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SPACES ADMITTING TOPOLOGICALLY TRANSITIVE MAPS

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ABSTRACT. In this paper, after observing that on certain topological spaces there are no topologically transitive maps at all, we characterize all those locally compact subspaces of the real line that admit a topologically transitive map.

We prove that any locally compact subspace X of $\mathbb{R}$ admitting a topologically transitive map, must be one of the following up to homeomorphism:
  (1) Finite discrete space
  (2) Finite union of nontrivial compact intervals
  (3) Finite union of nontrivial noncompact intervals
  (4) Cantor set $\mathcal{K}$
  (5) $\mathcal{K} + \mathbb{N}$.

1. INTRODUCTION

A self map f on a topological space X is said to be topologically transitive if for every pair of non-empty open sets U and V in X, there exists x in U and a natural number n such that $f^n(x) \in V$, i.e., $f^n(U) \cap V$ is non-empty.

Examples of transitive maps on the unit circle S^1, the Cantor set $\mathcal{K}$ and the compact interval $[0,1]$ are well known. See [5] or [7]. Some explicit examples of transitive maps on the real line $\mathbb{R}$ were constructed in [1] and [12], and they must have infinitely many critical points [2].

It is also known already that the following topological spaces admit topologically transitive maps (satisfying some additional properties) (see [8]):

Any connected region in $\mathbb{R}^n$. (see Sidorov [11])

The Torus $S^1 \times S^1$. (see Devaney [5])

The cylinder $S^1 \times \mathbb{R}$. (see Besicovitch [3])

The Banach Space l_1. (see Read [10])

The square $I \times I$. (see Xu [13])

Therefore, the following is one of the natural questions: What are all the subspaces of $\mathbb{R}$ that admit topologically transitive maps? We provide a complete answer, among locally compact spaces. We prove that among infinite, zero-dimensional subspaces there are exactly two such (up to homeomorphism), on which topologically transitive continuous self-maps exist: (i) the Cantor set $\mathcal{K}$ and (ii) $\mathcal{K} + \mathbb{N}$.

Similarly, we prove that there are only countably many subspaces of $\mathbb{R}$ (up to homeomorphism) that are locally connected and that admit a topologically transitive map. These are:

(i) $N_n = \{1, 2, ..., n\}$ for each positive integer n.

(ii) $[0, \frac{1}{2}] + N_n$, where again n is a positive integer.

(iii) $([0, \frac{1}{2}) + N_n) \cup ((n, n + \frac{1}{2}) + N_m)$ where n and m are positive integers.

Apart from these, there is nothing else; every locally connected subspace of $\mathbb{R}$ admitting a topologically transitive map must be homeomorphic to one of these.

Moreover, every locally compact subspace of $\mathbb{R}$ admitting a topologically transitive map must fall in one of the above two lists. This is our main theorem.

2. SUBSPACES OF $\mathbb{R}$

We restrict ourselves to subspaces of $\mathbb{R}$ presently.

Proposition. *Let X be a locally compact subspace of $\mathbb{R}$, and let $f : X \longrightarrow X$ be topologically transitive. Let some element of X have a connected neighbourhood. Then X has only finitely many connected components.*

Proof: Let $\tilde{X}$ be the space of all connected components of X, and $q : X \longrightarrow \tilde{X}$ be the natural quotient map. Let $\tilde{f} : \tilde{X} \longrightarrow \tilde{X}$ be

the map induced by f. Then $\tilde{f}q = qf$, i.e., q is a semi-conjugacy between f and $\tilde{f}$, and $\tilde{f}$ is topologically transitive on $\tilde{X}$.

Let x have a connected neighborhood W in X. Then, because f is transitive, there is a $y \in W$ having a dense f-orbit in X. Then $q(y)$ has a dense $\tilde{f}$-orbit in $\tilde{X}$. Also, since f is transitive, there is a positive integer k such that $f^k(W) \cap W$ is nonempty. Then $\tilde{f}^k(q(y)) = q(y)$. Thus, $q(y)$ is a periodic point, and the orbit of $q(y)$ is both finite and dense in $\tilde{X}$. Therefore, $\tilde{X}$ should be finite and so X has only finitely many connected components. $\square$

Theorem 2.1. *Let X be any locally compact subspace of $\mathbb{R}$ admitting a topologically transitive map. Then X must be homeomorphic to one of the following:*

(1) $\{1, 2, ..., m\}$, for some $m \in \mathbb{N}$

(2) $[0, 1] \cup [2, 3] \cup \cdots \cup [2n, 2n + 1]$, for some nonnegative integer n

(3) $([0, 1) \times \{1, 2, ..., k\}) \cup ((0, 1) \times \{k + 1, ..., n\})$, for some nonnegative integer $k \leq n$

(4) Cantor set $\mathcal{K}$

(5) $\mathcal{K} + \mathbb{N}$.

Proof: We first prove that each of the locally compact subspaces mentioned above admits a topologically transitive map.

(1) Let $X = \{1, 2, ..., m\}$ and $f : X \longrightarrow X$ be such that
$f(i) \ = i + 1, i = 1, 2, ..., m - 1$
$f(m) = 1$.
Then clearly f is topologically transitive on X and is nothing else but the m-cyclic permutation $\sigma = (1, 2, ..., m)$.

(2) Let X_n denote $(0,1] \ U \ldots U \ [2n, 2n+1]$. First, when $n = 0$, we have $X_0 = [0, 1]$. And it is already known that the tent map defined by

$$t(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

is topologically transitive on X_0. It can be proved that given any nontrivial subinterval J of X_0, there is a positive integer m such that $t^m(J) = X_0$. It follows that $t^p(J) = X_0$ for all $p \geq m$. This fact will be used in constructing a transitive map on X_n for all other values of n.

Let $n \geq 1$. Define $f : X_n \longrightarrow X_n$ by
$f(x) = t(x - 2k) + 2(k + 1) \ (mod \ 2(n + 1)), \ x \in [2k, 2k + 1],$
$k = 0, 1, 2, ..., n$.

Then it can be easily verified that $f_{|_{[2k,2k+1]}}$ is nothing but the composite $\theta_k.t.h_k$ where h_k is the map $x \longrightarrow x - 2k$, t is the tent map on $[0, 1]$, and θ_k is the map $x \longrightarrow x + 2(k + 1)(mod \ 2(n + 1))$.

To prove the transitivity of f, let I be a nonempty open interval in X. Then $I \subset [2k, 2k + 1]$, for some k. Consider $h_k(I)$. By a known property of the tent map t, there exists $m \in \mathbb{N}$ such that $t^m(h_k(I)) = [0, 1]$. For the same m, it is not hard to see that $h^m(I) \supset [2r, 2r + 1]$ for some r. It follows that the finite union $\cup_i = 0$ to $n \ h^{m+i}$ contains the whole X.

Note : We note that on $[0, 1] \times \{1, 2, ..., n\}$, the map $t \times \sigma$ is topologically transitive, where t is the tent map on $[0, 1]$ and σ is the cyclic permutation $(1, 2, ..., n)$. But it is not true in general that if g is any topologically transitive map on a space X, it induces in this way a topologically transitive map on $X \times \{1, 2, ..., n\}$.

(3) Any finite union of nontrivial noncompact intervals is homeomorphic to $X = \prod_{k=1}^n X_k$, where each X_k is either $[0, 1)$ or $(0, 1)$, for some $n \in \mathbb{N}$.

Define $g : [0, 1) \longrightarrow (0, 1)$, as

$$g(x) = \begin{cases} \frac{3}{10}(3 - x), & x \in [0, \frac{1}{2}] \\ \frac{1}{2k+2}, & x = \frac{2k}{2k+1} \\ \frac{2k+1}{2k+2}, & x = \frac{2k-1}{2k}, \end{cases}$$

and g is linear on $[\frac{2k-1}{2k}, \frac{2k}{2k+1}]$ and $[\frac{2k-2}{2k-1}, \frac{2k-1}{2k}]$, $k \in \mathbb{N}$ (see Fig. 1).

Then any interval, under iterations of g, expands to contain some $(\frac{2k-1}{2k}, \frac{2k}{2k+1})$ in finite number of steps which eventually expands to cover the whole of $(0, 1)$.

Define $h : X \longrightarrow X$ as $h = (g \times \sigma)|_X$ where σ is the cyclic permutation $(1, 2, ..., n)$, and $g \times \sigma : [0, 1) \times \{1, 2, ..., n\} \longrightarrow (0, 1) \times \{1, 2, ..., n\}$. Then, clearly h is topologically transitive on X.
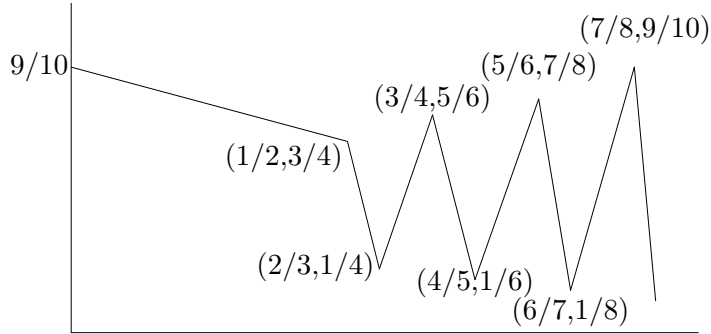
FIGURE 1. Graph of the function g.

(4) The modified *tent map*, $t : \mathcal{K} \longrightarrow \mathcal{K}$ defined as

$$t(x) = \begin{cases} 3x & x \in [0, \frac{1}{2}] \cap \mathcal{K} \\ 3 - 3x & x \in [\frac{1}{2}, 1] \cap \mathcal{K}. \end{cases}$$

is topologically transitive on $\mathcal{K}$.

(5) Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Let $X = \mathcal{K} + 2\mathbb{N}_0$. Define first $f : X \longrightarrow \mathbb{R}$, for each $n \in \mathbb{N}_0$, as

$$f(x) = \begin{cases} 3^{n+2}(x - 2n) & x \in [2n, \frac{6n+1}{3}] \\ 3^{n+2}(2n + 1 - x) & x \in [\frac{6n+2}{3}, 2n + 1]. \end{cases}$$

We see that any $x \in X$ under f first comes to the Cantor set in the unit interval $[0, 1]$ and then, due to multiplication by a power of 3, goes to a Cantor set with even integral part. And so, actually $f : X \longrightarrow X$.

Let us call $\mathcal{K}_n = X \cap [2n, 2n + 1]$, $n \in \mathbb{N}_0$. Then we see that, $f(\mathcal{K}_0) = \mathcal{K}_0 \cup \mathcal{K}_1$, $f(\mathcal{K}_1) = \mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$, and so on. And any subset of X expands to cover $\mathcal{K}_0$ in finite time. This proves the topological transitivity of f.

Thus, X and hence $\mathcal{K} + \mathbb{N}$ admits a transitive map.

We now prove the second part, that if X is any locally compact subspace of $\mathbb{R}$ admitting a topologically transitive self-map f, then X must be one of the spaces mentioned in the above list.

First, we claim that f cannot map a nontrivial interval to a singleton. For if J is an open interval such that $f(J) = \{x_0\}$, let k

be the least positive integer such that $f^k(J) \cap J$ is nonempty. Then, because $f^k(J)$ is a singleton, $f^k(J) \subset J$ and therefore $f^{k+1}(J) = \{x_0\}$. Thus, x_0 is a periodic point, the orbit of x_0 is finite, and there is a nonempty open subinterval J_1 of J disjoint from this finite set. But it is seen that $f^n(J)$ never meets J_1, whatever n may be. This contradicts the topological transitivity of f.

Secondly, the same argument proves that if J is an open singleton $\{y\}$, then the orbit of y is finite, and the transitivity of f will force that X is finite. This second claim will be repeatedly used in what follows.

Thirdly, the first claim implies that f maps nontrivial intervals in X to nontrivial intervals in X. Therefore, if Y is the union of all nontrivial intervals in X, then $f(Y) \subset Y$ and $f_{|Y}$ is topologically transitive on Y.

The transitivity of $(Y, f_{|Y})$ implies that if J_1 and J_2 are interiors of any two components of Y, then $f^n(J_1) \cap J_2$ is nonempty for some positive integer n. Correspondingly, in the quotient space $(\tilde{Y}, \tilde{f_{|Y}})$, this means that for any two elements $y_1, y_2 \; \exists \; m, n \in \mathbb{N}$ such that $f^m(y_1) = y_2$ and $f^n(y_2) = y_1$. This would mean that all elements are periodic and that there is a single orbit. This can happen only when $\tilde{Y}$ is finite.

This implies that X has only finitely many nontrivial connected components. Since every connected component is closed, Y is closed in X.

On the other hand, Y is f-invariant, and unless it is empty, it has nonempty interior. Therefore, Y is dense in X, unless Y is empty.

Combining these two observations, we conclude that $Y = X$ or $Y = \phi$. This means: Either X is a finite union of nontrivial intervals, or X is totally disconnected. Let us consider these two cases separately.

Case 1 : Let $X = Y = I_1 \cup I_2 \cup \cdots \cup I_n$ be the decomposition of X into connected components. We claim that if one of these components (say I_1) is compact, then X itself is compact.

In this case, the space $\tilde{X}$ of components is finite. The induced map $\tilde{f}$ on $\tilde{X}$ is also transitive, and is, therefore, a cyclic permutation. Therefore, $f^n(I_1) \subset I_1$, and for all positive integer k, $f^{n+k}(I_1) = f^k(f^n(I_1)) \subset f^k(I_1)$. This implies that $\cup_{m=1}^{\infty} f^m(I_1) =$

$\cup_{m=1}^{n} f^m(I_1)$ = a compact subset of X. But the transitivity of f implies that this is a dense subset of X. Therefore, X is compact.

Thus in this case, X is the union of finitely many compact intervals, or X is the union of finitely many noncompact intervals.

Case 2 : Let X be totally disconnected. First we claim that X is either finite or uncountable. The local compactness of X implies that X is a Baire space. If X were countable, writing X as the union of its singletons and applying Baire Category theorem, some singleton in X must be open. But, as already seen, this would imply that X is finite.

Now let X be uncountable. The locally compact space X may or may not be compact. Consider two subcases:

a: Let X be compact. Then X is an uncountable compact, totally disconnected space; also X has no isolated points, because the only spaces having isolated points and admitting transitive maps, are the finite spaces; also X is a subspace of $\mathbb{R}$ and hence metrizable. Therefore, by a classical theorem of Hausdorff (see [6]), X is homeomorphic to the Cantor set.

b: Let X be noncompact. First note that X is zero-dimensional, as every totally disconnected subset of $\mathbb{R}$ is; X is locally compact by assumption; X is second countable because X is a subspace of $\mathbb{R}$. Therefore, $X = \cup_{n=1}^{\infty} V_n$, where each V_n is compact and open in X. Defining $W_n = V_n - (V_1 \cup \cdots \cup V_{n-1})$, we can write $X = \cup_{n=1}^{\infty} W_n$, where each W_n is compact and open and these W_n's are pairwise disjoint. By deleting the empty sets among them, we may assume each W_n is nonempty. As already seen, X does not have an isolated point, and so W_n also does not. By Baire-Category theorem once again, each W_n is uncountable. The same classical theorem of Hausdorff implies that each W_n is homeomorphic to the Cantor set $\mathcal{K}$. It follows that X is the disjoint topological sum of countably many homeomorphic copies of $\mathcal{K}$. In other words, X is homeomorphic to $\mathcal{K} \times \mathbb{N}$ or equivalently to the subset $\mathcal{K} + \mathbb{N}$ of $\mathbb{R}$.

This completes the proof of our theorem. □

3. REMARKS

Remark 1. In the course of proving our main theorem, we have actually proved some minor results that hold for more general spaces,

not necessarily embeddable in $\mathbb{R}$. We mention some of them below.

Let X be a topological space admitting a topologically transitive map. Then

(1) If some singleton is open, then every singleton is open (and X is finite).

(2) If some connected component is open, then every connected component is open (and X has finitely many components).

(3) If some connected component has nonempty interior, then every connected component is open.

(4) If some connected component is compact and open, then the whole space is compact.

Remark 2. The ideas used in the proof of our main theorem can be used to provide complete answers to the following questions:

(A) What are all the countable metric spaces admitting topologically transitive maps?

> Only the finite spaces and the space Q of rational numbers. We make use of a theorem of [9]: Every countable metric space without isolated points is homeomorphic to Q.

(B) What are all locally compact, totally-disconnected, second countable spaces admitting topologically transitive maps?

> Apart from finite spaces, there are only two others, namely $\mathcal{K}$ and $\mathcal{K} \times \mathbb{N}$. We make use of a known theorem that every second countable zero-dimensional space can be embedded in the real line.

(C) What are all the homogenous G_δ - subspaces of the real line that admit topologically transitive maps?

> These are precisely the following:
>   (i) The Cantor set $\mathcal{K}$
> (ii) The space $\mathcal{K} \times \mathbb{N}$
> (iii) Finite union of disjoint open intervals
> (iv) Finite spaces
>   (v) The space P of irrational numbers.
> To prove this, we use our main theorem and the classical theorem of Hausdorff: *Every nowhere*

> *locally compact zero dimensional completely metriz-*
> *able separable space is homeomorphic to the space*
> *P of irrational numbers.*

Remark 3. Let X be a space admitting a topologically transitive map. Let there exist a countably infinite open subset of X. Then X is nowhere locally compact.

Reformulation: Let X be an infinite space having two points, one admitting a compact neighborhood and another admitting a countable neighborhood. Then X cannot admit a topologically transitive map.

Another similar result: Let V and W be two nonempty open subsets of an infinite space X such that V is countable and W is completely metrizable. Then X cannot admit a topologically transitive map.

Outline of proof: If $f : X \longrightarrow X$ were topologically transitive, $\exists$ a positive integer n such that $f^n(W) \cap V \neq \phi$. Let $W_1 = W \cap f^{-n}(V)$. Then W_1 is also a space for which Baire Category theorem is applicable.

Now $W_1 = \cup\{W \cap f^{-n}(x) : x \in f^n(W) \cap V\}$ is a representation of W_1 as a countable union of nowhere dense sets. Here we use the known result: A transitive map cannot collapse an infinite open set to a single point. It can be proved in a similar manner that a power f^n of a transitive map f cannot collapse an infinite open set to a single point; therefore, sets of the form $f^{-n}(x)$ are nowhere dense. This contradicts the Baire Category theorem.

Example. *Let $X = \{x \in \mathbb{R} : x$ is rational and positive$\} \cup \{x \in \mathbb{R} : x$ is irrational and negative$\}$ be provided with relative topology from $\mathbb{R}$. Then X does not admit a topologically transitive map because of the above remark.*

Remark 4. We leave the following open: Find all G_δ subspaces of $\mathbb{R}$ that admit a topologically transitive map. In this context it may be noted that the following result is known: A subspace X of $\mathbb{R}$ is locally compact if and only if $X = A \cap B$ where A is an open set and B is a closed set. Therefore, all locally compact subspaces of $\mathbb{R}$ are G_δ subspaces of $\mathbb{R}$.

Remark 5. It follows from the above results that the following spaces do not admit topologically transitive map:

(1) $X \times \mathbb{N}$, where X is a connected space, e.g., $\{x \in \mathbb{R}: [\mathrm{x}]$ is even$\}$
(2) $[0,1] \cup (2,3]$
(3) $\{0\} \cup [1,2]$
(4) $Q \cup [0,1]$
(5) $\mathcal{K} \cup Q$
(6) countably infinite compact-Hausdorff spaces.

References

[1] A. Nagar and S. P. Sesha Sai, *Some classes of transitive maps on* $\mathbb{R}$, Jour. of Anal., **8** (2000), 103–111.

[2] A. Nagar, V. Kannan, and S. P. Sesha Sai, *Some properties of topologically transitive maps on* $\mathbb{R}$, Real Anal. Exchange, **27/1** (2001/2002), 1–10.

[3] A. S. Besicovitch, *A problem on topological transformation of the plane*, Fund. Math., **28** (1937), 61–65.

[4] L. S. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, 1513, Springer-Verlag, New York, 1992.

[5] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, Reading, Mass., 1989.

[6] J. S. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.

[7] R. A. Holmgren, *A First Course in Discrete Dynamical Systems*, Springer-Verlag, New York, 1996.

[8] S. Kolyada and L. Snoha, *Some aspects of topological transitivity - a survey*, Grazer Math. Ber., **334** (1997), 3–35.

[9] S. Mazurkiewicz and W. Sierpinski, *Contribution a la topologic des ensembles demombrables*, Fund. Math. **1** (1920), 17–27.

[10] C. J. Read, *A solution to the invariant subspace problem on the space* l_1 , Bul. London Math. Soc., **17** (1985), 305–317.

[11] Ye. A. Sidorov, *Smooth topologically transitive dynamical systems*, Mat. Zametki **4 no. 6** (1968), 441–452.

[12] S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain Jour. of Math., **22 no. 1** (1992), 353–375.

[13] X.-Q. Xu, *Explicit transitive automorphisms of the closed unit square*, Proc. Amer. Math. Soc. **109** (1990), 571–572.

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