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TOPOLOGICALLY TORSION ELEMENTS OF TOPOLOGICAL GROUPS

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ABSTRACT. An element x of a topological group G is topologically torsion (topologically p-torsion) if the sequence $n!x$ (resp., $p^n x$) converges to 0 in additive notation. These notions, carrying relevant information about the structure of G, were introduced independently by Braconnier and Vilenkin and played a key role in the development of the structure theory of LCA groups and profinite groups. This survey offers a general unifying approach to these notion motivated by several applications.

1. Introduction

The relation between the algebraic and the topological structure of topological groups is displayed in most intermediate way by the topologically torsion elements. This is why their role in the structure theory of topological groups can hardly be exaggerated (see [39, 49] for the Sylow subgroup of profinite groups, or [5, 6, 46] for the structure theory of locally compact abelian (briefly, LCA) groups, further generalizations were found for locally compact nonabelian groups by Glushkov, Platonov and Ushakov).

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At the first stages of the development of the theory of topological groups the attention was naturally paid to generalize the notion of a *discrete* torsion (p-torsion) group. This is why the property of being topologically torsion (defined in one way or another, see §2) was imposed first on *all* elements of the group. The "local" approach (i.e., considering single elements x of a topological group such that the sequences $n!x$ or $p^n x$ converge to 0) appeared on a later stage (cf. [22, 43, 1]). A unifying approach to both notions was proposed in [19, §4.4.2], where for every increasing sequence $\underline{m} = (m_n)$ of naturals such that $m_n | m_{n+1}$ for every n, *topologically* $\underline{m}$*-torsion* elements of a topological abelian group G were defined as those $x \in G$ such that $m_n x \to 0$ in G.

This survey covering author's talk at the 16th Summer Topology Conference (New York, July 2001) does not obviously pretend for completeness. In particular, we do not pay enough attention to the historical aspect and the classical results in the area. We concentrate instead on the "local" point of view of torsion, with particular emphasis on the calculation of the topologically $\underline{m}$-torsion elements of locally bounded groups. Some of the proofs are ommitted, but hints or due references are given in all cases. In §2 we start by giving and comparing the different notions of topological torsion, including a moderate categorical approach to torsion (in the framework of preradicals) in §2.4. It permits to obtain easy uniform proofs of the permanence properties of the subgroup of topologically torsion elements (and to look at connectedness and at (pre)compactness as a special form of torsion). In §2.3 we recall three stronger notions of "topological torsion" introduced in [22, 43] because of their numerous applications in the theory of minimal groups and elsewhere. (A Hausdorff topological group (G, τ) is *minimal* if τ is a minimal element of the partially ordered, with respect to inclusion, set of Hausdorff group topologies on the group G [41].) Along with §2.4, this subsection can be skipped at first reading if the reader is interested only in topologically $\underline{m}$-torsion elements (i.e., §§2.1-2.2 are enough for understanding the rest of the paper). In §3 we compute the topologically $\underline{m}$-torsion elements of the circle group $\mathbb{T}$, in §3.1 we consider the case of sequences m_n such that $m_n | m_{n+1}$ which allows for a complete description. In §4 some applications are given (precompact topologies on $\mathbb{Z}$ that make a given sequence converge to 0, discrete subsets of $\mathbb{Z}$ equipped with the Bohr topology, a topological game on LCA groups).

Notation and terminology. The symbols $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are used for the set of primes, the set of positive integers, the group of integers and the group of rationals, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. The cyclic group of order n is denoted by $\mathbb{Z}(n)$. The p-adic integers are denoted by $\mathbb{Z}_p$.

Let G be a group. The cyclic subgroup of G generated by $g \in G$ is denoted by $\langle g \rangle$. The set of torsion (p-torsion) elements of G is denoted by $t(G)$ (resp., $t_p(G)$) (it is a subgroup of G when G is abelian). Abelian groups are written additively. For an abelian group G and $n \in \mathbb{N}$ we put $G[n] = \{g \in G : ng = 0\}$ and $nG = \{ng : g \in G\}$. We say that G is *p-divisible* if $G = pG$, G is *divisible* if G is p-divisible for every prime p.

All topological groups, unless otherwise specified, are Hausdorff. For a topological group G and a prime number p we denote by $t_{\underline{p}}(G)$ the set of all topologically p-torsion elements of G. For a LCA group G we denote by $\widehat{G}$ the Pontryagin dual of G.

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum, so $\mathfrak{c} = 2^{\aleph_0}$. For undefined terms see [19, 26, 31].

2. The various definitions of topological torsion

2.1. Topologically torsion and topologically p-torsion elements

Vilenkin [46] called a topological group G *topologically primary* with respect to a prime p if $x^{p^n} \to 1$ for all $x \in G$. The groups with the same property are later called *topological p-groups* by Robertson [40]. Let us stress the fact that the property $x^{p^n} \to 1$ was required *for all elements x of the group.*

In algebra one checks whether an element x of an abstract group G is torsion by looking at the cyclic subgroup $\langle x \rangle$ of G. This is why it is reasonable to ask the property for a single element $x \in G$. In this form the property appeared *explicitly* in Armacost [1]:

Definition 2.1. Let G be a topological group and $p \in \mathbb{P}$. An element $x \in G$ is *topologically p-torsion* if $x^{p^n} \to 1$.

Example 2.2.

(a) For the p-adic topology $(\mathbb{Z}, \tau_p)$ every $x \in \mathbb{Z}$ is topologically p-torsion.

(b) If G is locally compact, then x is topologically p-torsion iff either x is p-torsion or $\langle x \rangle \cong (\mathbb{Z}, \tau_p)$. Indeed, assume that x is topologically p-torsion and $\langle x \rangle$ is infinite. Then $\langle x \rangle$ cannot be discrete, so that $\langle x \rangle$ must be precompact. Now for every continuous character $\chi : \langle x \rangle \to \mathbb{T}$ the element $\chi(x) \in \mathbb{T}$ is topologically p-torsion, hence p-torsion (see item (c)). Hence $\langle x \rangle$ is topologically isomorphic to an infinite subgroup of the product $\prod_p \mathbb{Z}(p^n)$, so $\langle x \rangle \cong (\mathbb{Z}, \tau_p)$.

(c) The second case in b) does not occur in a Lie group G since Lie groups (and their subgroups) have no small subgroups, so $\langle x \rangle \ncong (\mathbb{Z}, \tau_p)$. In particular, the topologically p-torsion elements of $\mathbb{T}^n$ and $\mathbb{R}^n$ are p-torsion (so 0 in $\mathbb{R}^n$). For an alternative proof see the comments after Corollary 3.2.

According to Braconnier [5], a topological group G is *primary* (relative to the prime p), if for each x in G the homomorphism $n \to x^n$ of the integers $\mathbb{Z}$ into G admits a continuous extension to the group of p-adic integers $\mathbb{Z}_p$. If G is locally compact, then this notion coincides with "topologically p-torsion" (cf. 2.13).

A topological group G is called *topologically torsion* if $x^{n!} \to 1$ for all $x \in G$. It is easy to see that every profinite group is topologically torsion. It was proved in the forties that every topologically torsion LCA group is a local direct product of topologically primary groups ([46], see also [6], [40]).

Definition 2.3. An element x of a topological group G is *topologically torsion* if $x^{n!} \to 1$ (so torsion elements are topologically torsion).

Example 2.4. [1] $\mathbb{T} = (\mathbb{R}/\mathbb{Z}, +)$ has a non-torsion, topologically torsion element (take $e = \sum_{n=0}^{\infty} 1/n! \bmod \mathbb{Z}$ and note that $n!e - [n!e] \leq \frac{2}{n+1}$, where $[y]$ denotes, as usual, the largest integer below y).

This left open the following

Problem 2.5. [1, p.34] *Determine all topologically torsion elements of* $\mathbb{T}$.

For every $x \in [0,1)$ there exist integers $0 \le c_n < n$ with

$$x = \sum_{n=0}^{\infty} c_n/n!, \tag{1}$$

such that the equality $c_n = n-1$ for all $n > n_0$ is not allowed for any $n_0 \ge 0$ (indeed, in such a case, if $c_{n_0} < n_0 - 1$, then x has also another representation as a *finite* sum, namely $x = \sum_{n=0}^{n_0-1} \frac{c_n}{u_n} + \frac{c_{n_0}+1}{u_{n_0}}$). The integers c_n are uniquely determined by these properties.

Theorem 2.6. [19, Chap.4], [17] $x \in [0,1)$ *with representation (1) gives rise to a topologically torsion element of* $\mathbb{T}$ *iff* $\lim_n c_n/n = 0$ *in* $\mathbb{T}$.

The proof exploits the inequality $c_n/n \le n!x - [n!x] \le (c_n+1)/n$.

Corollary 2.7. *The topologically torsion elements of* $\mathbb{T}$ *form a (Haar) measure zero subgroup of* $\mathbb{T}$ *of size* $\mathfrak{c}$.

It is obvious that the topologically torsion elements of $\mathbb{T}$ form a subgroup of $\mathbb{T}$. It is clear from the description given in Theorem 2.6 that it has size $\mathfrak{c}$. The measure property is due to the following result of Comfort, Trigos and Wu [9] applied to $G = \mathbb{T}$.

Lemma 2.8. ([9, Lemma 3.10]) *Let* G *be a compact abelian group. Let* $(u_n)_{n<\omega}$ *be a faithfully indexed sequence in* $\widehat{G}$. *Then* $\{x \in G : u_n(x) \to 0$ *in* $\mathbb{T}\}$ *is a Haar measure zero subgroup of* G.

2.2. Topologically $\underline{m}$-torsion elements

In order to unify the notions of topologically p-torsion element and topologically torsion element let $\mathcal{Z} = \mathbb{Z}^{\mathbb{N}}$ and consider the following

Definition 2.9. For a topological group G and $\underline{m} = (m_n) \in \mathcal{Z}$ call $x \in G$ *topologically* $\underline{m}$*-torsion* if $x^{m_n} \to 1$ in G.

For a prime p and the sequence $\underline{p} = (p^n)$ this gives "topologically p-torsion element", while the topologically torsion elements are obtained with $m_n = n!$. For sequences m_n with $m_n | m_{n+1}$ this definition can be found in [19, §4.4.2]. Denote by $t_{\underline{m}}(G) := \{x \in G : x^{m_n} \to 1\}$ the sub*set* of G of all topologically $\underline{m}$-torsion elements. It is a subgroup of G, if G is abelian.

In the next example we see that the only topologically $\underline{m}$-torsion element may be the neutral element if the sequence $\underline{m}$ is not sufficiently "lacunary" in $\mathbb{Z}$ (see also [7]). A typical example of a lacunary set in $\mathbb{Z}$ is a *Hadamard* set $\{m_1, \ldots, m_n, \ldots\}$ (such that there exists some $q > 1$ with $m_{n+1} \geq q m_n$ for every n).

Example 2.10.

(a) $t_{\underline{m}}(G)$ consists of only torsion elements if $m_n = P(n)$, where $P(x) \in \mathbb{Z}[x]$ is a *fixed* polynomial function (this can be easily proved by induction on the degree of the polynomial $P(x)$).

(b) For every $q > 1$ there exists a (Hadamard) set $\{m_1, \ldots, m_n, \ldots\}$ in $\mathbb{Z}$ such that $\frac{m_{n+1}}{m_n} \geq q$ for every $n \geq q$ and $t_{\underline{m}}(G) = \{1\}$ for every topological group G. (Take $m_{2n} = (2n-1)!!(q+1)^{n-1}$, $m_{2n+1} = (2n-1)!!(q+1)^n + 1$.)

(c) Item (a) may leave the wrong feeling that *any* sequence m_n in $\mathbb{Z}$ with polynomial growth may fail to give rise to non-zero topologically $\underline{m}$-torsion elements of $\mathbb{Z}$. For an easy counterexample take r_n to be the residue of n^2 modulo $2^{[\log n]}$. Then the sequence $m_n = n^2 - r_n$ converges to 0 in the 2-adic topology of $\mathbb{Z}$ and obviously satisfies $n^2 - n \leq m_n \leq n^2$.

(d) [50] If $\lim m_{n+1}/m_n = \infty$ then there exists a non-torsion $x \in t_{\underline{m}}(\mathbb{T})$.

(e) [38] If $m_{n+1}/m_n \geq n + 1$ there exist $\mathfrak{c}$ many real numbers x such that $m_n x \to 0 \mod \mathbb{Z}$ (i.e., $x + \mathbb{Z} \in t_{\underline{m}}(\mathbb{T})$). For a stonger result see Theorem 3.8.

We show in Example 2.15 that topologically $\underline{m}$-torsion elements can be defined also by means of an appropriate topology on $\mathbb{Z}$. This allows for an easy verification (in §2.4) of the properties of the subgroups $t_{\underline{m}}(G)$ we give now. For every $\underline{m} \in \mathcal{Z}$ every continuous homomorphism $f : G \to H$ sends $t_{\underline{m}}(G)$ into $t_{\underline{m}}(H)$. The discussion of this important property (*functoriality*), along with that of (a) and (b) below, will be deferred to §2.4.

If G is abelian then $t_{\underline{m}}(H)$ is a subgroup of G whenever H is a subgroup of G. Moreover, one can easily prove that

(a) if $M \subseteq H$, then $t_{\underline{m}}(M) = t_{\underline{m}}(H) \cap M$.

(b) if G_i is a topological group for every $i \in I$, then $t_{\underline{m}}(\prod_i G_i) = \prod_i t_{\underline{m}}(G_i)$.

Let G be a precompact abelian group. Since G can be considered as a subgroup of $\mathbb{T}^\lambda$, one gets

$$t_{\underline{m}}(G) = G \cap t_{\underline{m}}(\mathbb{T}^\lambda) = G \cap t_{\underline{m}}(\mathbb{T})^\lambda = \bigcap_{\chi \in \widehat{G}} \chi^{-1}(t_{\underline{m}}(\mathbb{T})). \qquad (2)$$

For a precompact abelian group G this formula means $a \in t_{\underline{m}}(G)$ iff $\chi(a) \in t_{\underline{m}}(\mathbb{T})$ for every $\chi \in \widehat{G}$. It reduces the computation of the subgroup $t_{\underline{m}}(G)$ for precompact abelian G to that of $t_{\underline{m}}(\mathbb{T})$. On the other hand, the computation of $t_{\underline{m}}(G)$ in an arbitrary topological groups involves only the cyclic subgroup $\langle x\rangle$. Hence this property of precompact abelian groups extends to LCA groups G. Indeed, if $x \in G$ is a non-torsion element, then the closed subgroup L of G generated by x is either discrete and $\cong \mathbb{Z}$, or L is compact (so x is a *compact* element, according to the current terminology [31]). In the first case x cannot be topologically torsion. In the second case we have the formula (2) applied to the group L (not G !!). Let $B(G)$ denote the sum of all compact subgroups of G. Then $t_{\underline{m}}(G) \leq B(G)$, i.e.,

$$t_{\underline{m}}(G) = t_{\underline{m}}(B(G)) = \bigcup\{t_{\underline{m}}(K) : K \leq G,\ \text{compact}\}.$$

Clearly, this approach extends further to locally bounded abelian groups. Recall that a topological group G is called *locally bounded*, if G has a totally bounded non-empty open set. It was proved by A. Weil [47] that a group is locally bounded iff it is isomorphic to a subgroup of a locally compact group.

Example 2.11. It is important to note that $t_{\underline{m}}(\mathbb{T}) = \mathbb{T}$ for a sequence $\underline{m}$ implies that the sequence $\underline{m}$ is eventually null. This follows from a more general result of Flor [27] (see also the end of §2.4), we offer here a short direct argument. Clearly, for such a sequence $\underline{m}$ the definition of $t_{\underline{m}}(\mathbb{T})$ implies that $m_n x \to 0$ in $\mathbb{T}$ for every $x \in \mathbb{T}$, hence $\mathbb{T} = \bigcup_n F_n$, where

$$F_n = \{y \in \mathbb{T} : (\forall k \geq n)\ m_k y \in U\}$$

and U is the neighborhood $\left[-\frac{1}{6}, \frac{1}{6}\right] + \mathbb{Z}$ of 0 in $\mathbb{T}$. By the Baire category theorem there exists n_0 such that Int $F_{n_0} \neq \emptyset$. Let V be a neighborhood of 0 such that $z + V \subseteq F_{n_0}$. Since $m_n z \to 0$ we have $m_k V \subseteq U + U$ for all $k \geq n_0$. This may occur only for finitely many integer values m_k (depending on V). As every constant subsequence of m_n must be necessarily null, we conclude that $m_n = 0$ for all sufficiently large n.

2.3. Quasi-torsion and quasi-p-torsion elements

The following *stronger* version of "topologically torsion element" of a compact abelian group G was defined in [22]: an element $x \in G$

is *quasi-torsion* (following the terminology of [43]) if x is contained in a closed totally disconnected subgroup of G. Following [22] denote by $td(G)$ the sum of all closed totally disconnected subgroups of G. This subgroup coincides with the set of all quasi-torsion elements of G. Obviously, every quasi-torsion element is topologically torsion, while $td(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, so that (according to Example 2.4 and Corollary 2.7) the former property is essentially stronger than the latter one. The functor $td(-)$ applied to the category of compact abelian groups preserves surjective continuous homomorphisms (compare with Proposition 2.18) and commutes with direct products, i.e.,

$$td(\prod_{i\in I} G_i) = \prod_{i\in I} td(G_i) \tag{2}$$

for every family of compact abelian groups G_i, $i \in I$ [22, Ex. 10] (in particular, $td(\mathbb{T}^\alpha) = (td(\mathbb{T}))^\alpha = (\mathbb{Q}/\mathbb{Z})^\alpha$ for every cardinal α [22, Th. 11]). These nice properties of the subgroups $td(G)$ were exploited essentially in [22] to resolve problems of Arhangel′skii and Stephenson [42] on products of minimal groups (the latter asked whether the power $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$ is minimal, for other solutions of this problem see also [25, 30]).

The functor td was extended to arbitrary topological groups G by Stoyanov [43] (see also [19, Chap.4]) by setting $x \in td(G)$ for some $x \in G$ iff the cyclic subgroup $\langle x\rangle$ is either finite or carries a non-discrete topology generated by open subgroups of $\langle x\rangle$ (see §2 for an equivalent definition). Clearly, every quasi-torsion element is topologically torsion. Property (2) holds for arbitrary topological abelian groups G_i.

Here is also a stronger version of "topologically p-torsion element", introduced by Stoyanov [43].

Definition 2.12. An element x of a topological group G is *quasi p-torsion* if either $\langle x\rangle$ is a finite p-group, or $\langle x\rangle \cong (\mathbb{Z}, \tau_p)$ when equipped with the induced by G topology.

For a prime p denote by $td_p(G)$ the subset of all quasi p-torsion elements of a topological group G. If G is abelian, then $td_p(G)$ is a subgroup of G. The properties of this subgroups were studied in detail by Stoyanov [43] and in [11, 12, 19]. In particular, the counterpart of (2) holds true for the subgroups $td_p(-)$.

Clearly, quasi-p-torsion elements are always topologically p-torsion. Let us see that these notions coincide in locally bounded groups.

Proposition 2.13. [19, §4.4.1, Theorem] *Let G be a locally bounded group. Then for every prime p the topologically p-torsion elements of G are quasi-p-torsion.*

For reader's convenience we give a sketch of the proof here. For $G = \mathbb{T}$ this follows from item (b) of Example 2.2. By the counterpart of (2) for the subgroups $td_p(-)$, one can extend it to all compact abelian groups (and consequently, for all precompact ones). In the general case for every non-torsion element x of G the subgroup $\langle x \rangle$ is either discrete or precompact.

Local boundedness is essential in this proposition (for an example of a topologically 2-torsion element that is not quasi 2-torsion see [19, Exer. 4.5.1 (d)], or Example 2.15 (a)).

Let G be a compact abelian group. When G is totally disconnected, then $G = \prod_p td_p(G)$ by Braconnier-Vilenkin's theorem and by the above proposition. This property extends to sequentially closed subgroups of compact abelian groups, in particular to countably compact abelian groups [24]. On the other hand, when the compact abelian group G is connected (i.e., divisible), then $td_p(G)$ is divisible and dense in G for every prime p [11]. Therefore, for an arbitrary compact abelian group G the closure $T_p(G)$ of $td_p(G)$ is a closed subgroup of G containing the connected component $c(G)$ of G and such that $T_p(G)/c(G)$ coincides with $td_p(G/c(G))$. One can prove also that $T_p(G)$ coincides with the intersection of all subgroups of G of the form nG, where $n \in \mathbb{N}$ is coprime to p (or dually, identifying G and $\widehat{\widehat{G}}$, $T_p(G)$ coincides with the annihilator of the subgroup $\bigoplus_{q \neq p} t_q(\widehat{G})$ of $\widehat{G}$; recall that $c(G)$ coincides with the annihilator of $t(\widehat{G})$ [31]). The quasi p-torsion elements have been very successfully exploited in various questions regarding the structure of topological groups and especially, the minimal topological groups ([4, 8, 11, 12, 13, 18, 19, 22, 24, 43]).

As mentioned above, the topologically torsion elements of $\mathbb{T}$ are not quasi-torsion, so that Proposition 2.13 does not extend to topologically torsion elements even for the compact group $\mathbb{T}$.

An alternative way to define "topologically torsion elements" was adopted by Stoyanov [43], who considered the subgroup $wtd(G)$ of a topological group G generated by the family $\{td_p(G)\}_{p\in\mathbb{P}}$. The elements of this subgroup were called *weakly periodic* in [43] and the following important property was established:

Theorem 2.14. *For every compact group G the subgroup $wtd(G)$ is dense.*

It was shown in [23] that $wtd(G)$ is actually *sequentially* dense in G. Since every weakly periodic element of a topological group is quasi-torsion, this theorem implies that the subgroup $td(G)$ of a compact topological group G is dense. Let us note also that while the subgroups $wtd(G)$ and $td(G)$ of a compact abelian group are always covered by *compact* subgroups, this fails to be true for the subgroup of topologically torsion elements, even in the simplest case of the group $\mathbb{T}$. For further details on these subgroups see [4, 19, 24, 43].

In the diagram below we give all implications between the six levels of "topological torsion" we introduced so far. According to Proposition 2.13 the implication (1) becomes an equivalence for locally bounded groups. The implication (2) holds only for those $\underline{m}$ that converge to 0 in the group topology ν of $\mathbb{Z}$ that has as basic open neighborhoods of 0 all non-zero subgroups of $\mathbb{Z}$. The rest of the paper (with exception of §2.4) is dedicated to the three notions from the right half of the diagram.

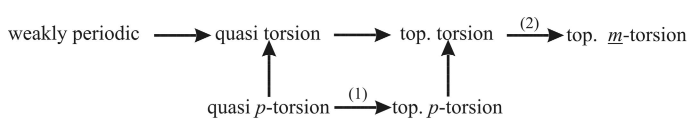

Diagram 1.

2.4. The "abstract" notion of topological torsion

A *functorial subgroup* (or *preradical*, cf. [11]-[15], [23]) in the class of topological abelian groups is defined by assigning to every group G a subgroup $r(G)$ such that $f(r(G)) \subseteq r(H)$ for every continuous homomorphism $f : G \to H$. Following the terminology from [28], a functorial subgroup $r(G)$ is said to be

(i) *hereditary*, if $x \in r(G)$ iff $x \in r(\langle x \rangle)$ (i.e., if $x \in r(G)$ can be decided for $x \in G$ within the cyclic subgroup $\langle x \rangle$ of G),

(ii) *jansian* if $r(\prod_i G_i) = \prod_{i \in I} r(G_i)$ for every family of topological abelian groups $\{G_i\}_{i \in I}$.

We shall see here that the five functorial subgroups we defined (with exception of $wtd(-)$) have both properties. In order to do it more effectively we need a unified treatment.

Using the group $(\mathbb{Z}, \tau_p)$ as a sample group for determining topological p-torsion (in the spirit of the definition of Braconnier), one can see that an element x of a topological group G is quasi p-torsion iff the unique homomorphism $f : (\mathbb{Z}, \tau_p) \to G$ with $f(1) = x$ is continuous. Indeed, if x has the latter property and $\ker f \neq 0$, then $\ker f = p^n\mathbb{Z}$ (being a closed, hence open, subgroup of $(\mathbb{Z}, \tau_p)$) and consequently $\langle x \rangle$ is a finite p-group. If $\ker f = 0$, the continuous group isomorphism $f : (\mathbb{Z}, \tau_p) \to \langle x \rangle$ is a homeomorphism by the minimality of $(\mathbb{Z}, \tau_p)$ ([19]). The other implication is obvious.

Let us see that *all* other notions of "topological torsion" given so far (with exception of $wtd(-)$)) can be obtained in this way. Namely, for a group topology τ on $\mathbb{Z}$ and a topological group G define $x \in G$ to be τ-*torsion* if the unique homomorphism f : $(\mathbb{Z}, \tau) \to G$ with $f(1) = x$ is continuous. We have just seen that "quasi p-torsion" coincides with "τ_p-torsion". Before doing this for the remaining notions of topological torsion we denote by $r_\tau(G)$ the set of all τ-torsion elements of G. This is a functorial subgroup when G is abelian. Moreover, it is easily verified that $r_\tau(G)$ satisfies both (i) and (ii). If we want τ-torsion to imply torsion in the usual sense, we have to ask τ to be finer than ν.

Example 2.15. Following [50] call a sequence $\underline{m} \in \mathcal{Z}$ a T-*sequence* if $\mathbb{Z}$ admits a Hausdorff group topology such that $m_n \to 0$ in that topology. It is clear, that if $\underline{m}$ is not a T-sequence, then every topologically $\underline{m}$-torsion element of a Hausdorff topological group is necessarily torsion. For a T-sequence $\underline{m}$ in $\mathbb{Z}$ there exists a finest group topology $\lambda_{\underline{m}}$ on $\mathbb{Z}$ such that $m_n \to 0$ in $\lambda_{\underline{m}}$ (for the important properties of this topology see [50], where this topology is denoted by $\mathbb{Z}\{m_n\}$). Clearly, $x \in G$ is topologically $\underline{m}$-torsion iff x is $\lambda_{\underline{m}}$-torsion in the above sense.

(a) Topologically p-torsion elements can be defined, in analogy to the case of quasi p-torsion ones, by means of the topology $\lambda_{\underline{p}}$ on $\mathbb{Z}$ (as before, $\lambda_{\underline{p}}$ denotes the finest group topology on

$\mathbb{Z}$ such that $p^n \to 0$), i.e., $x \in G$ is topologically p-torsion iff x is $\lambda_{\underline{p}}$-torsion. Note that every element of $(\mathbb{Z}, \lambda_{\underline{p}})$ is topologically p-torsion, whereas no element of $\mathbb{Z}$ beyond 0 is quasi p-torsion.

(b) An element x of a topological group G is quasi-torsion iff x is ν-torsion.

Let us see now that the functorial subgroups $r_\tau(G)$ can be characterized with the natural properties (i) and (ii).

Proposition 2.16. *A functorial subgroup $r(-)$ in the category of abelian topological groups coincides with $r_\tau(-)$ for some topology τ on $\mathbb{Z}$ iff $r(-)$ is hereditary and jansian.*

Proof. The necessity was already mentioned before. Assume now that $r(-)$ is hereditary and jansian. We have to find a group topology τ on $\mathbb{Z}$ such that for every abelian topological group G the subgroups $r(G)$ and $r_\tau(G)$ of G coincide. Let τ be the supremum of all, not necessarily Hausdorff, group topologies κ on $\mathbb{Z}$ with the property

$$C = \mathbb{Z}/\overline{\{0\}}^{\kappa} \quad \text{satisfies} \quad r(C) = C. \tag{3}$$

To see that this is the desired topology take a topological group G. Fix $x \in r(G)$. Then the cyclic subgroup $C = \langle x \rangle$ satisfies $r(C) = C$ by (i). Consider the initial topology κ induced on $\mathbb{Z}$ under the homomorphism $f : \mathbb{Z} \to C$, defined with $f(1) = x$. Since G is Hausdorff, $\ker f = \overline{\{0\}}^{\kappa}$ and $C \cong \mathbb{Z}/\ker f$. Hence, it follows from (3) that κ is coarser than τ, so that f is continuous when $\mathbb{Z}$ is equipped with τ, thus $x \in r_\tau(G)$. On the other hand, if $x \in G$ is τ-torsion, then $f : (\mathbb{Z}, \tau) \to G$ with $f(1) = x$ is continuous, so that the functoriality of r implies $x \in r(G)$. □

Note that if τ in the above proposition is not Hausdorff, then $r(G) = G[n]$ for every G, where $n\mathbb{Z} = \overline{\{0\}}^{\tau}$.

If (i) or (ii) fails, then this need not be true any more. For example, $wtd(-)$ fails to satisfy both (i) and (ii) (see also Example 2.17 (b) below, where only (ii) fails).

A more general approach (in module categories and without the restraint ii)) can be found in [14].

Example 2.17. (a) Call an element x of a topological group G *precompact* if the subgroup $\langle x \rangle$ is precompact. If G is complete, then every precompact element x is contained in a compact subgroup (so x is a compact element). If τ is the Bohr topology on $\mathbb{Z}$ we get as $r_\tau(G)$ the subset (subgroup if G is abelian) of precompact elements of G.

(b) An element $x \in G$ is *metrizable*, if the subgroup $\langle x \rangle$ is metrizable (Wilcox [49]). The set $M(G)$ of all metrizable precompact elements of an abelian group G is a functorial subgroup ([20]) that cannot be obtained in the way described above, i.e., as $r_\tau(G)$ for some appropriate τ.

Contrary to the case of td_p and td, the functor $t_{\underline{m}}$ need not preserve epis when restricted on the category of compact abelian groups.

Proposition 2.18. *The functor $t_{\underline{m}}$ from the category of compact abelian groups to the category of all topological abelian groups preserves epis iff $t_{\underline{m}}(\mathbb{T})$ is divisible.*

Proof. Since the epis in both categories are precisely the continuous surjective homomorphisms, we have to show that $t_{\underline{m}}$ preserves surjective morphisms iff $t_{\underline{m}}(\mathbb{T})$ is divisible.

If $t_{\underline{m}}(\mathbb{T})$ is divisible, then for a surjective continuous homomorphism $f : G \to G/N$, with G compact, consider an element $a \in t_{\underline{m}}(G/N)$. Let X be the discrete Pontryagin dual of G, so that the annihilator Y of the subgroup N is isomorphic to the Pontryagin dual of G/N. So the adjoint of f is the inclusion of Y in X. In these terms a can be considered as a character $a : Y \to \mathbb{T}$ such that $a(Y) \leq t_{\underline{m}}(\mathbb{T})$. Now the divisibility of $t_{\underline{m}}(\mathbb{T})$ permits to extend a to a character $a' : X \to \mathbb{T}$ with $a'(X) \leq t_{\underline{m}}(\mathbb{T})$, so that a' can be identified with an element of $t_{\underline{m}}(G)$. By the choice of a', we have $f(a') = a$.

Viceversa, if $t_{\underline{m}}$ preserves epis, then $t_{\underline{m}}(\mathbb{T})$ is divisible. Indeed, for every $n \in \mathbb{N}$ the multiplication by n is a surjective homomorphism $f : \mathbb{T} \to \mathbb{T}$. Hence

$$nt_{\underline{m}}(\mathbb{T}) = f(t_{\underline{m}}(\mathbb{T})) = t_{\underline{m}}(f(\mathbb{T})) = t_{\underline{m}}(\mathbb{T}). \qquad \square$$

Since the subgroup T of topologically torsion elements of $\mathbb{T}$ is not divisible (according to [21] it is not p-divisible for any p), for every $n \neq 0$ the (surjective) multiplication by n in $\mathbb{T}$ does not induce a surjective endomorphism of T.

For more examples of this categorical interpretation of "topological torsion" see [15].

Another possible categorical approach to topological torsion is offered by the so called limit laws in topological groups and topological algebras ([44]). A *sequential limit law* is a map $f : \mathbb{N} \to F(X)$ to the free group $F(X)$ over some set X. A topological group G *satisfies* the sequential limit law f provided that for every group homomorphism $\pi : F(X) \to G$ from $F(X)$ to G the sequence $\pi(f(n)) \to 1$ in G [32, 34]. Let us call the rank of $F(X)$ (i.e., the cardinality of X) *rank* of the sequential limit law f. Now it is easy to see that a topological group G satisfies a sequential limit law $f : \mathbb{N} \to \mathbb{Z}$ of rank one iff all elements of G are topologically $\underline{m}$-torsion, where $\underline{m} = (f(n))$. More generally, for a rank-one sequential limit law f and an arbitrary topological abelian group G the subgroup $t_{\underline{m}}(G)$ (with $\underline{m} = (f(n))$ as before) is the largest subgroup of G that satisfies the limit law f. In this setting, Example 2.11 follows also from the general fact that $\mathbb{T}$ admits no non-trivial sequential limit laws [32].

Let us close this discussion with a few words about *closedness* of the subgroup of (appropriately defined) topologically torsion elements. The subgroups $t_{\underline{m}}(G)$ need not be closed and the same applies to $td_p(G)$ as mentioned above. This can be considered as a disadvantage since it may lead out of the category, if one works with complete groups. On the other hand, one can always pass from a preradical $r(G)$ to its closure $\overline{r}(G) = \overline{r(G)}$ that gives the smallest closed preradical above r. There is obviously a loss of information in the passage $r \mapsto \overline{r}$. This makes clear that r has the advantage to be richer from the point of view of information.

3. Computation of $t_{\underline{m}}(\mathbb{T})$

In the previous section we showed that the computation of the subgroups $t_{\underline{m}}(G)$ can be reduced to that of $t_{\underline{m}}(\mathbb{T})$, whenever G is a locally bounded group. Here we propose various ways how to compute $t_{\underline{m}}(\mathbb{T})$ for certain sequences $\underline{m}$.

3.1. A complete description of $t_{\underline{m}}(\mathbb{T})$ when $m_n|m_{n+1}$

Let $b_n = \frac{m_{n+1}}{m_n} \in \mathbb{Z}$ and assume $b_n > 1$ for every n. Then again for every $x \in [0, 1)$ there exist integers $0 \leq c_n < b_n$ for every n such that

$$x = \sum_{n=0}^{\infty} \frac{c_n}{m_n}, \tag{4}$$

and $c_n < b_n - 1$ for infinitely many n. The integers c_n are uniquely determined with these properties.

Let supp $x = \{n \in \mathbb{N} : c_n \neq 0\}$. Call an infinite set A of naturals *b-bounded* if the sequence $\{b_n : n \in A\}$ is bounded.

Theorem 3.1. *For $x \in [0, 1)$ with representation (4) the following are equivalent:*

(a) *x gives rise to a topologically $\underline{m}$-torsion element of $\mathbb{T}$;*
(b) *if supp x is infinite, then for every infinite subset $A \subseteq$ supp x the following holds true:*
 (b1) *if A is b-bounded, then*

$$\lim_{n\in A} \frac{c_n + 1}{b_n} = \lim_{n\in A} \frac{c_{n+1} + 1}{b_{n+1}} = 1.$$

 (b2) *if $\lim_{n\in A} b_n = \infty$, then $\lim_{n\in A} \frac{c_n}{b_n} = 0$ (or, equivalently, $\lim_{n\in A} \frac{c_n+1}{b_n} = 0$) in $\mathbb{T}$.*

Note that the limit in (b1) is taken in $\mathbb{R}$, while that in (b2) – in $\mathbb{T}$. An incomplete proof of a different version of this theorem can be found in [19], a complete proof of the theorem is given in [17].

Due to its general character, Theorem 3.1 is somewhat heavy to apply. This is why we give now a series of corollaries where, under additional natural conditions, the description of the topologically $\underline{m}$-torsion elements of $\mathbb{T}$ becomes more transparent.

Corollary 3.2. *Let $x \in [0, 1)$ and let $A =$ supp x. If A is b-bounded, then $x \in t_{\underline{m}}(\mathbb{T})$ iff $c_n = 0$ for almost all $n \in \mathbb{N}$ (in particular, x is torsion).*

Proof. Assume that A is infinite. Now $\lim_{n\in A} \frac{c_n+1}{b_n} = 1$ implies $c_n = b_n - 1$ for almost all $n \in A$ as b_n is bounded. Next, $\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1$ implies that $c_{n+1} \neq 0$ for sufficiently large n (as $b_{n+1} \geq 2$), hence $n+1 \in A$ for all sufficiently large $n \in A$. This means that A is co-finite in $\mathbb{N}$ and hence cofinitely many $c_n = b_n - 1$, a contradiction. □

A similar result can be found also in [19, Chap. 4].

In particular, this corollary implies that all topologically p-torsion elements of $\mathbb{T}$ are p-torsion for every prime p (cf. Example 2.2).

In the next corollary (proved in [17]) we determine when $x \in t_{\underline{m}}(\mathbb{T})$ for $x \in [0,1)$ such that supp x splits into a b-bounded part and a part where $b_n \to \infty$.

Corollary 3.3. *Let $x \in [0,1)$. Suppose that $A = supp\, x \setminus I$ is b-bounded for some set $I \subseteq \mathbb{N}$ that is either finite or satisfies* $\lim_{n\in I} b_n = \infty$. *Then $x \in t_{\underline{m}}(\mathbb{T})$ iff the following conditions hold:*

(a) *$c_n = b_n - 1$ for almost all $n \in A$;*
(b) *if A is infinite, then also $C = \{n \in A, n+1 \notin A\}$ is infinite and*

$$\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1 \quad \textit{and} \quad \lim_{n\in C} \frac{c_{n+1}}{b_{n+1}} = 1;$$

(c) *if I is infinite, then* $\lim_{n\in I} \frac{c_n}{b_n} = 0$ *in* $\mathbb{T}$.

Note that (b) implies $\lim_{n\in C} b_{n+1} = \infty$ when $x \in t_{\underline{m}}(\mathbb{T})$ and A is infinite.

Corollary 3.4. *Let $x \in [0,1)$ and let $I = supp\ x$. If* $\lim_{n\in I} b_n = \infty$, *then $x \in t_{\underline{m}}(\mathbb{T})$ iff* $\lim_n \frac{c_n}{b_n} = 0$ *in* $\mathbb{T}$.

Proof. Follows directly from Corollary 3.3 with $A = \emptyset$. For a direct proof exploit the inequality $c_n/b_n \le m_n x - [m_n x] \le (c_n+1)/b_n$. □

Corollary 3.5. *If b_n is not bounded, then* $|t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$.

Proof. Let $I \subseteq \mathbb{N}$ be infinite with $b_n \to \infty$ when $n \in I$. For every $\xi = (\xi_n) \in \{0,1\}^{\mathbb{N}}$ consider $x_\xi \in [0,1)$ having a representation (3) with integers c_n satisfying $c_n = 0$ when $n \notin I$ and $c_n = \xi_n$ for all $n \in I$. By Corollary 3.4, the hypothesis $b_n \to \infty$ for $n \in I$ implies that each $x_\xi \in [0,1)$ gives rise to a topologically $\underline{m}$-torsion element of $\mathbb{T}$ (as $\lim_n c_n/b_n = 0$ in $\mathbb{T}$). Obviously, there are $\mathfrak{c}$ many elements x_ξ of $\mathbb{T}$ of this form. □

Example 2.10 (b) (with $t_{\underline{m}}(\mathbb{T}) = 0$ and $\frac{m_{n+1}}{m_n}$ unbounded, but not an integer) shows that the hypothesis $m_n | m_{n+1}$ cannot be removed for the proof of the implication d) ⇒ a) in the next theorem.

Corollary 3.6. *For a strictly increasing $\underline{m} \in \mathbb{Z}^{\mathbb{N}}$ with $m_n | m_{n+1}$ for each n the following are equivalent:*

(a) $|t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$;
(b) $t_{\underline{m}}(\mathbb{T})$ *is uncountable;*
(c) $t_{\underline{m}}(\mathbb{T})$ *contains non-torsion elements;*
(d) $b_n = \frac{m_{n+1}}{m_n}$ *is unbounded;*
(e) $|t_{\underline{m}}(\mathbb{T})/pt_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$ *for every prime* p.

Proof. The implications (e) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. The implication (c) $\Rightarrow$ (d) follows from Corollary 3.2. The implication (d) $\Rightarrow$ (a) follows from Corollary 3.5. The implication (d) $\Rightarrow$ (e) is proved in [21]. □

Remark 3.7. The equivalence a) $\Leftrightarrow$ b) of Corollary 3.6 can be proved without the assumption $m_n|m_{n+1}$ for each n (as $t_{\underline{m}}(\mathbb{T})$ is a Borel set of $\mathbb{T}$). We shall see later that in general the implication c) $\Rightarrow$ d) fails (cf. Example 3.10 (a)), while b) $\Rightarrow$ d) remains valid (cf. Theorem 3.9). Therefore the implication c) $\Rightarrow$ b) fails too in general.

In connection to Proposition 2.18 we shall briefly discuss now when $t_{\underline{m}}(\mathbb{T})$ is divisible. It follows from Corollary 3.6 (e) that when $m_n|m_{n+1}$ for each n and b_n is unbounded, then $t_{\underline{m}}(\mathbb{T})$ is not divisible. On the other hand, it is obvious that when $t_{\underline{m}}(\mathbb{T})$ contains $\mathbb{Q}/\mathbb{Z}$, then non-divisibility of $t_{\underline{m}}(\mathbb{T})$ yields $t_{\underline{m}}(\mathbb{T}) \not\subseteq \mathbb{Q}/\mathbb{Z}$ and consequently, under the condition $m_n|m_{n+1}$ for each n, b_n is unbounded by virtue of the above corollary. The restraint $t_{\underline{m}}(\mathbb{T}) \geq \mathbb{Q}/\mathbb{Z}$ in the last argument cannot be removed. Indeed, for distinct primes p, q and the sequence $m_n = pq^n$ one has $t_{\underline{m}}(\mathbb{T}) = \mathbb{Z}(p) \times \mathbb{Z}(q^\infty)$ non-divisible, while $\frac{m_{n+1}}{m_n} = q$ is bounded.

3.2. When $m_n|m_{n+1}$ fails

It turns out that the assymptotic behavior of the ratio $\frac{m_{n+1}}{m_n}$ may determine the size of the subgroup $t_{\underline{m}}(\mathbb{T})$ in two important cases. For $t = 0$ the next theorem is proved in [2]:

Theorem 3.8. *If* $\lim_n \frac{m_{n+1}}{m_n} = \infty$ *then for every* $t \in \mathbb{T}$ *there exist* $\mathfrak{c}$ *many elements* $x \in \mathbb{T}$ *such that* $m_n x \to t$ *in* $\mathbb{T}$. *In particular,* $|t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$.

Proof. Assume without loss of generality that $m_1 \geq 3$ and $\varepsilon_n = m_n/m_{n+1} \leq 1/4$ for every n. Then build for every $\xi = (\xi_n) \in \{0,1\}^{\mathbb{N}}$, a descending chain of intervals $I_k(\xi) = [u_k(\xi), v_k(\xi)]$ $(k \in \mathbb{N})$, such that:

(1) the length $\lambda(I_k(\xi)) = 3/m_{k+1}$;
(2) $\exists b_k(\xi) \in t + \mathbb{Z}$ such that $I_k(\xi)$ coincides with $[b_k(\xi)/m_k - \varepsilon_k/m_k, b_k(\xi)/m_k + \varepsilon_k/m_k]$;
(3) for $\eta \neq \xi$ there exists $k \in \mathbb{N}$ with $I_k(\xi) \cap I_k(\eta) = \emptyset$.

By (1) $\lim_k \lambda(I_k(\xi)) = 0$ for every ξ, hence there exists a (unique) point $x_\xi \in \bigcap_k I_k(\xi)$. By (2) one has $b_k(\xi) - \varepsilon_k < m_k x_\xi < b_k(\xi) + \varepsilon_k$. This proves that $\lim_k m_k x_\xi = t$ in $\mathbb{T}$ (in particular, $x_\xi \in t_{\underline{m}}(\mathbb{T})$ when $t = 0$) for every ξ. To see that there exist $\mathfrak{c}$ many elements $x \in \mathbb{T}$ with $m_n x \to t$ it suffices to observe that $x_\eta \neq x_\xi$ for $\eta \neq \xi$ by (3). □

If one relaxes the condition $\lim_n \frac{m_{n+1}}{m_n} = \infty$ in Theorem 3.8, then no $x \in \mathbb{T}$ with $m_n x \to t$ need exist (take $m_n = p^n$ for a fixed prime p and any non-torsion $t \in \mathbb{T}$).

The following recent general result from [2] shows that the implication b) ⇒ d) in Corollary 3.6 holds without the restraint $m_n | m_{n+1}$:

Theorem 3.9. ([2]) *If the ratio $\frac{m_{n+1}}{m_n}$ is bounded for a sequence $\underline{m}$, then $t_{\underline{m}}(\mathbb{T})$ is countable.*

Easy examples show that unboundedness of the ratio $\frac{m_{n+1}}{m_n}$ need not imply any definite assertion on the size of the subgroup $t_{\underline{m}}(\mathbb{T})$ when $m_n | m_{n+1}$ fails (Example 2.10 (b)). Hence the implication in the above theorem cannot be inverted.

Now we consider strictly increasing sequences $\underline{m} \in \mathbb{Z}^{\mathbb{N}}$ with another relevant divisiblity condition, namely

$$m_n | m_{n+1} - m_{n-1} \text{ for each } n. \tag{5}$$

This means that $m_{n+1} = a_n m_n + m_{n-1}$ for some positive $a_n \in \mathbb{Z}$. Now the continued fraction

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{6}$$

defines an *irrational* number α that satisfies $m_n \alpha \to 0$ in $\mathbb{T}$. Indeed, the n-th approximation of α is p_n/m_n with $|\alpha - p_n/m_n| < 1/m_n^2$, hence $|m_n\alpha - p_n| < 1/m_n$. On the other hand, every irrational number $\alpha \in (0,1)$ has a continuous fraction (6), hence α determines a topologically $\underline{m}$-torsion element of $\mathbb{T}$. This shows that *every element of $\mathbb{T}$ is topologically $\underline{m}$-torsion for an appropriate non-trivial* *sequence $\underline{m}$ satisfying (5).* However this cannot be extended to *every* power of $\mathbb{T}$. Indeed, one can choose an approariate element $x \in G = \mathbb{T}^{\mathfrak{c}}$ (namely, the "diagonal element" of $\mathbb{T}^{\mathfrak{c}} = \mathbb{T}^{\mathbb{T}}$), such that if $x \in t_{\underline{m}}(G)$, then $\underline{m} \in \mathcal{Z}_0$ by Example 2.11 (in other words, the ciclic subgroup $\langle x \rangle$ is $\mathfrak{t}$-dense in G in terms of §4).

Example 3.10. Let $b_n = m_{n+1}/m_n$ and note that this number is not an integer. Under condition (5) the torsion elements in $t_{\underline{m}}(\mathbb{T})$ can be only finitely many and hence form a cyclic subgroup. This subgroup is trivial precisely when m_1 and m_2 are coprime.

(a) It is easy to see, from the equality $b_n = a_n + 1/b_{n-1}$, that b_n is bounded iff a_n is bounded (while $b_n \to \infty$ iff $a_n \to \infty$). In particular, for every bounded sequence (a_n) we get an example of a sequence $\underline{m}$ with *bounded* b_n and a non-torsion element $\alpha \in t_{\underline{m}}(\mathbb{T})$.

(b) When a_n is eventually constant we get a sequence $\underline{m}$ with $b_n \to \frac{a+\sqrt{a^2+4}}{2}$. As far as the subgroup $t_{\underline{m}}(\mathbb{T})$ is concerned, one can assume that a_n is constant. Conversely, if b_n converges then the sequence a_n is eventually constant. Indeed, let $\lim_n b_n = \rho$. Then a_n is bounded by (a). Since a_n can take only finitely many values, there is one, say c, taken for infinitely many indices n_k. Since $\lim b_{n_k} = \lim b_{n_k-1} = \rho$, we get the equation $\rho = c + 1/\rho$. This determines $\rho = \frac{c+\sqrt{c^2+4}}{2}$. To prove that $a_n \to c$ it suffices to show that if $a_{n_m} = d$ for an infinite set of indices n_m, then $d = c$. Indeed, as above we conclude that $\rho = \frac{d+\sqrt{d^2+4}}{2}$. The function $f(x) = \frac{x+\sqrt{x^2+4}}{2}$ is strictly monotone for positive x, hence $d = c$. Therefore, all a_n coincide with c for $n \geq n_0$.

(c) In the general case of a bounded sequence (a_n) one should consider the cluster points of the sequence b_n in order to obtain useful information about $t_{\underline{m}}(\mathbb{T})$ (compare with $\lim_n b_n \in t_{\underline{m}}(\mathbb{T})$ in the case when b_n converges).

With $a = 1$ in Example 3.10 (b) we get the Fibonacci sequence $\underline{f} = (f_n)$ defined by $f_{n+1} = f_n + f_{n-1}$.

Question 3.11. Describe $t_{\underline{f}}(\mathbb{T})$, where $\underline{f}$ is the Fibonacci sequence.

4. Some applications

4.1. The finest precompact topology on $\mathbb{Z}$ with $m_n \to 0$

A counterpart of the notion of a T-sequence was proposed in [2] (see also Example 2.15): a sequence (m_n) in an abelian group G is a *TB-sequence* if there exists a precompact group topology τ on G such that $m_n \to 0$ in (G, τ). The next lemma offers an easy description of the TB-sequences of $\mathbb{Z}$.

Lemma 4.1. [2] *A fixed sequence $\underline{m} = (m_n)$ in $\mathbb{Z}$ is a TB-sequence iff $t_{\underline{m}}(\mathbb{T})$ is infinite.*

Proof. Since precompact group topologies on $\mathbb{Z}$ are determined by their continuous characters, one will have $\tau = T_H$, where T_H is the precompact group topology on $\mathbb{Z}$ generated by the subgroup H of the character group $\mathbb{T} = \widehat{\mathbb{Z}}$ [10]. As $m_n \to 0$ in $(\mathbb{Z}, T_H)$ we conclude that $m_n\alpha \to 0$ in $\mathbb{T}$ for every $\alpha \in H$. Hence $H \leq t_{\underline{m}}(\mathbb{T})$. Hausdorffness of T_H implies that H separates the points of $\mathbb{Z}$, hence H is infinite. Therefore, $t_{\underline{m}}(\mathbb{T})$ is infinite too. Viceversa, if $H = t_{\underline{m}}(\mathbb{T})$ is infinite, then T_H is Hausdorff and obviously $m_n \to 0$ in $(\mathbb{Z}, T_H)$ by the definition of T_H. □

The topologies $\tau = T_H$ as in the lemma correspond to infinite (Haar measure 0) subgroups H of $t_{\underline{m}}(\mathbb{T})$ (cf. Lemma 2.8). Clearly, $\tau_{\underline{m}} := T_{t_{\underline{m}}(\mathbb{T})}$ is the *finest* precompact topology on $\mathbb{Z}$ with $m_n \to 0$. Since always $w(\mathbb{Z}, T_H) = |H|$, we get $w(\mathbb{Z}, \tau_{\underline{m}}) = |t_{\underline{m}}(\mathbb{T})|$.

Theorem 4.2. (Raczkowski [37]) *If $\frac{m_{n+1}}{m_n} \geq n+1$, then there exists a precompact group topology τ of weight $\mathfrak{c}$ on $\mathbb{Z}$ such that $m_n \to 0$ in $(\mathbb{Z}, \tau)$.*

Applying Theorems 3.6, 3.8 and the above argument with $w(\mathbb{Z}, \tau_{\underline{m}}) = |t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$ one can conclude (cf. [2]) that

- if $\frac{m_{n+1}}{m_n} \to \infty$, then there exists a finest precompact group topology $\tau_{\underline{m}}$ of weight $\mathfrak{c}$ on $\mathbb{Z}$ such that $m_n \to 0$ in $(\mathbb{Z}, \tau_{\underline{m}})$;
- if $m_n | m_{n+1}$ and $\underline{m}$ is strictly increasing, then the following are equivalent:
  - (i) $\frac{m_{n+1}}{m_n}$ is bounded;
  - (ii) $w(\mathbb{Z}, \tau_{\underline{m}}) < \mathfrak{c}$;
  - (iii) $\tau_{\underline{m}}$ is metrizable;
  - (iv) $\tau_{\underline{m}}$ has a local base at 0 of open subgroups.

Therefore the restraint $\frac{m_{n+1}}{m_n} \geq n+1$ in Theorem 4.2 can be replaced by $\frac{m_{n+1}}{m_n} \to \infty$. On the other hand, if the latter condition fails, then $t_{\underline{m}}(\mathbb{T})$ may be trivial (see Example 2.10 where $\frac{m_{2n+1}}{m_{2n}}$ is bounded whereas $\frac{m_{2n}}{m_{2n-1}} \to \infty$).

4.2. The finest precompact topology on $\mathbb{Z}$

We denote by $\mathbb{Z}^{\#}$ the group $\mathbb{Z}$ equipped with the Bohr topology, namely the initial topology of all homomorphisms $\mathbb{Z} \to \mathbb{T}$. Kunen and W. Rudin [33] proved that $\mathbb{Z}^{\#}$ contains a subset A such that the only limit point of $A-A$ in $\mathbb{Z}^{\#}$ is 0 and $A+A$ has no limit points in $\mathbb{Z}^{\#}$. Actually, every Hadamard set A of $\mathbb{Z}$ has these properties [33, Th. 2.3], but this is not relevant for the proof of their main theorem which requires only the *existence* of such a set A. As an application of Theorem 3.8 we show here an easy way to construct such a set A with stronger properties.

Theorem 4.3. *Let $A = \{m_n : n \in \mathbb{N}\}$ be a subset of $\mathbb{Z}$ enumerated in a strictly monotone way, such that $m_0 > 0$ and $lim_n m_{n+1}/m_n = \infty$. Then there exist $\mathfrak{c}$ many elements $\theta \in \mathbb{T}$ such that for the topology τ induced on $\mathbb{Z}$ by the embedding $\chi : \mathbb{Z} \to \mathbb{T}$, defined by $\chi(n) = n\theta$, the following hold*

(a) $A^{(k)} = \underbrace{A + \ldots + A}_{k}$ *has no limit points in $(\mathbb{Z}, \tau)$ for every $k \in \mathbb{N}$,*

(b) *the only limit point of $A - A$ in $(\mathbb{Z}, \tau)$ is 0.*

Proof. Fix a non-torsion element $t \in \mathbb{T}$. By Theorem 3.8 there exist $\mathfrak{c}$ many elements $\theta \in \mathbb{T}$ satisfying $m_n\theta \to t$ in $\mathbb{T}$. Choose among them $\mathfrak{c}$ many non-torsion ones with $\langle t \rangle \cap \langle \theta \rangle = 0$. Then $\theta \in \mathbb{T}$ has the required properties. □

Corollary 4.4. *Let $A = \{m_n : n \in \mathbb{N}\}$ be a subset of $\mathbb{Z}$ enumerated in a strictly monotone way, such that $m_0 > 0$ and $lim_n m_{n+1}/m_n = \infty$. Then:*

(a) $A^{(k)} = \underbrace{A + \ldots + A}_{k}$ *has no limit points in $\mathbb{Z}^{\#}$ for every $k \in \mathbb{N}$,*

(b) *the only limit point of $A - A$ in $\mathbb{Z}^{\#}$ is 0.*

Question 4.5. Let $A \subseteq \mathbb{Z}$ be a Hadamard set of $\mathbb{Z}$.

(a) Is then $A + A + A$ closed discrete in $\mathbb{Z}^{\#}$?

(b) Is the set $A^{(k)}$ closed and discrete in $\mathbb{Z}^{\#}$ for every k?
(c) In case b) has negative answer, is there any relation between the ratio of A and the maximum number k such that $A^{(k)}$ is closed discrete ?

4.3. The Galois closure associated to topological torsion

Here we define for every subset H of a topological group G a largest subset $\mathfrak{t}(H)$ of G containing H that has "the same topologically $\underline{m}$-torsion elements as H" in the following sense:

$$\mathfrak{t}(H) = \bigcap\{x \in G : (\forall \underline{m} \in \mathcal{Z})[(\forall h \in H) h^{m_n} \to 1] \Rightarrow x^{m_n} \to 1]\},$$

in other words,

$$\mathfrak{t}(H) = \bigcap_{\underline{m}\in\mathcal{Z}} \{t_{\underline{m}}(G) : H \leq t_{\underline{m}}(G)\}.$$

We say

- H is $\mathfrak{t}$-*closed* if $H = \mathfrak{t}(H)$;
- H is $\mathfrak{t}$-*dense* if $\mathfrak{t}(H) = G$.

Consequently, for a topological group G and $\underline{m} \in \mathcal{Z} = \mathbb{Z}^{\mathbb{N}}$ the sub*set* $t_{\underline{m}}(G)$ as a typical $\mathfrak{t}$-closed subobject. It is easy to see that $\mathfrak{t}$ is a monotone idempotent closure operator in the power set of G (the lattice of all subgroups of G, in case G is abelian). One can define an appropriate Galois correspondence between subsets of G and subsets of $\mathcal{Z} = \mathbb{Z}^{\mathbb{N}}$, so that the $\mathfrak{t}$-closed subsets of G are precisely the Galois closed subobjects of G. Here we do not give explicitly this correspondence. The reader can find details concerning this correspondence and proofs of the theorems given below in [16]. Let us mention here that $\mathfrak{t}$-closedness and $\mathfrak{t}$-density have nothing to do with closedness and density w.r.t. the *topology* of the group G (cf. Example 4.11).

For a cyclic subgroup $\langle x\rangle$ of G write for brevity $\mathfrak{t}(x)$ instead of $\mathfrak{t}(\langle x\rangle)$. If G is abelian then $\mathfrak{t}(H)$ is a subgroup of G whenever H is a subgroup of G. Moreover, if $M \subseteq H$, then $\mathfrak{t}_H(M) = \mathfrak{t}_G(M) \cap H$. If H_i is a subgroup of G_i for every $i \in I$, then $\mathfrak{t}(\prod_i H_i) = \prod_i \mathfrak{t}(H_i)$.

Example 4.6. All subgroups of $\mathbb{Q}/\mathbb{Z}$ are $\mathfrak{t}$-closed, when $\mathbb{Q}/\mathbb{Z}$ is equipped with any Hausdorff group topology.

Let us see now an interesting property of the circle group $\mathbb{T}$.

Theorem 4.7. *All cyclic subgroups of* $\mathbb{T}$ *are* $\mathfrak{t}$*-closed.*

Proof. If $o(x) = n$ in $\mathbb{T}$, then $\mathfrak{t}(x) = \mathbb{T}[n] \cong \mathbb{Z}(n)$ so $\mathfrak{t}(x) = \langle x \rangle$ is the group of n-th roots of unity.

If $o(x) = \infty$ and $y \in \mathfrak{t}(x)$ consider the algebraic homomorphism $f : \langle x \rangle \to \langle y \rangle$ defined by $f(x) = y$. Now $y \in \mathfrak{t}(x)$ means $y^{m_n} \to 1$ in $\mathbb{T}$ if $x^{m_n} \to 1$ in $\mathbb{T}$. This proves that f is continuous at 1. So f is a continuous homomorphism, hence f extends to a continuous endomorphism of the completions $\widehat{\langle x \rangle} = \mathbb{T} \xrightarrow{\widehat{f}} \mathbb{T} = \widehat{\langle y \rangle}$. Since the continuous automorphism of $\mathbb{T}$ have the form $x \mapsto x^n$ with $n = \pm 1$, we conclude $y \in \langle x \rangle$. This proves $\mathfrak{t}(x) = \langle x \rangle$. □

Let $\mathcal{G}_c = \{G : \text{ every cyclic } \langle x \rangle \leq G \text{ is } \mathfrak{t}\text{-closed}\}$. Hence, $\mathbb{T} \in \mathcal{G}_c$ (by Theorem 4.7) and $\mathbb{Q}/\mathbb{Z} \in \mathcal{G}_c$ along with all its subgroups (by Example 4.6). The following surprising theorem shows that these are the only locally compact groups with this property.

Theorem 4.8. [16] *If $G \in \mathcal{G}_c$ is locally compact, then either $G \cong \mathbb{T}$ or G is isomorphic to a subgroup of the* discrete *group $\mathbb{Q}/\mathbb{Z}$.*

Note that according to this theorem $G \in \mathcal{G}_c$ alone implies G is *abelian* for a locally compact group G (for the failure of some familiar LCA groups to satisfy $G \in \mathcal{G}_c$ see Example 4.11).

Following the line of the proof of Theorem 4.7 one can prove that all *countable* subgroups of $\mathbb{T}$ are $\mathfrak{t}$-closed ([16]). Moreover, there are many uncountable proper $\mathfrak{t}$-closed subgroups of $\mathbb{T}$ according to Corollary 3.6 (indeed, the $\mathfrak{t}$-closed subgroups $t_{\underline{m}}(\mathbb{T})$ are proper and have size $\mathfrak{c}$ when $\frac{m_{n+1}}{m_n}$ is an unbounded sequence of integers). Nevertheless, the following question remains open.

Question 4.9. Are *all* subgroups of $\mathbb{T}$ $\mathfrak{t}$-closed?

This question should be compared to another question raised by Raczkowski [38, Question 1] that can be formulated as follows in these terms:

Question 4.10. Is it true that a measure zero subgroup of $\mathbb{T}$ is never $\mathfrak{t}$-dense ?

In other words, does there exists a measure-zero subgroup H of $\mathbb{T}$ such that H is not contained in any subgroup $t_{\underline{m}}(\mathbb{T}) \neq \mathbb{T}$? Note that by Lemma 2.8 every proper subgroup of $\mathbb{T}$ of the form $t_{\underline{m}}(\mathbb{T})$ has measure zero.

Now we consider the locally compact abelian groups with $\mathfrak{t}$-dense cyclic subgroups. Note that:

- $\mathfrak{t}(x) = \mathbb{R}$ for all $x \in \mathbb{R} \setminus \{0\}$,
- $\mathfrak{t}(x) = \mathbb{Z}_p$ for $x \in \mathbb{Z}_p \setminus \{0\}$.

Let $\mathcal{G}_d = \{G : \text{ every cyclic } \langle x \rangle \leq G \text{ is } \mathfrak{t}\text{-dense}\}$. It is clear, that $G \in \mathcal{G}_d$ if and only if G has no proper $\mathfrak{t}$-closed subgroups. It was kindly noted by Markus Stroppel that the class $\mathcal{G}_d$ contains the class of topological groups G such that the group of continuous automorphisms of G acts transitively on $G \setminus \{1\}$. However, the latter class is much smaller than $G \in \mathcal{G}_d$ even in the framework of compact abelian groups as the second example above shows (see also the examples below).

Example 4.11.

(1) $G = \mathbb{R}^n \times D \in \mathcal{G}_d$ for every discrete torsion-free group D and every $n \in \mathbb{N}$;
(2) $G = \mathbb{Z}_p^\alpha \in \mathcal{G}_d$ and $G = \mathbb{Z}(p)^\alpha \times (\bigoplus_\beta \mathbb{Z}(p)) \in \mathcal{G}_d$ for all cardinals α, β, where $\mathbb{Z}_p^\alpha$ and $\mathbb{Z}(p)^\alpha$ carry their compact product topology, while $\bigoplus_\beta \mathbb{Z}(p)$ is discrete;
(3) $G \in \mathcal{G}_d$ if G is torsion-free and G has an open subgroup $K \cong \mathbb{Z}_p^\alpha$ for some α, such that G/K is torsion (so that $\mathbb{Z}_p^\alpha \leq G \leq D(\mathbb{Z}_p^\alpha) = \mathbb{Z}_p^\alpha \otimes_\mathbb{Z} \mathbb{Q}$).

Theorem 4.12. ([16]) *If $G \in \mathcal{G}_d$ is locally compact abelian, then one of the cases (1)-(3) in Example 4.11 occurs.*

For compact abelian groups G distinct from $\mathbb{T}$ one can try to obtain a countepart of Theorem 4.7 by an appropriate change of the domain of the sequences (m_n). Indeed, for the group $\mathbb{T}$ the integers $\mathbb{Z} = \widehat{\mathbb{T}}$ arise as a natural object via the Pontryagin duality. For an arbitrary compact abelian group G one may consider a sequence $\underline{u} = (u_n)$ in the discrete Pontyagin dual $\widehat{G}$ of G. Now the subgroup $t_{\underline{u}}(G)$ can be defined as the set of elements $x \in G$ such that $u_n(x) \to 0$ in $\mathbb{T}$ (as in the statement of Theorem 2.8). We do not know if this countepart of Theorem 4.7 remains true for every compact abelian group G:

Question 4.13. Is every cyclic subgroup $\langle x \rangle$ of a compact abelian group G intersection of the subgroups $t_{\underline{u}}(G)$ containing x (i.e., if for some $y \in G$ $u_n(y) \to 0$ in $\mathbb{T}$ for every sequence u_n in $\widehat{G}$, such that $u_n(x) \to 0$ in $\mathbb{T}$, is it true that $y = kx$ for some $k \in \mathbb{Z}$)?

The answer to this question is positive for the tori $\mathbb{T}^n$, for all pro-p abelian groups and for all torsion-free profinite abelian groups.

This approach will necessarily lead to a different Galois closure, but we are not going to discuss this matter here. The following example showing that the Galois closure $\mathfrak{t}$ has not good functorial properties may serve as a motivation to carry out such a programme.

Example 4.14. Let $f : \mathbb{R} \to \mathbb{T}$ be the canonical quotient map. Then for every irrational number $\alpha \in \mathbb{R}$ the cyclic subgroup $H = \langle\alpha\rangle$ is $\mathfrak{t}$-dense in $\mathbb{R}$, while $f(H)$ is $\mathfrak{t}$-closed in $\mathbb{T}$ by Theorem 4.7. Hence $f(\mathfrak{t}(H)) \not\subseteq \mathfrak{t}(f(H))$.

Note added July 2002. Questions 4.9 and 4.10 are answered negatively in [2], under the assumption of Martin Axiom. A complete descrtiption of the subgroups $t_{\underline{m}}(\mathbb{T})$, under confition (5), is given in [3]. This answers completely Question 3.11 (the group $t_{\underline{f}}(\mathbb{T})$ is shown to be infinite cyclic).

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