (T_2-space)	$+$
G_δ^*-diagonal	?	$+$
regular G_δ-diagonal	$+$	$+$
Moore space	$+$	?
Moore space with a regular G_δ-diagonal	$+$	$+$

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